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On the Poincaré series associated to the p-adic points on a curve

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1. Introduction. Let p denote a prime number, $\boldsymbol{Z}_p$ the ring of p-adic integers and $\boldsymbol{Q}_p$ the field of p-adic numbers. Let $f(x, y)$ denote a p-adic power series in two variables x and y over $\boldsymbol{Z}_p$. For $n \in \boldsymbol{N}$ we define N_n to be the number of elements in the set

$$\{(x, y) \bmod p^n |\ (x, y) \in (\boldsymbol{Z}_p)^2 \text{ and } f(x, y) = 0\}.$$

We define the following Poincaré series:

$$P(T) = \sum_n N_n T^n.$$

We will prove that $P(T)$ is a rational function and give a set of candidates for the poles of $P(T)$ for special cases of the singularity of $f(x, y)$, namely:

1. if the singularity is analytically irreducible, and
2. if the singularity is non-degenerate with respect to its Newton diagram.

The concept of non-degeneracy is introduced by Varchenko in [22]. We will give its short description in the following paragraph. Let

$$f(x, y) = \sum_{n,m} a_{(n,m)} x^n y^m.$$

Suppose $f(0, 0) = 0$ and define

$$S = \{(n, m) \in \boldsymbol{N}^2 |\ a_{(n,m)} \neq 0\}.$$

Then we define *Newton's polyhedron* as the convex hull of the set

$$\bigcup_{n,m \in S} \{(n, m) \in \boldsymbol{N}^2\}.$$

Newton's diagram is defined as the union of all compact faces of that polyhedron, and we denote it by $\Delta(f)$. Let γ be a face of $\Delta(f)$, and

$$f_\gamma = \sum_{(n,m) \in \gamma} a_{(n,m)} x^n y^m.$$

If, for every face γ, there exists no $(x, y) \in (C_p^\times)^2$ (C_p being the completion of the algebraic closure of Q_p) such that the two partial derivatives

$$f'_{y,x}(x, y) = 0 \quad \text{and} \quad f'_{y,y}(x, y) = 0$$

then we call the point with coordinates (0, 0) a singularity which is *non-degenerate with respect to its Newton diagram.* We prove the following theorems:

THEOREM 1. *Let* $f(x, y) \in Z_p[[x, y]]$ *converge over* Z_p^2. *Suppose* f *has only a singularity at* (0, 0) *and is analytically irreducible. Then* $P(T)$ *is a rational function and the absolute values of the poles can only be of the form* 1, p^{-1}, $p^{\alpha-1}$ *where* $\alpha = \varkappa^{-1}$, $\varkappa$ *being a characteristic exponent of the curve.*

THEOREM 2. *Let* $f(x, y) \in Z_p[[x, y]]$ *converge over* Z_p^2. *Suppose* f *has only a singularity at* (0, 0) *which is non-degenerate with respect to its Newton diagram. Then* $P(T)$ *is rational and the absolute values of the poles can only be of the form* 1, p^{α}, p^{-1} *where* $\alpha = \text{slope}(\gamma)-1$ *if* $\text{slope}(\gamma) < 1$ *and* $\alpha = \text{slope}(\gamma)^{-1}-1$ *otherwise.* Slope(γ) *is the slope of a face* γ *of the Newton diagram.*

An analogous idea was used by Driggs in his thesis [7]: we investigate the curve $f(x, y)$ in the neighbourhood of the singular point with the use of Newton–Puiseux series, and in the neighbourhood of a regular point with the use of the usual Hensel lifting procedure.

We remark that for a general Q_p-rational variety we can define a power series $P(T)$ in an analogous way. The question of the rationality of $P(T)$ was posed by Serre in [19] and answered by Denef [3]. We refer to those authors for exact definitions in the general case. We can also define an analogous Poincaré series in the archimedean case. For the results in that direction we refer to Loeser [14].

2. Some general theorems. Let S be an open compact subset of $(Z_p)^2$. Let us denote by $N_i(S)$ the number of points in S mod p^i that can be lifted to a Z_p-rational point on $f \in Z_p[[x, y]]$.

LEMMA 2.1. *Let* $S \subset (Z_p)^2$ *be an open compact region in which* $f(x, y) \in Z_p[[x, y]]$ *has no singular* Z_p*-rational point. Then* $P_S(T) = \sum_i N_i(S)\, T^i$ *is a rational function of* T *with denominator* $1-pT$.

The proof of the lemma is essentially Hensel's lifting procedure. We refer to Serre [19], p. 147.

We now turn our attention to the singularity. We remark that Z_p^2 without a circle neighbourhood around (0, 0) can be covered by compact open neighbourhoods. That case is dealt with in the previous lemma. In order to use the Weierstrass preparation lemma we make the following construction: we rewrite f as

$$f = \sum_{i=d}^{\infty} f_i \; (d > 1, f_d \neq 0)$$

where the f_i are homogeneous polynomials of degree i. We can write f_d in the following form:

$$f_d = \sum_{i=0}^{d} a_i x^i y^{d-i}.$$

Because this form is not identically zero, we can find a Z_p-rational point $(b, 1)$ so that

$$\sum_i a_i b^i = 0.$$

We are perfectly allowed to set $x' = x+by$ and $y' = y$ because the number of $(x', y') \bmod p^i$ equals the number of $(x, y) \bmod p^i$ and if a pair $(x', y') \bmod p^i$ can be lifted then so can the corresponding pair (x, y) mod p^i, and vice versa.

So we make this change of coordinates and we may assume that y^d appears with a non-zero coefficient.

We are now going to refer to a few seemingly unrelated lemmas. They will all be used in our theorem.

LEMMA 2.2 (Weierstrass preparation th.). *Suppose* $f(x, y) \in Q_p[[x, y]]$ *is a power series converging in a neighbourhood of* (0, 0). *Then if*

$$f(0, y) = \sum_i a_i y^i \qquad (a_d \neq 0),$$

where d *is the first exponent in the infinite sum with non-zero coefficient, then we can write*

$$f(x, y) = u(x, y)(y^d + \lambda_{d-1}(x) y^{d-1} + \ldots + \lambda_0(x))$$

where $u(x, y)$ *converges in a neighbourhood of* (0, 0) *and* $u(0, 0) \neq 0$ *and the* $\lambda_i(x)$ *converge in a neighbourhood of* 0.

For a proof we refer to [10], p. 404.

LEMMA 2.3 (Newton–Puiseux expansion theorem). *If* $f(x, y)$ *is a monic irreducible polynomial in* $Q_p((x))[y]$ *and if* $\deg_y f(x, y) = n$, *then*

$$f(x, y) = \prod_{\omega} (y - \sum_i a_i \omega^i x^{i/n}).$$

The product is over the set $\{\omega \mid \omega^n = 1\}$ *and the* a_i *are in a finite algebraic extension field* k *of* Q_p. *If* $f(0, 0) = 0$ *then the Puiseux series* $\sum_i a_i \omega^i x^{i/n}$ *converges in a neighbourhood of* 0.

