Let now $g_j(\xi^p) \neq 0$ for $j \in S$, where S is a set of cardinality c_1+1. We have for $j \in S$

$$(y-\xi^p)^{(n-c_1)/p} \| f_j(y),$$

hence by the inductive assumption $f_j(y)$ has at least as many terms as $(y-\xi^p)^{(n-c_1)/p}$, i.e. by Lemma 5 and by (32) at least $\prod_{i=2}^{k}(c_i+1)$ terms. It follows that $f(x)$ has at least $\prod_{i=1}^{k}(c_i+1)$ terms, but this is exactly by Lemma 5 the number of terms of $(x-\xi)^n$.

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Perfect powers in products of integers from a block of consecutive integers

by

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To Professor P. Erdős on his 75th birthday

1. Erdős and Selfridge [5] confirmed an old conjecture by proving that the product of two or more consecutive positive integers is never a power. We consider a more general question. For an integer $v > 1$, we define $P(v)$ to be the greatest prime factor of v and we write $P(1) = 1$. Let $m \geq 0$ and $k \geq 2$ be integers. Let $d_1, \ldots, d_t$ with $t \geq 2$ be distinct integers in the interval $[1, k]$. For integers $l \geq 2$, $y > 0$ and $b > 0$ with $P(b) \leq k$, we consider the equation

$$(m+d_1)\ldots(m+d_t) = by^l. \tag{1}$$

For $l \geq 2$, let v_l be a real number satisfying $0 < v_l \leq 1$. If $\alpha > 1$ and $k^\alpha < m \leq k^l$, then equation (1) implies that $P(m+d_i) \leq k$ for $1 \leq i \leq t$ and hence

$$t < \alpha^{-1}k + \pi(k).$$

See Erdős and Turk [6], Lemma 2.1. For $m > k^l$, we have

THEOREM 1. *Let $\varepsilon > 0$ and $0 \leq u < 1$. Suppose that equation* (1) *with*

$$l \geq 3, \quad m > k^l, \quad t \geq v_l k \tag{2}$$

is satisfied. Then the inequalities

$$v_l \geq \frac{1}{2-u}+\varepsilon, \quad v_l \geq \frac{1}{2}\left(1+\frac{2l-3+u}{(2l-4+u)(l-1)}\right) \tag{3}$$

imply that k is bounded by an effectively computable number depending only on ε.

We observe that (3) with an optimal choice of u is somewhat stronger than

$$v_l \geq \frac{1}{2}\left(1+\frac{1}{l-1}\right).$$

We apply Theorem 1 together with Lemma 6 of [9] to derive

COROLLARY 1. *Equation* (1) *with* (2) *and*

$$v_l \geqslant \frac{1}{2}\left(1+\frac{4l^2-8l+7}{2(l-1)(2l^2-5l+4)}\right) \tag{4}$$

implies that k is bounded by an effectively computable absolute constant.

This is an improvement of Theorem 1 of [9] where we showed that equation (1), with

$$l \geqslant 4, \quad m > k^l, \quad t \geqslant \frac{1}{2}\left(1+\frac{1}{l-2}+\varepsilon\right)k$$

or

$$l = 3, \quad m > k^3, \quad m \notin \left(k^4 (\log k)^3, k^{4+\varepsilon}\right), \quad t \geqslant Ck$$

for some effectively computable positive number $C < 1$, implies that k is bounded by an effectively computable number depending only on ε. If $l = 3$, we see that Corollary 1 is a considerable improvement of the result mentioned above. The improvement depends on the method of Roth [8] as elaborated in Halberstam and Roth [7] on the difference between consecutive l-free integers. If l exceeds a sufficiently large effectively computable number C_1, it is shown in [9] that (4) can be replaced by

$$v_l k \geqslant kl^{-1/11}+\pi(k)+2.$$

The proof of this result depends on the theory of linear forms in logarithms and the value of C_1 turns out to be very large. Further, in view of this result, we remark that Corollary 1, in fact, improves on earlier results of [9] only for $l \leqslant C_1$. The proofs of Theorem 1 and Corollary 1 also depend on the above mentioned result for l sufficiently large together with a theorem of Baker [1] on the approximations of certain algebraic numbers by rationals.

If $l = 2$, we prove

THEOREM 2. *Let $\varepsilon > 0$. Equation* (1) *with*

$$l = 2, \quad m > k^2, \quad t \geqslant k-(1-\varepsilon)k\frac{\log\log k}{\log k} \tag{5}$$

implies that k is bounded by an effectively computable number depending only on ε.

