ACTA ARITHMETICA
L (1988)

Arithmetical properties of gap series with algebraic coefficients

by

ZHU YAOCHEN* (Beijing, China)

1. Introduction. The Liouville numbers $\sum_{k=0}^{\infty} g^{-k!}$ ($g \in Z$, $\geqslant 2$) are formed by rapidly converging series. Generally, we consider the gap series

$$\sigma(z) = \sum_{k=0}^{\infty} c_k z^{e_k}, \tag{1}$$

where $c_k \in \bar{Q}^{\times}$ and $e_k \in N$, and the radius of convergence r is positive. Cijsouw and Tijdeman [2] proved the transcendence of $\sigma(\alpha)$ for $\alpha \in \bar{Q}$ with $0 < |\alpha| < r$ under global growth conditions depending only on c_k and e_k. Bundschuh and Wylegala [1] considered the algebraic independence of values of $\sigma(z)$ at algebraic points. The author [6] gave some further results about the algebraic independence of values of $\sigma(z)$ at algebraic points.

Furthermore, Mahler [4] considered the more general gap series

$$F_0(z) = \sum_{k=0}^{\infty} f_k z^k, \tag{2}$$

where there are increasing sequences of natural numbers $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ satisfying

$$0 = \lambda_1 \leqslant \mu_1 < \lambda_2 \leqslant \mu_2 < \ldots < \lambda_n \leqslant \mu_n < \ldots$$

such that

$$f_k = 0 \quad (\mu_n < k < \lambda_{n+1}) \quad \text{but} \quad f_{\mu_n} \neq 0, \quad f_{\lambda_{n+1}} \neq 0 \quad (n = 1, 2, \ldots),$$

and the radius of convergence R_0 is positive. Assuming $f_k \in Z$ ($k = 1, 2, \ldots$), Mahler gave a result about the transcendence of values of $F_0(z)$ at algebraic points.

In this paper we consider the transcendence and the algebraic independence of values of certain gap series, which have the form of (2) and algebraic

* This work was supported by an Alexander von Humboldt Fellowship.

coefficients, at algebraic points. In Section 2 we prove a fundamental theorem about the algebraic independence of values of gap series at algebraic points. In Section 3 we give some applications of the fundamental theorem. We establish the transcendence and the algebraic independence of values of $F_0(z)$ at algebraic points, and the algebraic independence of values of $\sigma(z)$ and its derivatives at algebraic points. In particular, all the results of [1], [2], [4] and [6] are corollaries of our fundamental theorem.

It is a pleasure to acknowledge my indebtedness to Professor R. Tijdeman for his helpful suggestions.

2. The Fundamental Theorem

2.1. Formulation of the Fundamental Theorem. Let $P \in C[z_1, \ldots, z_t]$. We put $\partial(P) = \sum_{i=1}^{t} \deg_{z_i}(P)$, and denote the length of P, i.e. the sum of the absolute values of the coefficients of P, by $L(P)$, and the height of P, i.e. the maximum of the absolute values of P, by $H(P)$. For $\alpha \in \bar{Q}$, if $P(z)$ is its minimal polynominal, then $d(P)$, $L(P)$, $H(P)$ are called the degree, the length, the height of α, respectively. We denote them by $d(\alpha)$, $L(\alpha)$ and $H(\alpha)$, respectively. If $\alpha = \alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(d)}$ are all the conjugates of α, we put

$$\overline{|\alpha|} = \max(1, |\alpha^{(1)}|, \ldots, |\alpha^{(d)}|).$$

For $\alpha, \beta \in \bar{Q}$, we have

$$\overline{|\alpha+\beta|} \leqslant \overline{|\alpha|} + \overline{|\beta|} \quad \text{and} \quad \overline{|\alpha\beta|} \leqslant \overline{|\alpha|}\,\overline{|\beta|}.$$

For $\alpha_1, \ldots, \alpha_t \in \bar{Q}$, the number $M \in N$ is called a *denominator of* $\alpha_1, \ldots, \alpha_t$, if $M\alpha_1, \ldots, M\alpha_t$ are all algebraic integers. We denote the least denominator of $\alpha_1, \ldots, \alpha_t$ by $\operatorname{den}(\alpha_1, \ldots, \alpha_t)$. We denote the positive constants by $N_0, C_1, C_2, \ldots$ These constants and the constants in $\ll$ are all independent of n.

Let $s, t \in N, \geqslant 1$. Suppose that

(3) $$F_\nu(z) = \sum_{k=0}^{\infty} f_{\nu,k} z^k \quad (\nu = 1, 2, \ldots, s)$$

are s power series satisfying the following conditions:

For any ν $(1 \leqslant \nu \leqslant s)$,

(i) $f_{\nu,k} \in \bar{Q}$ $(k = 0, 1, 2, \ldots)$.

(ii) There exist increasing sequences of natural numbers $\{\lambda_{\nu,n}\}_{n=1}^{\infty}$ and $\{\mu_{\nu,n}\}_{n=1}^{\infty}$ satisfying

$$0 = \lambda_{\nu,1} \leqslant \mu_{\nu,1} < \lambda_{\nu,2} \leqslant \mu_{\nu,2} < \ldots < \lambda_{\nu,n} \leqslant \mu_{\nu,n} < \ldots$$

such that

$$f_{\nu,k} = 0 \ (\mu_{\nu,n} < k < \lambda_{\nu,n+1}), \quad \text{but} \quad f_{\nu,\mu_{\nu,n}} \neq 0,\ f_{\nu,\lambda_{\nu,n+1}} \neq 0 \quad (n = 1, 2, \ldots).$$