Proofs can be found in [1], p. 25 and in [8], p. 118.

3. A partition of a neighbourhood of $(0, 0)$. In order to control the rationality of $\sum_i a_i x^{i/n}$ we want to have information about the field $Q_p\{x^{1/n}\}$, i.e. Q_p extended with an nth root $x^{1/n}$. The choice of the nth root will be explained later. We will divide a neighbourhood of zero into disjoint classes so that $Q_p\{x^{1/n}\}$ is the same field for all x in each class. Let us fix a p-adic unit u_0 with p-adic expansion

$$u_0 = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{m-1} p^{m-1},$$

where α_0 is in $\{1, 2, \ldots, p-1\}$ and the other α_j are in $\{0, 1, 2, \ldots, p-1\}$. Let us take a look at all x so that $x = up^{ln+k}$ with $u \equiv u_0 \bmod p^m$ and both n and $k\,(0 \leqslant k < n)$ fixed. We can choose an nth root of x as follows: we write u as

$$u = u_0(1+z) \qquad (z \in p^m Z_p)$$

so that

$$u^{1/n} = u_0^{1/n}(1+z)^{1/n} \qquad (z \in p^m Z_p).$$

If the constant m is chosen large enough, then for $(1+z)^{1/n}$ we can choose the nth root obtained by binomial expansion (which is Q_p-rational), and for $u_0^{1/n}$ we choose a fixed nth root. We obtain also $(p^{ln+k})^{1/n} = p^l (p^{1/n})^k$ and take as nth root of p a fixed number. We collect this information in the following lemma.

LEMMA 3.1. *In a class $C(u_0, k)$ in a neighbourhood of zero consisting of x of the form $x = up^{ln+k}\,(l \geqslant 0)$ so that $u \equiv u_0 \bmod p^m$ we have, with the convention of taking n-th roots as above, that*

$$Q_p\{x^{1/n}\} = Q_p\{u_0^{1/n} p^{k/n}\}$$

is a fixed field.

In the following sections we will always refer to this partitioning of the neighbourhood of zero.

4. Lemma about the rationality of a Newton–Puiseux series. We turn our attention to a neighbourhood of zero and count the number of points mod p^i on a series $y - \sum_i a_i x^{i/n}$. Because we will count only points in Z_p mod p^i and the Puiseux series are only defined in a finite algebraic extension of Q_p, we will need a lemma which controls the rationality of $\sum_i a_i x^{i/n}$.

LEMMA 4.1. *Let C be a class as described in Section 3 and let $x^{1/n}$ be an n-th root as described in the previous section. Then the image of the map*

$$C \to Q_p^a \ (\text{algebraic closure of } Q_p)$$
$$x \mapsto \sum_i a_i \omega^i x^{i/n}$$

is a subset of Q_p or has a finite intersection with Q_p.

Proof. Suppose the field $Q_p\{a_i\omega^i, x^{1/n}\}$ is strictly larger than $Q_p\{x^{1/n}\}$. Then take as basis of $Q_p\{a_i\omega^i, x^{1/n}\}$ over $Q_p\{x^{1/n}\}$ the elements $\lambda_1 = 1, \lambda_2, \lambda_3, \ldots, \lambda_q$ $(q > 1)$. We write the Newton–Puiseux series as follows:

$$\sum_i a_i \omega^i x^{i/n} = \sum_i \Big(\sum_j c_{ij}\lambda_j\Big) x^{i/n} = \sum_j \Big(\sum_i c_{ij} x^{i/n}\Big)\lambda_j \qquad (c_{ij} \in Q_p\{x^{1/n}\}).$$

The summation is not identically zero for a j with $2 \leqslant j \leqslant q$, thus there are only finitely many $Q_p\{x^{1/n}\}$-rational values because a non-zero power series in one variable on a compact set has only finitely many zeros. So we may suppose that the field $Q_p\{a_i\omega^i\}$ is contained in $Q_p\{x^{1/n}\}$. Take a basis $\lambda_1, \ldots, \lambda_q$ for $Q_p\{x^{1/n}\}$ over Q_p. Now

$$x = up^{ln+k} = p^k\Big(\sum_{r=0}^{m-1} u_r p^r\Big)(1+p^m z)\, p^{ln} \qquad (z \in Z_p).$$

So

$$x^{1/n} = p^{k/n}\Big(\sum_{r=0}^{m-1} u_r p^r\Big)^{1/n}(1+p^m z)^{1/n} p^l.$$

By putting

$$(1+p^m z)^{1/n} p^l = v$$

and

$$a_i \omega^i = \sum_j c_{ij}\lambda_j \quad \text{with } c_{ij} \in Q_p,$$

$$\Big(p^k\Big(\sum_{r=0}^{m-1} u_r p^r\Big)\Big)^{i/n} = \sum_j e_{ij}\lambda_j \quad \text{with } e_{ij} \in Q_p$$

we obtain

$$\sum_i a_i \omega^i x^{i/n} = \sum_i \Big(\sum_j c_{ij}\lambda_j\Big)\Big(\sum_j e_{ij}\lambda_j\Big) v^i.$$

We see that we obtain

$$\sum_j \lambda_j p_j(v)$$

where the p_j are power series in the variable v. Either $p_2(v), \ldots, p_q(v)$ are identically zero and in that case all y-values are Q_p-rational, or at least one $p_j(v) \neq 0$, $2 \leqslant j \leqslant q$, and then only finitely many y-values are Q_p-rational.

5. Lemmas about the number of elements mod p^i **on a Newton–Puiseux series.** In the following, let us denote the order function in Q_p by ord. We suppose that e is the ramification index of a finite-dimensional field extension k of Q_p and denote the order function in k by ord_k. We know that for rational $x \in Q_p$, $\mathrm{ord}_k(x) = e\,\mathrm{ord}(x)$.