This is an improvement of Theorem 2 of [9] where we assumed that

$$t \geqslant k-(1-\varepsilon)k\frac{\log\log\log k}{\log k}.$$

Theorem 2 answers a question of Erdős [4], p. 88. The proof of Theorem 2 depends on a theorem of Baker [2] on the integer solutions of a hyperelliptic equation together with elementary arguments of Erdős [3] and [4]. For earlier results on equation (1), see Erdős and Selfridge [5]. For a given k, it follows from Theorem 2 of [10] and a theorem of Baker [2] that equation (1) with $l \geqslant 3$ and $y > 1$ implies that $\max(m, y, l)$ is bounded by an effectively computable number depending only on k. Also, by Baker [2], equation (1) with $l = 2$ and $t \geqslant 3$ implies that $\max(m, y)$ is bounded by an effectively computable number depending only on k.

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2. In this section, we shall prove Theorem 1. Let $0 < \varepsilon < 1$. Suppose that equation (1) with (2) and (3) is satisfied. Then, by Lemma 6 of [9], we may assume that l is bounded by an effectively computable number depending only on ε. Let $l \geqslant 3$ be fixed. Let $\varepsilon_1 = \varepsilon/(500\, l^3)$. We assume that k exceeds a sufficiently large effectively computable number depending only on l and ε_1. Denote by $c_1, c_2, \ldots, c_7$ effectively computable positive numbers depending only on l and ε_1. We put

$$\tau = \left(1+\frac{\varepsilon_1}{4}\right)v_l^{-1} < 2, \quad \tau_1 = (v_l^{-1}-1)/(l-1), \quad \tau_2 = l/(2l-1)$$

and

$$\tau_3 = \tau/(2l-1), \quad \tau_4 = (2l-1)/(l-1), \quad f = k^{1-\tau_3-\varepsilon_1}.$$

Notice that

$$2-\tau_1-(1-\tau_3)\tau_4 = \frac{\varepsilon_1 v_l^{-1}}{4(l-1)} < \frac{\varepsilon_1}{4},$$

since $l \geqslant 3$. Further observe that

$$(1-\tau_3-\varepsilon_1)(\tau_4-\varepsilon_1) > (1-\tau_3)\tau_4-\varepsilon_1(1+\tau_4)$$

and $\tau_4 \leqslant 5/2$, since $l \geqslant 3$. Therefore

$$(1-\tau_3-\varepsilon_1)(\tau_4-\varepsilon_1) > 2-\tau_1-4\varepsilon_1. \tag{6}$$

Let N be the least positive integer such that

$$1+\tau_2+\ldots+\tau_2^{N-1} \geqslant \tau_4-\varepsilon_1. \tag{7}$$

Notice that N is bounded by an effectively computable number depending only on l and ε_1.

We see from equation (1) that

$$m+d_i = a_i x_i^l, \quad 1 \leqslant i \leqslant t, \tag{8}$$

where a_i and x_i are positive integers satisfying

$$P(a_i) \leqslant k, \quad \Big(x_i, \prod_{p\leqslant k} p\Big) = 1. \tag{9}$$

Observe that $a_1, \ldots, a_t$ are distinct, since $m > k^l$. Put $S = \{a_1, \ldots, a_t\}$. For every prime $p \leqslant k$, we choose an $f(p) \in S$ such that p does not appear to a higher power in the factorisation of any other element of S. Denote by S_1 the subset of S obtained by deleting from S all $f(p)$ with $p \leqslant k$. For a given $p \leqslant k$ and an integer $e' \geqslant 1$, observe that there are at most $[k/p^{e'}]$ multiples of $p^{e'}$ in S_1. Then

$$\prod_{a_i \in S_1} a_i \leqslant \prod_{p \leqslant k} p^{[k/p]+[k/p^2]+\ldots} = k!$$

(This fundamental argument is due to Erdős [4], Lemma 3.) Hence

$$\prod_{a_i \in S_1} a_i \leqslant k^k, \qquad |S_1| \geqslant t - \pi(k).$$

Consequently there exists a subset S_2 of S_1 such that $|S_2| \geqslant c_1 k$ with $c_1 = \varepsilon_1/16$ and

$$a_i \leqslant k^{\tau}, \qquad a_i \in S_2. \tag{10}$$

Now we see from (8), (2) and (10) that $x_i > 1$ and hence, by (9), $x_i > k$ with $a_i \in S_2$. Further we prove

LEMMA 1. *There exists a subset S_3 of S_2 such that $|S_3| \geqslant c_2 k^{1-\varepsilon_1}$ and*

$$x_i > k^{2-\tau_1-5\varepsilon_1}$$

whenever $a_i \in S_3$.