(iii) The radius of convergence R_ν of series F_ν is positive. We put

$$D_n = [Q(f_{1,0}, \ldots, f_{1,\mu_{1,n}}, \ldots, f_{s,0}, \ldots, f_{s,\mu_{s,n}}) : Q],$$

$$A_n = \max_{\substack{0 \leqslant k \leqslant \mu_{\nu,n} \\ 1 \leqslant \nu \leqslant s}} \overline{|f_{\nu,k}|},$$

$$M_n = \operatorname{den}(f_{1,0}, \ldots, f_{1,\mu_{1,n}}, \ldots, f_{s,0}, \ldots, f_{s,\mu_{s,n}}),$$

and for any ν $(1 \leqslant \nu \leqslant s)$ put

$$P_{\nu,k}(z) = \sum_{h=\lambda_{\nu,k}}^{\mu_{\nu,k}} f_{\nu,h} z^h,$$

$$p_{\nu,k}(z) = \sum_{h=0}^{\mu_{\nu,k}-\lambda_{\nu,k}} f_{\nu,h+\lambda_{\nu,k}} z^h = P_{\nu,k}(z)/z^{\lambda_{\nu,k}} \quad (k = 1, 2, \ldots).$$

Then we have

(4) $$F_\nu(z) = \sum_{k=1}^{\infty} P_{\nu,k}(z) = \sum_{k=1}^{\infty} p_{\nu,k}(z) z^{\lambda_{\nu,k}} \quad (1 \leqslant \nu \leqslant s).$$

FUNDAMENTAL THEOREM. *Suppose that series* (3) *satisfy*

(5) $$\lim_{n\to\infty} \big(\max_{1 \leqslant \nu \leqslant s} \mu_{\nu,n} + \log A_n + \log M_n\big) D_n \big/ \min_{1 \leqslant \nu \leqslant s} \lambda_{\nu,n+1} = 0,$$

and that $\alpha_1, \ldots, \alpha_t \in \bar{Q}$ *with* $0 < |\alpha_1| < \ldots < |\alpha_t| < \min_{1 \leqslant \nu \leqslant s} R_\nu$. *If there exists an increasing sequence of natural numbers* $\{k_n\}_{n=1}^{\infty} = \{k_n(\alpha_1, \ldots, \alpha_t)\}_{n=1}^{\infty}$ *such that*

(6) $$P_{\nu,k_n}(\alpha_\mu) \neq 0 \quad (n = 1, 2, \ldots;\ \nu = 1, \ldots, s;\ \mu = 1, \ldots, t),$$

and for any nonempty subset $\mathcal{T}$ *of the set* $\mathcal{S} = \{(\nu, \mu)(1 \leqslant \nu \leqslant s,\ 1 \leqslant \mu \leqslant t)\}$ *there exists* $(\nu_0, \mu_0) = (\nu_0(\mathcal{T}), \mu_0(\mathcal{T})) \in \mathcal{T}$ *satisfying*

(7) $$\varliminf_{n\to\infty} \sum_{(\nu,\mu)\in\mathcal{T}} |P_{\nu,k_n}(\alpha_\mu)| / |P_{\nu_0,k_n}(\alpha_{\mu_0})| = 1,$$

then $F_\nu(\alpha_\mu)$ $(1 \leqslant \nu \leqslant s,\ 1 \leqslant \mu \leqslant t)$ *are algebraically independent.*

Remark. Our proof also yields the more refined assertion: Let $\mathcal{S}^*$ be a nonempty subset of $\mathcal{S}$. If for any nonempty subset $\mathcal{T}$ of $\mathcal{S}^*$ there exists $(\nu_0, \mu_0) = (\nu_0(\mathcal{T}), \mu_0(\mathcal{T})) \in \mathcal{T}$ satisfying (7), then $F_\nu(\alpha_\mu)$ $((\nu, \mu) \in \mathcal{S}^*)$ are algebraically independent.

2.2. A criterion of algebraic independence. For $P \in Z[z_1, \ldots, z_t]$, we put

$$\Lambda(P) = 2^{\partial(P)} L(P).$$

With any $(\theta_1, \ldots, \theta_l) \in C^l$ we associate an order function of $u \in N$ (see [3]

$$O(u|\theta_1, \ldots, \theta_l) = \operatorname{Sup} \log |P(\theta_1, \ldots, \theta_l)|^{-1},$$

where the supremum is taken over all $P \in Z[z_1, \ldots, z_l]$ such that

$$P(\theta_1, \ldots, \theta_l) \neq 0 \quad \text{and} \quad A(P) \leqslant u.$$

The following properties are obvious:

1° $O(u|\ \theta_1, \ldots, \theta_l) \leqslant O(u|\ \theta_1, \ldots, \theta_l, \varphi)$ for $(\theta_1, \ldots, \theta_l) \in C^l$ and $\varphi \in C$.

2° $O(u|\ \theta_1, \ldots, \theta_l) \leqslant O(v|\ \theta_1, \ldots, \theta_l)$ for $u, v \in N$, $u \leqslant v$.

3° $O(uv|\ \theta_1, \ldots, \theta_l) \geqslant O(u|\ \theta_1, \ldots, \theta_l) + O(v|\ \theta_1, \ldots, \theta_l)$ for $u, v \in N$.

LEMMA 1. *Suppose that* $\Theta = \{\theta_1, \ldots, \theta_L\} \subset C$ *has the following propert*) *For any subset* $T = \{\theta_{\mu_1}, \ldots, \theta_{\mu_l}\} \subset \Theta$ *consisting of* l $(1 \leqslant l \leqslant L)$ *element. there exist* l *infinite sequences of complex numbers*

$$\{\theta_{v,n}\}_{n=1}^{\infty} = \{\theta_{v,n}(T)\}_{n=1}^{\infty} \ (1 \leqslant v \leqslant l)$$

such that

(i) $\lim\limits_{n \to \infty} \theta_{v,n} = \theta_{\mu_v}$, $|\theta_{v,n} - \theta_{\mu_v}| > 0$ $(n \geqslant 1)$ $(v = 1, \ldots, l)$,

(ii) $\sum\limits_{v=1}^{l} |\theta_{v,n} - \theta_{\mu_v}| \sim \max\limits_{1 \leqslant v \leqslant l} |\theta_{v,n} - \theta_{\mu_v}|$ $(n \to \infty)$,

(iii) *There exists a sequence of natural numbers* $\{u_n\}_{n=1}^{\infty} = \{u_n(T)\}_{n=1}^{\infty}$ *wit* $u_n \to \infty$ *as* $n \to \infty$ *satisfying*

$$\max_{1 \leqslant v \leqslant l} |\theta_{v,n} - \theta_{\mu_v}| \leqslant \exp(-O(u_n|\ \theta_{1,n}, \ldots, \theta_{l,n})) \quad (n \geqslant n_0(T)).$$

Then $\theta_1, \ldots, \theta_L$ *are algebraically independent.*

Proof. See [7].