LEMMA 5.1. *Let $\sum_i a_i x^{i/n}$ be a Puiseux series with first non-zero exponent ξ/n. Let C be a class whose image under a Puiseux series is Q_p-rational. Let x and x' be in C with $\operatorname{ord}(x) = \operatorname{ord}(x')$. Then*

$$\operatorname{ord}\Big(\sum_i a_i x^{i/n} - \sum_i a_i x'^{i/n}\Big)$$
$$= \operatorname{ord}(x-x') - \operatorname{ord}(x) + \operatorname{ord}_k(a_\xi)/e + (\xi/n)\operatorname{ord}(x) + \operatorname{ord}(\xi/n).$$

Proof. Let k be the field generated by Q_p, $x^{1/n}$ and all $a_i\omega^i$. Suppose the ramification index of k over Q_p is e. Write

$$u_0 = \alpha_0 + \alpha_1 p + \ldots + \alpha_{m-1} p^{m-1},$$
$$x = u_0(1+z)p^{ln+k}, \quad x' = u_0(1+z')p^{ln+k}.$$

By binomial expanding $(1+z)^\alpha$, $z \in p^m Z_p$ (we suppose that m is large enough to allow the binomial expansion), we obtain

$$\operatorname{ord}(1+z)^\alpha = \operatorname{ord}(1+\alpha z).$$

So

$$\operatorname{ord}(x^\alpha - x'^\alpha) = \operatorname{ord}\big(\alpha(z-z')p^{(ln+k)\alpha}\big) = \operatorname{ord}(\alpha) + \operatorname{ord}(z-z') + \alpha\operatorname{ord}(x)$$
$$= \operatorname{ord}(\alpha) + \operatorname{ord}(x-x') - \operatorname{ord}(x) + \alpha\operatorname{ord}(x).$$

Thus

$$\operatorname{ord}_k\Big(\sum_i a_i x^{i/n} - \sum_i a_i x'^{i/n}\Big) = \operatorname{ord}_k\big(a_\xi(x^{\xi/n} - x'^{\xi/n})\big)$$
$$= \operatorname{ord}_k(a_\xi) + e\operatorname{ord}(\xi/n) + e\operatorname{ord}(x-x') - e\operatorname{ord}(x) + e(\xi/n)\operatorname{ord}(x).$$

The lemma follows. ∎

PROPOSITION 5.2. *Let C be a class whose image under a Puiseux series is Q_p-rational. Suppose the first non-zero exponent of the Puiseux series is ξ/n. The number of solutions of the congruence*

$$y - \sum_j a_j x^{j/n} \equiv 0 \bmod p^i$$

in the class C with x having prescribed fixed valuation $ln+k$ so that

$$ln+k+m < i$$

is

$$p^{i-(ln+k+m)+(1-\xi|n)(ln+k)-A} \quad \text{if } \xi/n < 1,$$
$$p^{i-(ln+k+m)-A} \quad \text{if } \xi/n = 1 \text{ and } A < 0,$$
$$p^{i-(ln+k+m)} \quad \text{if } (\xi/n = 1 \text{ and } A \geqslant 0) \text{ or } \xi/n > 1,$$

with A a constant depending on the first non-zero coefficient and the first non-zero exponent of the Puiseux series only.

Proof. Let us count the number of elements with fixed valuation mod p^i. We know that the $(ln+k)$-th p-adic digit up to the $(ln+k+m-1)$-th digit are uniquely determined by the definition of the class. The $(ln+k+m)$-th p-adic digit up to the $(i-1)$-th digit can be chosen freely. So there are $p^{i-(ln+k+m)}$ choices.

Let us now investigate how the number of y mod p^i is associated with the number of x mod p^i. We treat the case $\xi/n < 1$. The variable y is not completely determined for a particular choice of an element x mod p^i in this case. Put $x = u_0(1+z)p^{ln+k}$ and take the ξ/n-th power. Because the derivative to the variable z of $(1+z)^{\xi/n}$ is not zero for $z=0$ we can apply Hensel's lemma to guarantes surjectivity on a neighbourhood of 1. By the preceding lemma we have $p^{(1-\xi/n)(ln+k)-A}$ choices for $y \bmod p^i$. Multiplying the number of choices for x and y together we obtain

$$p^{i-(ln+k+m)+(1-\xi/n)(ln+k)-A}$$

choices. The other cases may be proved in a similar fashion. ∎

PROPOSITION 5.3. *Let F_i be the number of (x, y) mod p^i such that:*

1. *x is in a small neighbourhood of zero, i.e. $\operatorname{ord}(x) = ln+k > l_0 n+k$ (for $l_0 \geqslant 0$).*
2. *$\operatorname{ord}(x) = ln+k$ such that $ln+k+m < i$.*

Then the Poincaré series $\sum_i F_i T^i$ has possible poles

with absolute value $p^{\xi/n-1}$ and p^{-1} if $\xi/n < 1$,
with absolute value p^{-1} otherwise.

Proof. Let us consider the most difficult case, namely that the first non-zero characteristic exponent is $\xi/n < 1$. It is clear by condition 2 that to calculate F_i we must count all solutions (x, y) with x having a valuation $ln+k$ satisfying

$$ln+k+m < i.$$

So

$$l < i/n - k/n - m/n \quad \text{and} \quad l > l_0.$$

For technical reasons we are going to look at all $i \equiv \lambda \bmod n$, so $i = \lambda + jn$ with λ a fixed number between 0 and $n-1$. So

$$l < j + C$$

where C is a constant which is independent of the summation index in the series. In the following we use capital letters for constants independent of the summation index, without making this explicit every time. We define $B = C$ if C is an integer number and $B = [C]+1$ otherwise. The notation $[\ ]$ means as usual the 'largest integer equal to or smaller than'. So

$$l < j+B.$$

Therefore to determine F_i with $i = \lambda+jn$, we have to look at all x with valuation $ln+k$ such that

$$l_0 < l < j+B.$$

The preceding Proposition 5.2 says that for one such l we have

$$p^{i-(ln+k+m)+(1-\xi/n)(ln+k)-A}$$

solutions.

So

$$\sum_j F_{\lambda+jn} T^{\lambda+jn}$$

is

$$\sum_j \Big(\sum_l p^{(\lambda+jn)-(ln+k+m)+(1-\xi/n)(ln+k)-m-A}\Big) T^{\lambda+jn}$$

where the inner sum is over

$$l_0 < l < j+B.$$

We neglect all constants independent of j and l because they have no effect on the poles and the rationality to obtain

$$\sum_j \Big(\sum_l p^{jn-\xi l}\Big) T^{\lambda+jn}.$$

We neglect T^{λ} and obtain

$$\sum_j \Big(\sum_l p^{-\xi l}\Big)(p\,T)^{jn}.$$

The inner sum is easily calculated as

$$\frac{(p^{-\xi})^{l_0+1}}{p^{-\xi}-1} - \frac{(p^{-\xi})^{j+B}}{p^{-\xi}-1}.$$

The first summand clearly gives the denominator $1-p^n T^n$ with pole of absolute value p^{-1}, the second summand is more interesting, the denominator is essentially $1-p^{n-\xi} T^n$ which gives a pole of absolute value $p^{\xi/n-1}$. The proofs of the other cases run along the same lines and are easier. ∎

Proposition 5.4. *Let C be a class whose image under a Puiseux series is $\mathbf{Q}_p$-rational. Suppose the first non-zero exponent of the Puiseux series is ξ/n. The number of solutions of the congruence*

$$y - \sum_j a_j x^{j/n} \equiv 0 \bmod p^i$$

in the class C with x having prescribed fixed valuation $ln+k$ so that

$$ln+k < i \leqslant ln+k+m$$

is

$$\begin{array}{ll} p^{i-(\xi/n)(ln+k)-m-A} & \text{if } \xi/n < 1, \\ p^{i-(ln+k+m+A)} & \text{if } \xi/n = 1 \text{ and } A < i-1-(ln+k+m), \\ p^0 = 1 & \text{otherwise.} \end{array}$$

Proof. Let us first count the number of elements with fixed valuation $ln+k \bmod p^i$. Because the $(ln+k)$-th p-adic digit up to the $(ln+k+m-1)$-th digit are uniquely determined we have only one x with the valuation $ln+k$. But, for one x, again as before, y is not completely determined because the leading non-zero exponent is <1. Because $\operatorname{ord}(x-x')$ is at least $ln+k+m$ we have y determined from the $((\xi/n)(ln+k)+m+A)$-th p-adic digit up to the $(i-1)$-th digit. Thus we have for y in the case $\xi/n < 1$ exactly $p^{i-(\xi/n)(ln+k)-m-A}$ choices left. If $\xi = n$, so that the first non-zero exponent is 1, then y is determined by x up to the $(ln+k+m+A-1)$-th digit. This leaves us only free choices for y if $ln+k+m+A < i-1$, i.e. $A < i-1-(ln+k+m)$. Then we have $p^{i-(ln+k+m+A)}$ extra choices for y. ∎

Proposition 5.5. *Let S_i be the number of $(x, y) \bmod p^i$ such that:*

1. *x is in a small neighbourhood of zero, i.e.* $\operatorname{ord}(x) = ln+k > l_0 n+k$ *(for* $l_0 \gg 0$).

2. $ln+k < i \leqslant ln+k+m$.

Then the Poincaré series $\sum_i S_i T^i$ has possible poles

with absolute value p^{-1} and $p^{\xi/n-1}$ if $\xi/n < 1$,
with absolute value p^{-1} otherwise.

Proof. Let us again consider the most difficult case, namely when the leading non-zero exponent is $\xi/n < 1$. To calculate S_i, we must count all solutions (x, y) with x having a valuation $ln+k$ satisfying

$$ln+k < i \leqslant ln+k+m.$$

So by subtracting k and dividing by n, we obtain

$$l < i/n - k/n \leqslant l+m/n \quad \text{and} \quad l > l_0.$$

For technical reasons we are going to sum over all $i \equiv \lambda \bmod n$, so $i = \lambda+jn$ with λ a fixed number between 0 and $n-1$.

So
$$l < j+\lambda/n-k/n \leqslant l+m/n.$$
Therefore to count S_i with $i = \lambda+jn$, we have to look at all x with valuation $ln+k$ such that
$$l_0 < l < j+B \leqslant l+C.$$
This is only non-trivial if $C > 0$. In that case we define $D = B-C$ if $B-C$ is an integer number and $D = [B-C]+1$ otherwise. We define $F = B-1$ if B is an integer number and $F = [B]+1$ otherwise. Then l satisfies
$$j+D \leqslant l < j+F.$$
We remember that by Proposition 5.4 we have to calculate
$$\sum_j \big(\sum_l p^{\lambda+jn-(\xi/n)(ln+k)-m-A}\big)\, T^{\lambda+jn}$$
where the inner sum is over all l satisfying
$$j+D \leqslant l < j+F.$$
We neglect all constants independent of j and l because they have no effect on the poles and obtain
$$\sum_j \big(\sum_l p^{jn-\xi l}\big)\, T^{\lambda+jn}.$$
We neglect T^{λ} and obtain
$$\sum_j \big(\sum_l p^{-\xi l}\big)(pT)^{jn}.$$
The inner sum is easily calculated as
$$\frac{(p^{-\xi})^{(j+D)}-(p^{-\xi})^{(j+F)}}{1-p^{-\xi}}.$$
This gives us clearly the denominator $1-(p^{n-\xi})\,T^n$ with pole of absolute value $p^{\xi/n-1}$. The rest of the theorem follows in a similar fashion. ■

LEMMA 5.6. *Let $\sum a_i x^{i/n}$ be a Puiseux series. Suppose it maps a class C in Q_p. Then the Poincaré series of $y-\sum a_i x^{i/n}$ associated to the class C in a small enough neighbourhood $\{(x, y) \mid \operatorname{ord}(x) > N_1$ and $\operatorname{ord}(y) > N_2\}$ is rational and has possible poles*

with absolute value $p^{\xi/n-1}$, p^{-1} and 1 if $\xi/n < 1$,
with absolute value 1 and p^{-1} otherwise.

Proof. We know that the Newton–Puiseux series $f(x) = \sum_i a_i x^{i/n}$ is continuous at $x = 0$. Take a neighbourhood of $y = 0$ on the y-axis of the form $N = \{y \mid \operatorname{ord}(y) > N_2\}$. Now $f^{-1}(N)$ is an open set on the x-axis which contains zero. We now take a neighbourhood $\{x \mid \operatorname{ord}(x) > N_1\}$ which is completely contained in $f^{-1}(N)$. We compute our Poincaré series associated to the class C in the neighbourhood $\{(x, y) \mid \operatorname{ord}(x) > N_1$ and $\operatorname{ord}(y) > N_2\}$. We apply Propositions 5.3 and 5.5. The analogue of Proposition 5.3 where we count the number of (x, y) with $\operatorname{ord}(x) > i$ runs along the same lines of the proof of Proposition 5.3 and is easily seen not to obtain new poles. ■

6. A singularity which is non-degenerate with respect to its Newton diagram. Suppose we have a Newton diagram of the form described in Figure 1.

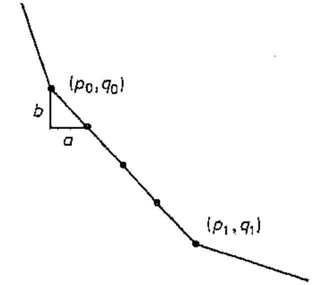

Fig. 1. A part of the Newton polygon of a curve with starting point (p_0, q_0) and end point (p_1, q_1) of a face γ

We use a reasoning which is described in Brieskorn's book about singularities ([2], p. 635), to obtain information about the singularity. The trick is to find new coordinates (u, v) so that the singularity is more easily described. We are going to take a look at the compact face γ. The endpoints of γ are as indicated in the figure: (p_0, q_0) and (p_1, q_1). Let (p_0+a, q_0-b) be the next lattice point on γ. So g.c.d.$(a, b) = 1$ and γ has equation $bp+aq = e$. The lattice points on γ are $(p_0+\varkappa a, q_0-\varkappa b)$, $\varkappa = 0, 1, \ldots, k$. Now g.c.d.$(a, b) = 1$ and consequently we can find natural numbers c and d so that $d/c > b/a$ and $ad-bc = 1$. So the line l with equation $dp+cq = f$ with $f = dp_0+cq_0$ lies completely below the Newton diagram (see Figure 2).