2.3. Auxiliary lemmas. Let $\alpha_1, \ldots, \alpha_t \in \bar{Q}$ with $0 < |\alpha_1| < \ldots < |\alpha$ $< \min\limits_{1 \leqslant v \leqslant s} R_v$. We put

$$\hat{F}_{v,\mu}(z) = \sum_{k=1}^{\infty} p_{v,k}(\alpha_\mu) z^{\lambda_{v,k}} \quad (1 \leqslant v \leqslant s;\ 1 \leqslant \mu \leqslant t), \tag{8}$$

and denote the radius of convergence by $\hat{R}_{v,\mu}$. We put

$$\hat{D}_n = [Q(p_{v,k}(\alpha_\mu)\ (1 \leqslant v \leqslant s,\ 1 \leqslant \mu \leqslant t,\ 1 \leqslant k \leqslant n)) : Q],$$

$$\hat{A}_n = \max_{\substack{1 \leqslant v \leqslant s \\ 1 \leqslant \mu \leqslant t \\ 1 \leqslant k \leqslant n}} \overline{|p_{v,k}(\alpha_\mu)|},$$

$$\hat{M}_n = \operatorname{den}(p_{v,k}(\alpha_\mu)\ (1 \leqslant v \leqslant s,\ 1 \leqslant \mu \leqslant t,\ 1 \leqslant k \leqslant n)).$$

LEMMA 2. $\hat{R}_{v,\mu} \geqslant R_v$ $(1 \leqslant v \leqslant s,\ 1 \leqslant \mu \leqslant t)$.

Proof. If $0 < |\beta| < R_v$, then

$$|p_{v,k}(\alpha_\mu)|\,|\beta|^{\lambda_{v,k}} < \sum_{h=\lambda_{v,k}}^{\mu_{v,k}} |f_{v,h}| \max(|\alpha_\mu|, |\beta|)^h.$$

Since $|\alpha_\mu|, |\beta| < R_v$, the lemma follows.

LEMMA 3. *If* (5) *holds, then*

$$\lim_{n \to \infty} \big(\max_{1 \leqslant v \leqslant s} \mu_{v,n} + \log \hat{A}_n + \log \hat{M}_n\big) \hat{D}_n / \min_{1 \leqslant v \leqslant s} \lambda_{v,n+1} = 0.$$

Proof. Since we have

$$\hat{D}_n \leqslant D_n,$$

$$\hat{A}_n \leqslant \max_{1 \leqslant v \leqslant s} \mu_{v,n} \cdot C_1^{\max\limits_{1 \leqslant v \leqslant s} \mu_{v,n}} \cdot A_n,$$

$$\hat{M}_n \leqslant C_2^{\max\limits_{1 \leqslant v \leqslant s} \mu_{v,n}} \cdot M_n,$$

the lemma holds.

Suppose that the sequence $\{k_n\}_{n=1}^{\infty}$ satisfies (6). We put

$$\Phi_{v,\mu,n} = \sum_{k=1}^{k_n - 1} p_{v,k}(\alpha_\mu) \alpha_\mu^{\lambda_{v,k}} = \sum_{k=1}^{k_n - 1} P_{v,k}(\alpha_\mu),$$

$$\Psi_{v,\mu,n} = \hat{F}_{v,\mu}(\alpha_\mu) - \Phi_{v,\mu,n} = \sum_{k=k_n}^{\infty} p_{v,k}(\alpha_\mu) \alpha_\mu^{\lambda_{v,k}} = \sum_{k=k_n}^{\infty} P_{v,k}(\alpha_\mu)$$

$$(n = 1, 2, \ldots;\ v = 1, \ldots, s;\ \mu = 1, \ldots, t).$$

LEMMA 4. *If* (5) *holds, then for sufficiently large* n,

$$|\Psi_{v,\mu,n}| \leqslant |P_{v,k_n}(\alpha_\mu)| \leqslant |\Psi_{v,\mu,n}| \quad (1 \leqslant v \leqslant s,\ 1 \leqslant \mu \leqslant t).$$

Proof. Since $\hat{M}_{k_n} \cdot p_{v,k_n}(\alpha_\mu)$ is a nonzero algebraic integer, we have

$$|\operatorname{Norm}(\hat{M}_{k_n} \cdot p_{v,k_n}(\alpha_\mu))| \geqslant 1,$$

therefore

$$|P_{v,k_n}(\alpha_\mu)| \geqslant |\alpha_\mu|^{\lambda_{v,k_n}} (\hat{M}_{k_n} \hat{A}_{k_n})^{-\hat{D}_{k_n}} \tag{9}$$

$$(n = 1, 2, \ldots;\ v = 1, \ldots, s;\ \mu = 1, \ldots, t).$$

Taking ϱ such that

$$|\alpha_t| < \varrho < \min_{1 \leqslant v \leqslant s} R_v,$$

by Lemma 2 we have

$$|p_{v,n}(\alpha_\mu)| \leqslant \varrho^{-\lambda_{v,n}}$$

for sufficiently large n. Since $0<\varrho^{-1}|\alpha_\mu|<1$, we get

$$(10)\qquad \Big|\sum_{k=n}^{\infty} p_{\nu,k}(\alpha_\mu)\cdot\alpha_\mu^{\lambda_{\nu,k}}\Big| \leqslant \sum_{k=n}^{\infty}(\varrho^{-1}|\alpha_\mu|)^{\lambda_{\nu,k}} \ll (\varrho^{-1}|\alpha_\mu|)^{\lambda_{\nu,n}} \quad (1\leqslant\nu\leqslant s,\ 1\leqslant\mu\leqslant t).$$

By Lemma 3, noticing $\lim\limits_{n\to\infty}\lambda_{\nu,n}/\lambda_{\nu,n+1}=0$, we obtain

$$\lim_{n\to\infty}(\varrho^{-1}|\alpha_\mu|)^{\lambda_{\nu,k_n+1}}/\big(|\alpha_\mu|^{\lambda_{\nu,k_n}}\cdot(\hat M_{k_n}\hat A_{k_n})^{-\hat D_{k_n}}\big)=0,$$

hence from (9) and (10), we have

$$\Big|\sum_{k=k_n+1}^{\infty} P_{\nu,k}(\alpha_\mu)\Big|<C_3|P_{\nu,k_n}(\alpha_\mu)| \quad (1\leqslant\nu\leqslant s;\ 1\leqslant\mu\leqslant t),$$

for sufficiently large n, where $0<C_3<1$. Therefore the lemma follows.