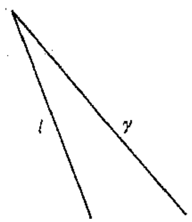

Fig. 2. Relative position of the line l and the face γ

Let U be the plane with coordinates (u, v) and let π be the mapping
$$\pi\colon U \to C_p^2$$

which maps (u, v) to $(x, y) = (u^d v^b, u^c v^a)$. The jacobian of π is
$$(ad-bc)(u^{c+d-1}\, v^{a+b-1}).$$
If $x \cdot y$ is not zero then $u \cdot v$ is not zero either. We calculate the pullback of f_γ in the coordinates (u, v):
$$f_\gamma(\pi(u, v)) = \sum_{(p,q)\in\gamma} a_{pq}\, u^{dp+cq}\, v^{bp+aq}.$$
$$f_\gamma(\pi(u, v)) = v^e \sum_{p,q} a_{pq}\, u^{dp+cq} = v^e u^f \big(\sum_{p,q} a_{pq}\, u^{d(p-p_0)+c(q-q_0)}\big)$$
$$= v^e u^f \big(\sum_{\varkappa} a_{(p_0+\varkappa a, q_0-\varkappa b)}\, u^{\varkappa}\big).$$
We define $g(u)$ as
$$g(u) = \sum_{\varkappa} a_{(p_0+\varkappa a, q_0-\varkappa b)}\, u^{\varkappa}.$$
The singular points (x, y) so that $x \cdot y$ does not equal zero on f_γ correspond to those (u, v) for which $u \cdot v$ does not equal zero and both the partial derivatives $(v^e u^f g(u))'_u = 0$ and $(v^e u^f g(u))'_v = 0$. This is equivalent to $g(u) = g'(u) = 0$. This means that $g(u)$ cannot have roots with multiplicity strictly larger than 1. We formulate this in a lemma.

LEMMA 6.1. *Let γ be a compact face of the Newton diagram and let f_γ be its associated polynomial. Then the polynomial $g(u)$ defined in this section has no roots with multiplicity strictly larger than one if the singularity is non-degenerate with respect to the Newton polygon.*

In the following, let us consider the Newton diagram of a compact face still in the non-degenerate case.

Let f_γ be the associated polynomial of the compact face γ. Define the polynomial $\tilde{f}_\gamma$ by writing $f_\gamma(x, y) = x^\lambda y^\mu \tilde{f}_\gamma(x, y)$ such that $x \nmid \tilde{f}_\gamma$, $y \nmid \tilde{f}_\gamma$.

THEOREM 6.2. *Let γ be a compact face of the Newton diagram with associated polynomial f_γ. The Newton expansion of $\tilde{f}_\gamma$ has the following form:*
$$\tilde{f}_\gamma = \prod_t (y - t x^{a/b}),$$
where the product runs over all t which are roots of $g(t^b)$ (the polynomial g is defined in the introduction of this section).

Proof. $\tilde{f}_\gamma$ has equation $\sum a_{pq} x^p y^q$. It is now understood that (p_0, q_0) is on the y-axis and (p_1, q_1) is on the x-axis. The equation of the line γ which passes through (p_0, q_0) and (p_1, q_1) is $bp + aq = e'$. We recall the first step of Newton's approximation algorithm (see [2], p. 494). Consider
$$\tilde{f}_\gamma(x, y) = \sum_{p,q} a_{pq}\, x^p y^q.$$

The summation is running over all lattice points (p, q) satisfying $p + (a/b) q = e'/b$. We then substitute $y = t x^{a/b}$ and obtain
$$\tilde{f}_\gamma(x, t x^{a/b}) = \sum_{p,q} a_{pq}\, t^q x^{e'/b}.$$
Because $(p, q) = (p_0 - \varkappa a, q_0 - \varkappa b)$ $(\varkappa = 0, \ldots, k)$, we obtain
$$\sum_q a_{pq}\, t^q = t^{q_0} g(t^{-b})$$
where $g(u)$ is the same polynomial as before. We know that $g(u)$ does not have double roots so $g(t^{-b})$ also has no roots with multiplicity larger than one. All the roots of $\tilde{f}_\gamma$ have the form $y = t_0 x^{a/b}$ with t_0 a root of $g(t^{-b})$. The theorem follows. ∎

The Newton diagram described in Figure 1 is also the Newton diagram of $\prod \tilde{f}_\gamma$ where the product is taken over all compact faces γ of the Newton diagram. This is because the fact that if $f = 0$ and $g = 0$ are the equations of two curves, then we can find the Newton diagram of $f \cdot g = 0$ by tying together the Newton diagrams of $f = 0$ and $g = 0$ (see Figure 2), thus the Newton diagram consists of the product of all $\tilde{f}_\gamma$. By following Brieskorn [2], p. 639, a sketch of proof based on induction is as follows:

1. We note that the monomial $x^p y^q$ of an endpoint (p, q) of a face of the tied diagram can be written in a unique way as a product of a monomial of f and a monomial of g.

2. Below the composed diagram there are no points of $\Delta(f \cdot g)$ (i.e. of the Newton diagram of $f \cdot g$), because $\Delta(f \cdot g) \subset \Delta(f) + \Delta(g)$ and $\Delta(f)$ and $\Delta(g)$ have no points of the composed diagram.

LEMMA 6.3. *The curve with equation $\prod \tilde{f}_\gamma = 0$, where the product is over all compact faces γ has the following property:*

1. *The Newton–Puiseux series which are solutions of $\tilde{f}_\gamma = 0$ have first non-zero exponent a/b where b/a is the slope of γ.*

2. *Any two solutions of $\tilde{f}_\gamma = 0$ differ by the coefficient of $x^{a/b}$.*

Proof. We use the preceding Lemma 6.1 and Theorem 6.2 and the information on composing a diagram consisting of different curves $\tilde{f}_\gamma$. ∎

We generalize this to an arbitrary curve.

LEMMA 6.4. *Suppose $(0, 0)$ is a singularity of $f(x, y) = 0$ which is non-degenerate with respect to its Newton diagram. The curve with equation $f(x, y) = 0$ has the following properties as regards the Newton–Puiseux series which are solutions of $f(x, y) = 0$:*

1. *Either the first non-zero exponents (i.e. slopes of the Newton diagram) are different, or*

2. *if they are equal then their associated coefficients are different.*

Proof. We have proved this for the case $\prod \tilde{f}_\gamma$. We remark that for the first (= left above) compact face γ_1 of the Newton diagram, we have by applying Newton's expansion theorem $\varkappa_1 b_1$ different solutions with first exponent a_1/b_1. The coefficients of these exponents are the roots of $g(t^b)$. We take the product of all these solutions, say $g(x, y) \in \boldsymbol{Q}_p[[x]][y]$, and remark that the Newton diagram of $f(x, y)/g(x, y)$ is the same as the Newton diagram of $(x^\lambda y^\mu \prod \tilde{f}_\gamma)/\tilde{f}_{\gamma_1}$ with the same coefficients a_{pq} of the monomials of the support of the compact faces. The Newton diagram in fact changes as indicated in Figure 3. ∎

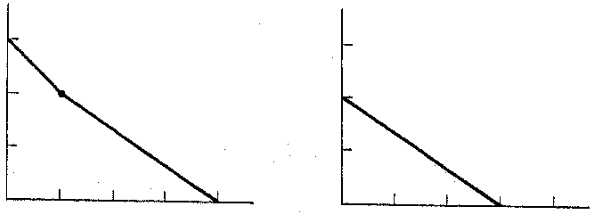

Fig. 3. On the left-hand side the Newton diagram of $f(x, y)$ and on the right-hand side the Newton diagram of $f(x, y)/g(x, y)$.

$$f(x, y) = (y^2 - x^3)(y - x), \qquad g(x, y) = y - x$$

7. A description of a singularity in the case of analytic irreducibility. We recall here a few important notions of the theory of Puiseux series. We refer to [23] for more details. A Puiseux series can be rewritten in the following form:

$$\sum_i a_{1,i} x^i + b_1 x^{m_1/n_1} + \sum_i a_{2,i} x^{(m_1+i)/n_1} + b_2 x^{m_2/(n_1 n_2)} + \ldots$$
$$+ \sum_i x^{(m_{g-1}+i)/n_1 n_2 \ldots n_{g-1}} + b_g x^{m_g/(n_1 n_2 \ldots n_g)} + \sum_i c_i x^{(m_g+i)/n}.$$

Let us define rational numbers called *characteristic exponents* as follows: $\varkappa_1 = m_1/n_1$, $\varkappa_2 = m_2/(n_1 n_2)$, $\ldots$, $\varkappa_g = m_g/(n_1 n_2 \ldots n_g)$ with $n = n_1 n_2 \ldots n_g$ and $(m_j, n_j) = 1$ for all j.

LEMMA 7.1. *If $f(x, y) \in \boldsymbol{Q}_p[[x]][y]$ is monic and irreducible and if we take two Puiseux series, say $\sum_i a_i \omega_1^i x^{i/n}$ and $\sum_i a_i \omega_2^i x^{i/n}$, from the Newton–Puiseux product expansion of $f(x, y)$, then the exponent of the first non-zero term of $\sum_i a_i \omega_1^i x^{i/n} - \sum_i a_i \omega_2^i x^{i/n}$ is a characteristic exponent.*

Proof. Suppose the coefficients of the first characteristic exponents are equal. Thus ω_1/ω_2 is an $m_1 n_2 \ldots n_g$-th root of unity. We claim that ω_1/ω_2 is an $n_2 n_3 \ldots n_g$-th root of unity. This is because $(m_1, n_1) = 1$, so there exist

integers a and b so that $an_1 + bm_1 = 1$. Thus because ω_1/ω_2 is an $n_1 n_2 \ldots n_g$-th root of unity we infer that ω_1/ω_2 is an $n_2 n_3 \ldots n_g$-th root of unity. This means that

$$\sum_i a_{2,i} x^{(m_1+i)/n_1} \omega_1^{(m_1+i)n_2 \ldots n_g} = \sum_i a_{2,i} x^{(m_1+i)/n_1} \omega_2^{(m_1+i)n_2 \ldots n_g}.$$

Suppose that the coefficients at the second characteristic exponent are also the same, i.e. ω_1/ω_2 is an $m_2(n_3 \ldots n_g)$-th root of unity. Then we can prove exactly as above that ω_1/ω_2 is an $n_3 \ldots n_g$-th root of unity. We can proceed further by induction. ∎

8. The number of solutions mod p^i **of a system of Puiseux series.** We stress the point that in this section we are going to calculate the Poincaré series $\sum N_i T^i$ where N_i is the number of solutions mod p^i of a system of Puiseux series where in contrast with the previous sections the points (x, y) do not have to be on the locus defined by the system of equations. We are going to look at the Poincaré series of a system

$$y - \sum_j a_j x^{j/n} = 0,$$
$$y - \sum_j b_j x^{j/n} = 0,$$
$$\ldots\ldots\ldots\ldots$$
$$y - \sum_j z_j x^{j/n} = 0$$

for x in one class C so that all Puiseux series are $\boldsymbol{Q}_p$-rational. We may suppose that the first equation has a leading exponent which is equal or smaller than the leading exponents of the following series.

We may also suppose that the coefficients of the series which begin with the same characteristic exponent are different and the order of the equations beginning with the same first characteristic exponent in the system is so that the p-adic absolute values are ranked from small to large. Then we can rewrite our system of equations by subtracting the first equation from the second up to the last to obtain

$$y - \sum_j a_j x^{j/n} = 0,$$
$$\sum_j (a_j - b_j) x^{j/n} = 0,$$
$$\ldots\ldots\ldots\ldots$$
$$\sum_j (a_j - z_j) x^{j/n} = 0.$$

We are first going to calculate N_i which is the number of solutions of

$$y-\sum_j a_j x^{j/n} \equiv 0 \bmod p^i,$$

$$\sum_j (a_j-b_j)\, x^{j/n} \equiv 0 \bmod p^i,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\sum_j (a_j-z_j)\, x^{j/n} \equiv 0 \bmod p^i.$$

We may suppose that the second equation has leading exponent ξ/n which is the smallest leading exponent of all Puiseux series in the system and also has a coefficient $a_\xi-b_\xi$ which is the coefficient with smallest p-adic absolute value among the Puiseux series with smallest exponent ξ/n. It is then clear that in order to calculate the Poincaré series of the system we have, by making an x-neighbourhood as small as necessary, to calculate the Poincaré series of the system of modular equations

$$(a_\xi-b_\xi)\, x^{\xi/n} \equiv 0 \bmod p^i$$

and

$$y-\sum_j a_j x^{j/n} \equiv 0 \bmod p^i.$$

PROPOSITION 8.1. *Let F_i be the number of* (x, y) mod p^i *with x in a class C such that*

$$(a_\xi-b_\xi)\, x^{\xi/n} \equiv 0 \bmod p^i$$

with x having a valuation satisfying the following conditions:

1. *x is in a small neighbourhood of zero; i.e.* $\operatorname{ord}_p(x) = ln+k$ *($l_0 \gg 0$ and $l > l_0$),*
2. $\operatorname{ord}_p(x) = ln+k$ *such that $ln+k+m < i$.*

Then the Poincaré series $\sum_i F_i T^i$ is rational and

if $\xi/n < 1$ there are no poles,

in all other cases the possible poles have absolute value $p^{n/\xi-1}$.