LEMMA 5. *If $\alpha\in\bar{\mathbf{Q}}^\times$, then*

$$|\alpha|\geqslant\big(\max(\operatorname{den}(\alpha),\ \overline{|\alpha|})\big)^{-2d(\alpha)}.$$

Proof. See [5], § 1.2.

LEMMA 6. *For $u\in N$ and sufficiently large n,*

$$O(u|\ \Phi_{1,1,n},\Phi_{1,2,n},\dots,\Phi_{1,t,n},\dots,\Phi_{s,1,n},\Phi_{s,2,n},\dots,\Phi_{s,t,n}) \ll \big(\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_n-1}+\log\hat A_{k_n-1}+\log\hat M_{k_n-1}\big)\hat D_{k_n-1}\cdot\log u.$$

Proof. Denoting $m=st$, we suppose that

$$P(z_1,\dots,z_m)=\sum_{i_1=0}^{N_1}\dots\sum_{i_m=0}^{N_m}p_{i_1,\dots,i_m}z_1^{i_1}\dots z_m^{i_m}$$

with $p_{i_1,\dots,i_m}\in\mathbf{Z}$, $\deg_{z_i}(P)=N_i$, and that

$$\Lambda(P)=2^{\partial(P)}L(P)\leqslant u.$$

We have

$$(11)\qquad \partial(P)=N_1+\dots+N_m\ll\log u,\qquad L(P)\leqslant u.$$

Suppose that

$$\alpha=P(\Phi_{1,1,n},\Phi_{1,2,n},\dots,\Phi_{1,t,n},\dots,\Phi_{s,1,n},\Phi_{s,2,n},\dots,\Phi_{s,t,n})\neq0.$$

We have

$$d(\alpha)\ll\hat D_{k_n-1},$$

$$\operatorname{den}(\alpha)\ll\big(C_4^{\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_n-1}}\cdot\hat M_{k_n-1}\big)^{\partial(P)},$$

$$\overline{|\alpha|}\ll L(P)\cdot\big((k_n-1)C_5^{\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_n-1}}\cdot\hat A_{k_n-1}\big)^{\partial(P)},$$

where $C_4, C_5>1$. Hence by Lemma 5 and (11) we get

$$-\log|\alpha|\ll\big(\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_n-1}+\log\hat A_{k_n-1}+\log\hat M_{k_n-1}\big)\hat D_{k_n-1}\cdot\log u.$$

Therefore the lemma is proved.

2.4. Proof of the Fundamental Theorem. It is enough to verify the conditions of Lemma 1 for

$$\Theta=\{F_\nu(\alpha_\mu)\ (1\leqslant\nu\leqslant s,\ 1\leqslant\mu\leqslant t)\}.$$

Put

$$\mathscr{S}=\{(\nu,\mu)\ (1\leqslant\nu\leqslant s,\ 1\leqslant\mu\leqslant t)\}.$$

Let T be any nonempty subset of Θ, and the set of suffixal tuples (ν,μ) of T be $\mathscr{T}$. By (7), there exist an infinite subsequence $\{k_{l_n}\}_{n=1}^{\infty}=\{k_{l_n}(T)\}_{n=1}^{\infty}$ of $\{k_n\}_{n=1}^{\infty}$ and a tuple $(\nu_0,\mu_0)\in\mathscr{T}$ such that

$$|P_{\nu,k_{l_n}}(\alpha_\mu)|=o\big(|P_{\nu_0,k_{l_n}}(\alpha_{\mu_0})|\big)\quad(n\to\infty)$$

for $(\nu,\mu)\in\mathscr{T}\setminus(\nu_0,\mu_0)$. By Lemma 4 we get

$$|\Psi_{\nu,\mu,l_n}|=o(|\Psi_{\nu_0,\mu_0,l_n}|)\quad(n\to\infty)$$

for $(\nu,\mu)\in\mathscr{T}\setminus(\nu_0,\mu_0)$. Taking $\theta_{\nu,\mu,n}=\theta_{\nu,\mu,n}(T)=\sum_{k=1}^{k_{l_n}-1}P_{\nu,k}(\alpha_\mu)$ for any $(\nu,\mu)\in\mathscr{T}$, condition (ii) of Lemma 1 is verified. Furthermore, from (6) and by Lemma 4, condition (i) of Lemma 1 is also verified.

In order to verify condition (iii) of Lemma 1, we put

$$\beta_n=\beta_n(T)=\big(\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_{l_n}-1}+\log\hat A_{k_{l_n}-1}+\log\hat M_{k_{l_n}-1}\big)\hat D_{k_{l_n}-1}/\min_{1\leqslant\nu\leqslant s}\lambda_{\nu,k_{l_n}},$$

and

$$u_n=u_n(T)=[\exp(1/\sqrt{\beta_n})].$$

By Lemma 3, $u_n\to\infty$ $(n\to\infty)$. From Lemma 6, noticing property 1° of Section 2.2, we get

$$(12)\qquad O\big(u_n|\ \Phi_{\nu,\mu,l_n}((\nu,\mu)\in\mathscr{T})\big) \ll\frac{1}{\sqrt{\beta_n}}\big(\max_{1\leqslant\nu\leqslant s}\mu_{\nu,k_{l_n}-1}+\log\hat A_{k_{l_n}-1}+\log\hat M_{k_{l_n}-1}\big)\hat D_{k_{l_n}-1}.$$

From (10), we have

$$(13)\qquad |\Psi_{\nu_0,\mu_0,l_n}|\ll C_6^{\lambda_{\nu_0,k_{l_n}}}\ll C_6^{\min_{1\leqslant\nu\leqslant s}\lambda_{\nu,k_{l_n}}}$$

where $0 < C_6 < 1$. By Lemma 3, we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{\beta_n}} \Big(\max_{1\leqslant v\leqslant s} \mu_{v,k_{l_n}-1} + \log \hat{A}_{k_{l_n}-1} + \log \hat{M}_{k_{l_n}-1} \Big) \hat{D}_{k_{l_n}-1} \Big/ \min_{1\leqslant v\leqslant s} \lambda_{v,k_{l_n}} = \lim_{n\to\infty} \sqrt{\beta_n} = 0.$$

Hence from (12) and (13), noticing $\log C_6 < 0$, we deduce that

$$|\Psi_{v_0,\mu_0,l_n}| \ll \exp\big(-O\big(u_n|\ \Phi_{v,\mu,l_n}((v,\mu)\in\mathcal{S})\big)\big)$$

for sufficiently large n. Therefore condition (iii) of Lemma 1 is verified. The Fundamental Theorem is proved.

3. Applications

3.1. Transcendence. For series (2) we suppose that all the coefficients $f_k \in \bar{Q}$. We put

$$D_n^{(0)} = [Q(f_0, \ldots, f_n): Q],$$
$$A_n^{(0)} = \max(\overline{|f_0|}, \ldots, \overline{|f_n|}),$$
$$M_n^{(0)} = \operatorname{den}(f_0, \ldots, f_n),$$

and

$$P_k(z) = \sum_{h=\lambda_k}^{\mu_k} f_h z^h,$$
$$p_k(z) = \sum_{h=0}^{\mu_k-\lambda_k} f_{h+\lambda_k} z^h \quad (k = 1, 2, \ldots).$$

Then we have

$$F_0(z) = \sum_{k=1}^{\infty} P_k(z) = \sum_{k=1}^{\infty} p_k(z) z^{\lambda_k}.$$

In the Fundamental Theorem taking $s = t = 1$, we deduce the following

THEOREM 3.1. *Suppose that series* (2) *has algebraic coefficients and satisfies*

$$\lim_{n\to\infty} (\mu_n + \log A_n^{(0)} + \log M_n^{(0)}) D_n^{(0)}/\lambda_{n+1} = 0 \tag{14}$$

and that $\alpha \in \bar{Q}$ with $0 < |\alpha| < R_0$. Then $F_0(\alpha) \in \bar{Q}$ if and only if there exists a constant $N = N(\alpha)$ such that

$$P_k(\alpha) = 0 \quad (k \geqslant N).$$

Remark 1. If all $f_h \in Z$, then $D_n^{(0)} = M_n^{(0)} = 1$, $A_n^{(0)} \ll C_7^{\mu_n}$, and the condition (14) becomes

$$\lim_{n\to\infty} (\mu_n/\lambda_{n+1}) = 0.$$

Hence Theorem 3.1 implies the result of Mahler [4].

Remark 2. If $\lambda_n = \mu_n$ $(n = 1, 2, \ldots)$, then series (2) becomes series (1). Hence Theorem 3.1 implies the result of Cijsouw and Tijdeman [2]. Inversely, noticing Lemma 3 and applying the theorem of [2] to the series

$$\hat{F}_0(z) = \sum_{n=1}^{\infty} p_{k_n}(\alpha) z^{\lambda_{k_n}},$$

we deduce Theorem 3.1.

3.2. Algebraic independence.

THEOREM 3.2. *Suppose that series* (2) *has algebraic coefficients and satisfies* (14), *and that $\alpha_1, \ldots, \alpha_t \in \bar{Q}$ with $0 < |\alpha_1| < \ldots < |\alpha_t| < R_0$. If there exists an increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty} = \{k_n(\alpha_1, \ldots, \alpha_t)\}_{n=1}^{\infty}$ such that*

$$P_{k_n}(\alpha_\mu) \neq 0 \quad (n = 1, 2, \ldots;\ \mu = 1, \ldots, t), \tag{15}$$

$$\log\left|\frac{p_{k_n}(\alpha_\mu)}{p_{k_n}(\alpha_\tau)}\right| < 0 \quad (n \geqslant n_0), \quad \text{or} \quad = o(\lambda_{k_n}) \quad (n\to\infty) \quad (1 \leqslant \mu < \tau \leqslant t), \tag{16}$$

then $F_0(\alpha_1), \ldots, F_0(\alpha_t)$ are algebraically independent.

Proof. In the Fundamental Theorem we put $s = 1$. It is enough to verify condition (7). For any τ $(2 \leqslant \tau \leqslant t)$ and $\mu < \tau$, since $|\alpha_\mu| < |\alpha_\tau|$, from (16) we obtain

$$\frac{|P_{k_n}(\alpha_\mu)|}{|P_{k_n}(\alpha_\tau)|} = \frac{|p_{k_n}(\alpha_\mu)|}{|p_{k_n}(\alpha_\tau)|} \cdot \left|\frac{\alpha_\mu}{\alpha_\tau}\right|^{\lambda_{k_n}} \to 0 \quad (n\to\infty).$$

Hence (7) is verified and the theorem is proved.

Remark 1. Taking $\lambda_n = \mu_n$ $(n = 1, 2, \ldots)$ in series (2), we deduce the theorem of [1] from Theorem 3.2.