Proof. Writing $a_\xi-b_\xi = d_\xi$ we see that x is a solution if (e still being the ramification index of k over Q_p)

$$\operatorname{ord} p^i - (\operatorname{ord}_k(d_\xi))/e \leqslant \operatorname{ord} x^{\xi/n}$$

or

$$\operatorname{ord} p^i - A \leqslant \operatorname{ord} x^{\xi/n};$$

equivalently, since $\operatorname{ord} x = ln+k$

$$i-A \leqslant (\xi/n)(ln+k)$$

or

$$i/\xi+B \leqslant l.$$

By condition 2

$$ln+k+m < i$$

or

$$l < i/n-k/n-m/n$$

or

$$l < i/n+C.$$

For technical reasons we are going to compute this Poincaré series for $i \equiv \lambda \bmod \xi n$. So $i = \lambda+j(\xi n)$. The foregoing inequalities combined become $jn+D \leqslant l < j\xi+E$. It is clear from this inequality that if the exponent $\xi/n < 1$, there is no l satisfying this inequality if j is very large. For the other cases we have to calculate the following Poincaré series by Proposition 5.2 of Section 5, case 2 (we neglect A) and case 3:

$$\sum_j \Big(\sum_l p^{\lambda+j(\xi n)-ln-k-m}\Big) T^{\lambda+j(\xi n)}$$

where the inner sum is over all l such that

$$jn+D \leqslant l < j\xi+E.$$

We put $F = D$ if D is an integer number and $F = [D]+1$ otherwise. We put $G = E$ if E is an integer number and $G = [E]+1$ otherwise. So by dropping all constants which are independent of λ and l and T^λ we have to calculate

$$\sum_j \Big(\sum_l p^{j(\xi n)-ln}\Big) T^{j(\xi n)} = \sum_j \Big(\sum_l p^{-ln}\Big)(pT)^{j(\xi n)}.$$

So the inner sum is easily calculated as

$$\frac{(p^{-n})^{(j\xi+G)}-(p^{-n})^{(jn+F)}}{p^{-n}-1}$$

The first summand gives the denominator $1-T^{\xi n}$ and the second gives the denominator $1-p^{-nn+\xi n}T^{\xi n}$ which has as consequence poles with absolute value $p^{n/\xi-1}$. ∎

PROPOSITION 8.2. *Let S_i be the number of* (x, y) mod p^i *with x in a class C such that*

$$(a_\xi-b_\xi)\, x^{\xi/n} \equiv 0 \bmod p^i$$

with x having a valuation satisfying the following conditions:

1. *x is in a small neighbourhood of zero;* $\operatorname{ord}(x) = ln+k$ $(l_0 \geqslant 0)$,
2. $ln+k < i \leqslant ln+k+m$.

Then the Poincaré series has possible poles with absolute value 1.

Proof. We obtain by repeating exactly the first lines of the proof of Proposition 8.1

$$i/\xi + B \leqslant l.$$

By condition 2, l is also constrained by

$$ln+k < i \leqslant ln+k+m.$$

Therefore

$$i/\xi + B \leqslant l < i/n - k/n \leqslant l + m/n.$$

We consider this for $i \equiv \lambda \bmod \xi n$, i.e. $i = \lambda + j(\xi n)$. Then the inequality becomes

$$jn + C \leqslant l < j\xi + D \leqslant l + E.$$

We put $F = C$ if C is an integer number, $F = [C]+1$ otherwise; $G = D$ if D is an integer number, $G = [D]+1$ otherwise; $H = E$ if E is an integer number, $H = [E]+1$ otherwise. So the inequality becomes

$$jn + F \leqslant l < j\xi + G \leqslant l + H.$$

From the first two inequalities we see that if $n > \xi$ (i.e. $\xi/n < 1$) there are no solutions for l. If $\xi = n$, then the solution set for l is possibly not empty if $F < G$ and $H > 0$. By the last inequality l satisfies also $j\xi + G - H \leqslant l$ and $l < j\xi + G$. So $F \leqslant G - H$ (otherwise we do not have to sum) and thus by Proposition 5.4 (take e.g. the second case of that proposition) the sum is equal to

$$\sum_j \Big(\sum_l p^{\lambda + j(\xi n) - ln - k - m - A}\Big) T^{\lambda + j(\xi n)}.$$

By neglecting all constants which are independent of j, l and T^λ we obtain

$$\sum_j \Big(\sum_l p^{j\xi n - ln}\Big) T^{j\xi n} = \sum_j \Big(\sum_l p^{-ln}\Big) (pT)^{j\xi n}$$

where the inner sum runs over all l satisfying

$$j\xi + G - H < l < j\xi + G.$$

The inner sum is easily calculated as

$$\frac{(p^{-n})^{(j\xi+G)}}{p^{-n}-1} - \frac{(p^{-n})^{(j\xi+G-H-1)}}{p^{-n}-1}.$$

Summing over the first and second summand we obtain the denominator $1 - T^{\xi n}$. If $\xi/n > 1$ we have to sum over

$$l < j\xi + G \leqslant l + H,$$

so l satisfies

$$j\xi + G - H \leqslant l < j\xi + G.$$

The number of l is constant thus we have as denominator again $1 - T^{\xi n}$. ∎

We collect the information of Proposition 8.1 and Proposition 8.2 in Lemma 8.3.

Suppose all x are in a class on which all Puiseux series are rational.

LEMMA 8.3. *Suppose*

$$y - \sum_j a_j x^{j/n} = 0,$$

$$y - \sum_j b_j x^{j/n} = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$y - \sum_j z_j x^{j/n} = 0$$

is a system of Puiseux series. Suppose the smallest beginning exponent which occurs in a Puiseux series in the system is ξ/n. Then the Poincaré series of a class C associated to the system is rational and

if $\xi/n < 1$ there are no poles,

in all other cases the possible poles have absolute value 1 *or* $p^{n/\xi - 1}$.

Proof. Combine Proposition 8.1 and Proposition 8.2. ∎

9. Intersections of different classes. It is possible that α different classes, say $C(u_1, k), \ldots, C(u_\alpha, k)$ have a non empty intersection $\bmod p^l$. Suppose q is a fixed natural number, $0 \leqslant q < m$, and that $u_1 \equiv u_2 \equiv \ldots \equiv u_\alpha \bmod p^q$. Consider the same system of equations as in the previous section. Suppose that the α classes $C(u_j, k)$, $1 \leqslant j \leqslant \alpha$ have a $\mathbf{Q}_p$-rational image on all Puiseux series in the system. Let (x, y) be a point such that $\operatorname{ord}(x) = ln+k$. Remark that we do not demand as in Section 8 that (x, y) lies on the locus defined by the system of equations. We are going to calculate the Poincaré series $\sum_l N_{ln+k+q} T^{ln+k+q}$. We have to compute as in the previous section the Poincaré series of $(a_\xi - b_\xi) x^{\xi/n} \bmod p^{ln+k+q}$. If $\xi/n < 1$, there is by analogy with the previous section nothing to count. If $\xi/n \geqslant 1$, then we obtain $N_{ln+k+q} = p^{ln+k+q-(ln+k+m+A)}$. The Poincaré series is

$$\sum_l p^{ln+k+q-(ln+k+m+A)} T^{ln+k+q}.$$

The denominator is $1-T^n$, so the absolute value of the poles is 1. We collect the information in the following lemma.