Remark 2. Put

$$\hat{F}_\mu(z) = \sum_{k=1}^{\infty} p_k(\alpha_\mu) z^{\lambda_k} \quad (\mu = 1, \ldots, t). \tag{17}$$

If $p_k(\alpha_\mu) = 0$ $(1 \leqslant \mu \leqslant t)$ for $k \neq k_n$ $(n = 1, 2, \ldots)$, then (15) and (16) imply that the radii of convergence of series (17) are equal. This fact suggests us the following

THEOREM 3.2′. *If condition* (16) *of Theorem 3.2 is replaced by*

(16a) *the radii of convergence of series* (17) *are equal and finite,*

then $F_0(\alpha_1), \ldots, F_0(\alpha_t)$ *are algebraically independent.*

Proof. As above it is enough to verify condition (7). Let $\hat{R}_0 > 0$ be the radius of convergence of $\hat{F}_\mu(z)$. For any τ $(2 \leqslant \tau \leqslant t)$ we take ϱ_1, ϱ_2 such that

$$\frac{|\alpha_{\tau-1}|}{|\alpha_\tau|}\hat{R}_0 < \varrho_1 < \hat{R} \tag{18}$$

and

$$\hat{R}_0 < \varrho_2 < \frac{|\alpha_\tau|}{|\alpha_{\tau-1}|}\varrho_1. \tag{19}$$

By Lemma 2, $|\alpha_\tau| < R_0 \leqslant \hat{R}_0$. Hence from (18) we have

$$|\alpha_1| < |\alpha_2| < \ldots < |\alpha_{\tau-1}| < \varrho_1 < \hat{R}_0. \tag{20}$$

From (20), for sufficiently large n,

$$|P_{k_n}(\alpha_\mu)| \leqslant (\varrho_1^{-1}|\alpha_\mu|)^{\lambda_{k_n}} \quad (\mu < \tau). \tag{21}$$

From (19), there exists an infinite subsequence $\{k_{l_n}\}_{n=1}^{\infty} = \{k_{l_n}(\tau)\}_{n=1}^{\infty}$ of $\{k_n\}_{n=1}^{\infty}$ such that

$$|P_{k_{l_n}}(\alpha_\tau)| \geqslant (\varrho_2^{-1}|\alpha_\tau|)^{\lambda_{k_{l_n}}}. \tag{22}$$

From (21) and (22), noticing (19), we get

$$\frac{|P_{k_{l_n}}(\alpha_\mu)|}{|P_{k_{l_n}}(\alpha_\tau)|} \leqslant \left(\frac{\varrho_2|\alpha_\mu|}{\varrho_1|\alpha_\tau|}\right)^{\lambda_{k_{l_n}}} \leqslant \left(\frac{\varrho_2|\alpha_{\tau-1}|}{\varrho_1|\alpha_\tau|}\right)^{\lambda_{k_{l_n}}} \to 0 \quad (n \to \infty)$$

for $\mu < \tau$. Thus we obtain (7), and the theorem follows.

3.3. Further results about the algebraic independence.

THEOREM 3.3. *Suppose that series* (3) *satisfy* (5) *and*

$$\lambda_{\nu,n} = \lambda_{1,n} + O(1) \quad (\nu = 1, 2, \ldots, s), \tag{23}$$

and that $\alpha_1, \ldots, \alpha_t \in \bar{\boldsymbol{Q}}$ *with* $0 < |\alpha_1| < \ldots < |\alpha_t| < \min_{1 \leqslant \nu \leqslant s} R_\nu$. *If there exists an increasing sequence of natural numbers* $\{k_n\}_{n=1}^{\infty}$ *satisfying*

$$P_{\nu,k}(\alpha_\mu) \neq 0 \quad (n = 1, 2, \ldots; \ \nu = 1, \ldots, s; \ \mu = 1, \ldots, t),$$

and

$$|p_{\nu,k_n}(\alpha_\mu)| = o(|p_{\tau,k_n}(\alpha_\mu)|) \quad (n \to \infty) \quad (\tau > \nu, \ \mu = 1, \ldots, t), \tag{24}$$

$$\log\left|\frac{p_{s,k_n}(\alpha_\mu)}{p_{1,k_n}(\alpha_\tau)}\right| < 0 \quad (n \geqslant n_0) \quad \text{or} \quad = o(\lambda_{1,k_n}) \quad (n \to \infty) \quad (\mu < \tau), \tag{25}$$

then $F_\nu(\alpha_\mu)$ $(1 \leqslant \nu \leqslant s, 1 \leqslant \mu \leqslant t)$ *are algebraically independent.*

Proof. We verify condition (7). For any μ $(1 \leqslant \mu \leqslant t)$, from (23) and (24) we obtain

$$|P_{\nu,k_n}(\alpha_\mu)| = o(|P_{\tau,k_n}(\alpha_\mu)|) \quad (n \to \infty) \tag{26}$$

for $\tau > \nu$. For any $\tau > 1$, from (23) and (25) we get

$$\frac{|P_{s,k_n}(\alpha_\mu)|}{|P_{1,k_n}(\alpha_\tau)|} \leqslant \frac{|p_{s,k_n}(\alpha_\mu)|}{|p_{1,k_n}(\alpha_\tau)|} \cdot \left(\frac{|\alpha_\mu|}{|\alpha_\tau|}\right)^{\lambda_{1,k_n}} \to 0 \quad (n \to \infty) \tag{27}$$

for $\mu < \tau$. From (26) and (27) we deduce that

$$\begin{aligned} |P_{1,k_n}(\alpha_1)| &\prec |P_{2,k_n}(\alpha_1)| \prec \ldots \prec |P_{s,k_n}(\alpha_1)| \\ &\prec |P_{1,k_n}(\alpha_2)| \prec |P_{2,k_n}(\alpha_2)| \prec \ldots \prec |P_{s,k_n}(\alpha_2)| \\ &\prec \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\prec |P_{1,k_n}(\alpha_t)| \prec |P_{2,k_n}(\alpha_t)| \prec \ldots \prec |P_{s,k_n}(\alpha_t)|, \end{aligned}$$

where the symbol $f(n) \prec g(n)$ means that $f(n)/g(n) \to 0$ $(n \to \infty)$. Thus (7) is verified. The proof is complete.

COROLLARY. *Suppose that series* (1) *satisfies*

$$\lim_{n\to\infty} (e_n + \log a_n + \log m_n) d_n / e_{n+1} = 0,$$

where $d_n = [\boldsymbol{Q}(c_0, \ldots, c_n): \boldsymbol{Q}]$, $a_n = \max(\overline{|c_0|}, \ldots, \overline{|c_n|})$, $m_n = \mathrm{den}(c_0, \ldots, c_n)$, *and that* $\alpha_1, \ldots, \alpha_t \in \bar{\boldsymbol{Q}}$ *with* $0 < |\alpha_1| < \ldots < |\alpha_t| < r$, *then* $\sigma^{(\nu)}(\alpha_\mu)$ $(0 \leqslant \nu \leqslant s-1, 1 \leqslant \mu \leqslant t)$ *are algebraically independent.*

Remark 1. If $s = 1$, then (25) becomes (16), hence Theorem 3.3 implies Theorem 3.2.

Remark 2. Similar to Theorem 3.2′, condition (25) of Theorem 3.3 may be replaced by the condition

(25a) The radii of convergence of series (8) are equal and finite.

THEOREM 3.4. *Suppose that series* (3) *satisfy* (5) *and*

$$\lambda_{\nu,n} \sim \lambda_{1,n} \quad (n \to \infty) \quad (\nu = 1, 2, \ldots, s) \tag{28}$$

and that $\alpha_1, \ldots, \alpha_s \in \bar{\boldsymbol{Q}}$, $0 < |\alpha_1| < \ldots < |\alpha_s| < \min_{1 \leqslant \nu \leqslant s} R_\nu$. *If there exists an increasing sequence of natural numbers* $\{k_n\}_{n=1}^{\infty} = \{k_n(\alpha_1, \ldots, \alpha_s)\}_{n=1}^{\infty}$ *satisfying*

$$P_{\nu,k_n}(\alpha_\nu) \neq 0 \quad (n = 1, 2, \ldots; \ \nu = 1, \ldots, s),$$

and

$$\log\left|\frac{p_{\nu,k_n}(\alpha_\nu)}{p_{\tau,k_n}(\alpha_\tau)}\right| < 0 \quad (n \geqslant n_0), \quad \text{or} \quad = o(\lambda_{1,k_n}) \quad (n \to \infty) \quad (\nu < \tau), \tag{29}$$

then $P_\nu(\alpha_\nu)$ $(1 \leqslant \nu \leqslant s)$ *are algebraically independent.*

Proof. From (28) and (29) we get

$$|P_{\nu,k_n}(\alpha_\nu)| = o\left(|P_{\tau,k_n}(\alpha_\tau)|\right) \quad (n\to\infty)$$

for $\nu<\tau$. Therefore we deduce the theorem from the remark of the fundamental theorem.

Remark. Condition (29) may be replaced by the condition

(29a) The radii of convergence of the series

$$\hat{F}_\nu(z) = \sum_{k=1}^{\infty} p_{\nu,k}(\alpha_\nu) z^{\lambda_{\nu,k}} \quad (\nu = 1, \ldots, s)$$

are equal and finite.

THEOREM 3.5. *Suppose that series* (3) *satisfy* (5) *and*

$$\sum_{l=1}^{\nu} \lambda_{l,n} \sim \lambda_{\nu,n} \quad (1\leqslant \nu\leqslant s) \tag{30}$$

and that $\alpha_1, \ldots, \alpha_t \in \bar{Q}$ *with* $0<|\alpha_1|<\ldots<|\alpha_t|<\min_{1\leqslant\nu\leqslant s} R_\nu$. *If there exists an increasing sequence of natural numbers* $\{k_n\}_{n=1}^{\infty} = \{k_n(\alpha_1, \ldots, \alpha_t)\}_{n=1}^{\infty}$ *such that*

$$P_{\nu,k_n}(\alpha_\mu) \neq 0 \quad (n = 1, 2, \ldots;\ \nu = 1, \ldots, s;\ \mu = 1, \ldots, t),$$

and if the radii of convergence of series (8) *satisfy*

$$0<\hat{R}_{\nu,1} = \ldots = \hat{R}_{\nu,t} = \hat{R}_\nu < \infty \quad (\nu = 1, \ldots, s), \tag{31}$$

then $P_\nu(\alpha_\mu)$ $(1\leqslant\nu\leqslant s,\ 1\leqslant\mu\leqslant t)$ *are algebraically independent.*

Proof. As above we verify condition (7). Let $\mathscr{T}$ be any nonempty subset of $\mathscr{S} = \{(\nu, \mu)\ (1\leqslant\nu\leqslant s,\ 1\leqslant\mu\leqslant t)\}$. We put

$$\nu_0 = \max\{\nu|\ (\nu, \mu)\in\mathscr{T}\},$$

$$\mu_0 = \max\{\mu|\ (\nu_0, \mu)\in\mathscr{T}\}.$$

We prove that there exists an infinite subsequence $\{k_{l_n}\}_{n=1}^{\infty} = \{k_{l_n}(\mathscr{T})\}_{n=1}^{\infty}$ of $\{k_n\}_{n=1}^{\infty}$ such that

$$|P_{\nu,k_{l_n}}(\alpha_\mu)| = o\left(|P_{\nu_0,k_{l_n}}(\alpha_{\mu_0})|\right) \quad (n\to\infty) \tag{32}$$

for $(\nu, \mu)\in\mathscr{T}\backslash(\nu_0, \mu_0)$.

Take $\varrho_3, \varrho_4, \varrho_5$, satisfying

$$\frac{|\alpha_{\mu_0-1}|}{|\alpha_{\mu_0}|}\hat{R}_{\nu_0} < \varrho_3 < \hat{R}_{\nu_0}, \tag{33}$$

$$\hat{R}_{\nu_0} < \varrho_4 < \frac{|\alpha_{\mu_0}|}{|\alpha_{\mu_0-1}|}\varrho_3, \tag{34}$$

$$\varrho_5 < \min_{\nu<\nu_0} \hat{R}_\nu. \tag{35}$$

By Lemma 2, $|\alpha_{\mu_0}| < R_{\nu_0} \leqslant \hat{R}_{\nu_0}$. Hence from (33) we have

$$|\alpha_\mu| < \varrho_3 < \hat{R}_{\nu_0} \tag{36}$$

for $\mu<\mu_0$. From (36), for sufficiently large n,

$$|P_{\nu_0,k_n}(\alpha_\mu)| \leqslant (\varrho_3^{-1}|\alpha_\mu|)^{\lambda_{\nu_0,k_n}} \tag{37}$$

for $\mu<\mu_0$. By (34), there exists an infinite subsequence $\{k_{l_n}\}_{n=1}^{\infty} = \{k_{l_n}(\mathscr{T})\}_{n=1}^{\infty}$ of $\{k_n\}_{n=1}^{\infty}$ such that

$$|P_{\nu_0,k_{l_n}}(\alpha_{\mu_0})| \geqslant (\varrho_4^{-1}|\alpha_{\mu_0}|)^{\lambda_{\nu_0,k_{l_n}}}. \tag{38}$$