LEMMA 9.1. *Suppose given a system of Puiseux series as in the previous section. Let* $C(u_1, k), \ldots, C(u_\alpha, k)$ *be* α *classes such that* $u_1 \equiv u_2 \equiv \ldots \equiv u_\alpha \bmod p^q$, $0 \leqslant q < m$ *and all Puiseux series have a* $\boldsymbol{Q}_p$*-rational image on each class. Let* N_{ln+k+q} *be the number of elements* $(x, y) \bmod p^{ln+k+q}$ *satisfying the system of equations* $\bmod p^{ln+k+q}$ *where* $x \in C(u_1, k)$ *and* $\operatorname{ord}(x) = ln+k$. *Then the Poincaré series* $\sum_l N_{ln+k+q} T^{ln+k+q}$ *is rational and has*

no poles if $\xi/n < 1$,
poles with absolute value 1 *in all other cases.*

10. Proof of the main theorems. Because our proof is very computational and requires many steps we now explain the subdivision of our reasoning:

1. Suppose we have a plane curve $f(x, y)$ with a singularity at (0, 0). $\boldsymbol{Z}_p^2$ minus a very small neighbourhood around (0, 0) is handled by Hensel's lemma (Lemma 2.1). The denominator is there $1-pT$.

2. We now take a look at our neighbourhood of (0, 0). The Weierstrass preparation lemma (Lemma 2.2) prepares us to make the Newton expansion (Lemma 2.3). We may replace our curve $f(x, y)$ with a finite polynomial in y, because $u(x, y)$ is a unit in the ring $\boldsymbol{Z}_p[[x, y]]$ and consequently the valuation of $u(x, y)$ is constant on our small neighbourhood. So the poles and the rationality will not be changed.

3. We make the Newton–Puiseux expansion and divide (Section 3) our neighbourhood of (0, 0) into disjoint classes to have control over the rationality of the Puiseux series which are only defined on a finite algebraic extension of $\boldsymbol{Q}_p$.

4. Let us fix one class C. For this class C we look at all Puiseux series which have a $\boldsymbol{Q}_p$-rational image on C (Section 4). For all these Puiseux series we calculate the Poincaré series according to Lemma 5.6. We repeat this for all classes. If $\xi/n < 1$, this gives poles with absolute value $p^{\xi/n-1}$, p^{-1} and 1, and in all other cases with absolute value 1. In the case of an analytic irreducible curve we can avoid the case that the first exponent is smaller than one by interchanging the roles of x and y.

5. Our analysis of a singularity in the case that the singularity is non-degenerate (Section 6) and the singularity is analytically irreducible (Section 7) tells us that if we have to compute the Poincaré series of a system of equations we have essentially to compute the Poincaré series of the modular equation $(a_\xi - b_\xi) x^{\xi/n}$. The coefficients a_ξ and b_ξ are different. So ξ/n is a leading exponent of a Puiseux series in the product expansion of $f(x, y)$. The leading exponent ξ/n is, in the case of analytic irreducibility, a characteristic exponent, and in the case of non-degeneracy the multiplicative inverse of a slope of a face γ of the Newton diagram. If in the case of non degeneracy the equations $x \equiv \bmod p^i$ or $y \equiv \bmod p^i$ occur in the system of equations, then the Poincaré series associated to these systems are easily seen not to give new absolute values of poles.

6. Let us again fix one class C. We have calculated our Poincaré series for a number of Puiseux series. We have counted too many points mod p^i. So we take all systems of two Puiseux factors, all systems of three Puiseux factors, and so on. Our Poincaré series becomes the sum of the Poincaré series according to the preceding point minus the Poincaré series of all systems of two equations plus the Poincaré series of all systems of three equations and so on. This gives poles according to Lemma 8.3 with absolute value $n/\xi - 1$. The number n/ξ is the inverse of a characteristic exponent in the case of analytic irreducibility, and the slope of a compact face of the Newton diagram in the case of a non-degenerate singularity.

7. Let $C(u_1, k), C(u_2, k), \ldots, C(u_\alpha, k)$ be α classes. Then their intersection mod p^i is possibly not empty for infinitely many i. By the definition of a class these intersections can only occur for those x with a valuation $ln+k+q = i$ where q runs from 0 to $m-1$ (Section 9). So the sum of the separate Poincaré series attached to the α classes as described in the previous points is not the Poincaré series of $f(x, y)$ with x in one of those classes. We have to subtract all Poincaré series of systems of two equations according to Lemma 9.1 where in every system of equations we consider all equations with Puiseux series rational on our two classes, then add all Poincaré series of systems of three equations and so on.

8. We did not take into account the zero solution mod p^i. Doing that gives a denominator $1-T$ with absolute value 1.

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Units in parametrized p-adatropic number fields

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0. Introduction. In [2]–[4] H. Cohn studied fields generated by polynomials which assumed values of powers of 2 at several consecutive integers. It was felt that these fields might yield independent units parametrically. We make the following generalization:

DEFINITION 1. Let p be a prime. Let $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_i \in \boldsymbol{Z}$, $0 \leqslant i < n$. The polynomial, f, is said to be *p-adatropic* if there exist $n+1$ consecutive rational integers, c_i, such that $|f(c_i)|$ is a power of p.

From finite differencing the following lemma is known:

LEMMA 1. *Let $f(x)$ be a monic polynomial of degree n and let $x_0 \in \boldsymbol{R}$. Let $y_k = f(x_0+k)$, $k = 0, 1, \ldots, n$. Then*

$$(*) \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} y_k = (-1)^n n!.$$

COROLLARY 1. *Every p-adatropic polynomial has degree greater than or equal to p.*

Proof. Since p divides each term on the left of $(*)$, it must also divide the degree, n.

THEOREM 1. *In a field generated by a p-adatropic polynomial of degree p, the prime ideal (p) must split completely.*

In what follows we will normalize against translation so that the powers of p occur with abscissas near 0. Specifically, $x_0 = -n/2$ if n is even and $x_0 = (1-n)/2$ otherwise. We will also avoid the symmetry $f_1(\theta) \leftrightarrow \pm f_2(-\theta)$ which gives rise to the same field since these polynomials have the same zeros.

It follows from Corollary 1 that there are no linear p-adatropic polynomials. Furthermore, this result dictates that the only p-adatropic quadratic polynomials are those where $p = 2$. These 2-adatropic polynomials were studied extensively by H. Cohn [3]. We summarize his results:

Let $v = (-1)^s 2^k$. The only parametrized family of 2-adatropic quadratic polynomials is the one given by $f(x) = x^2 + (1-v)x + v$. Let $f(\theta) = 0$. We