From (37) and (38), noticing (34), we obtain

$$|P_{\nu_0,k_{l_n}}(\alpha_\mu)| = o\left(|P_{\nu_0,k_{l_n}}(\alpha_{\mu_0})|\right) \quad (n\to\infty) \tag{39}$$

for $\mu<\mu_0$. Furthermore, from (35), for sufficiently large n,

$$|P_{\nu,k_n}(\alpha_\mu)| \leqslant (\varrho_5^{-1}|\alpha_\mu|)^{\lambda_{\nu,k_n}} \quad (n\to\infty) \tag{40}$$

for $\nu<\nu_0$ and $1\leqslant\mu\leqslant t$. Noticing (30) and $\varrho_4^{-1}|\alpha_{\mu_0}|<1$, from (38) and (40) we deduce that

$$|P_{\nu,k_{l_n}}(\alpha_\mu)| = o\left(|P_{\nu_0,k_{l_n}}(\alpha_{\mu_0})|\right) \quad (n\to\infty) \tag{41}$$

for $\nu<\nu_0$. By (39) and (41) we obtain (32). Thus the theorem is proved.

Remark. If $s=1$ and $\lambda_n = \mu_n$ $(n = 1, 2, \ldots)$, then we deduce the results of [6] from Theorem 3.5.

References

[1] P. Bundschuh and F.-J. Wylegala, *Algebraic independence of certain gap series*, Arch. Math. 38 (1982), pp. 438–442.

[2] P. L. Cijsouw and R. Tijdeman, *On the transcendence of certain power series of algebraic numbers*, Acta Arith. 23 (1973), pp. 301–305.

[3] M. A. Durand, *Fonction Θ-ordre et classification de C^p*, C. R. Acad. Sc. Paris, Ser. A., 280 (1975), pp. 1085–1088.

[4] K. Mahler, *Arithmetic properties of lacunary power series with integral coefficients*, J. Austr. Math. Soc. 5 (1965), pp. 56–64.

[5] M. Waldschmidt, *Nombres transcendents*, LNM 402, Springer, Berlin–Heidelberg New York 1974.
[6] Zhu Yaochen, *On the algebraic independence of certain power series of algebraic numbers*, Chin. Ann. Math. 5B (1) (1984), pp. 109–117.
[7] — *On a criterion of algebraic independence of numbers*, Kexue Tongbao (to appear).

INSTITUTE OF APPLIED MATHEMATICS
ACADEMIA SINICA
Beijing, China
MAX-PLANCK-INSTITUT FÜR MATHEMATIK
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Received on 4.3.1986
and in revised form on 19.8.1986 (1611)

ACTA ARITHMETICA
L (1988)

Diagonalizable indefinite integral quadratic forms

by

DONALD G. JAMES* (University Park, Penn.)

1. Introduction. Let L be a Z-lattice on an indefinite regular quadratic Q-space V, of finite dimension $n \geqslant 3$, with associated symmetric bilinear form $f\colon V \times V \to Q$. Assume, for convenience, that $f(L, L) = Z$, namely the scale of L is Z. Let $x_1, \ldots, x_n$ be a Z-basis for L and put $d = dL = \det f(x_i, x_j)$, the discriminant of the lattice L. We study a Hasse principle for diagonalization, that is, we investigate the set $\mathscr{D}$ of discriminants with the property that all indefinite lattices with discriminant in $\mathscr{D}$, which diagonalize locally at all primes, also diagonalize globally over Z. Since all lattices diagonalize locally at the odd primes (see O'Meara [5]), the local condition is only significant for the prime 2. A result of J. Milnor states that all odd lattices L with $dL = \pm 1$ have an orthogonal basis (see Serre [6] or Wall [7]). Thus $\pm 1 \in \mathscr{D}$. It is also shown in James [3] that $\pm 2q \in \mathscr{D}$ for all primes $q \equiv 3 \bmod 4$, but $2.41 \notin \mathscr{D}$. We prove here the following

THEOREM. *Let* $p \equiv 1 \bmod 4$, $p' \equiv 5 \bmod 8$, $q \equiv 3 \bmod 4$ *and* $q' \equiv 3 \bmod 8$ *be primes with Legendre symbols* $\left(\frac{q}{p}\right) = \left(\frac{p'}{p}\right) = -1$. *Then* $\pm d \in \mathscr{D}$ *for the following values of* d:

$$1,\ 2,\ 4,\ q,\ 2q,\ q^2,\ 2q^2,\ 2qq',\ 2p',\ pq,\ 2pq,\ 2pp',\ 2p'^2,\ 2p'q.$$

For each of the discriminants d considered in the above theorem, except $d = 4$, the local condition that L_2 diagonalizes is equivalent to the global condition that L is an odd lattice, namely the set $\{f(x, x) \mid x \in L\}$ contains at least one odd number. An exact determination of $\mathscr{D}$ appears very difficult. In fact we will exhibit $d \in \mathscr{D}$ with d containing arbitrarily many prime factors (see Proposition 2).

Let $i = i(L) = i(V)$ be the Witt index of V. Then $\mathscr{D}(i)$ denotes the set of discriminants of lattices L on spaces V with Witt index at least i which diagonalize over Z whenever the localization L_2 diagonalizes. It is also useful to introduce the stable version $\mathscr{D}(\infty)$ of discriminants where $dL \in \mathscr{D}(\infty)$

* This research was partially supported by the National Science Foundation.