Proof of Lemma 1. We may assume that $m \leqslant k^{3l}$, otherwise the lemma follows from (8) and (10). Denote by S_4 the subset of S_2 such that x_i is prime whenever $a_i \in S_4$. If x_i is composite, we see from (9) that $x_i > k^2$. Therefore we may assume that $|S_4| \geqslant c_1 k/2$. By prime number theory, there exists a subset S_5 of S_4 such that $s_5 := |S_5| \geqslant c_1 k/4$ and

$$x_i \geqslant c_3 k \log k, \qquad a_i \in S_5. \tag{11}$$

By permuting $d_1, \ldots, d_t$, we may assume that $a_1, \ldots, a_{s_5}$ are elements of S_5 and

$$x_1 < x_2 < \ldots < x_{s_5}. \tag{12}$$

For distinct integers i and j with $1 \leqslant i, j \leqslant s_5$, we see from (8) and (11) that

$$0 < |a_i x_i^l - a_j x_j^l| < k \leqslant \eta x_j$$

where η is the constant appearing in Lemma 1 of Halberstam and Roth [7]. For $1 \leqslant i < s_5$, we consider the inequalities

$$k^{\tau}(x_i - x_1)^{2l-1} \leqslant \tfrac{1}{2} x_1^l.$$

Now we see from (10) and Lemma 2 of Halberstam and Roth [7] that there exists μ_2 with $1 < \mu_2 \leqslant k^{\varepsilon_1}$ such that

$$x_{\mu_2} - x_{\mu_1} \geqslant c_4 x_{\mu_1}^{\tau_2} k^{-\tau_3}$$

where $\mu_1 = 1$. Proceeding inductively, we find integers $1 = \mu_1 < \mu_2 < \ldots < \mu_R < s_5$ such that $\mu_j \leqslant (j-1) k^{\varepsilon_1}$ for $2 \leqslant j \leqslant R$ with $R = [c_5 k^{1-\varepsilon_1}]$ and

$$x_{\mu_j} - x_{\mu_{j-1}} \geqslant c_4 x_{\mu_{j-1}}^{\tau_2} k^{-\tau_3}, \qquad 2 \leqslant j \leqslant R. \tag{13}$$

Put

$$v_r = [R/2^{N-r+1}], \qquad 0 \leqslant r \leqslant N,$$

and, for simplicity, we write $X_{v_r} = x_{\mu_{v_r}}$. By (12) and (13), observe that

$$X_{v_r} - X_{v_{r-1}} \geqslant c_6 f X_{v_{r-1}}^{\tau_2}, \qquad 1 \leqslant r \leqslant N.$$

In particular, we have

$$X_{v_r} \geqslant c_6 f X_{v_{r-1}}^{\tau_2}, \qquad 1 \leqslant r \leqslant N,$$

which, together with (7), implies that

$$X_{v_N} \geqslant c_7 f^{\tau_4 - \varepsilon_1}. \tag{14}$$

Now we see from (14), (6) and (12) that

$$x_{\mu_j} \geqslant k^{2-\tau_1-5\varepsilon_1}, \qquad v_N \leqslant j \leqslant R.$$

This completes the proof of Lemma 1.

Proof of Theorem 1. Let $0 < \varepsilon < 1$. Suppose that equation (1) with (2) and (3) is satisfied. Then, as already observed in the beginning of this section, we may assume that l is bounded by an effectively computable number depending only on ε. Let $l \geqslant 3$ be fixed. Put $\varepsilon_1 = \varepsilon/(500\, l^3)$. Denote by c_8, c_9, c_{10} and c_{11} effectively computable positive numbers depending only on l and ε_1. We may assume that $k \geqslant c_8$ with c_8 sufficiently large and we shall arrive at a contradiction.

Put $s_3 = |S_3|$. By permuting the suffixes of $d_1, \ldots, d_t$, there is no loss of generality in assuming that $a_1, \ldots, a_{s_3}$ are elements of S_3 and $a_1 < a_2 < \ldots < a_{s_3}$. By (10) and $\tau < 2$,

$$\sum_{j=1}^{s_3-1} \log\left(\frac{a_{j+1}}{a_j}\right) \leqslant 2 \log k.$$

Now we apply inequality $s_3 \geqslant c_2 k^{1-\varepsilon_1}$ of Lemma 1 to conclude that there exists μ with $1 \leqslant \mu < s_3$ such that

$$\log\left(\frac{a_{\mu+1}}{a_\mu}\right) \leqslant \frac{c_9 \log k}{k^{1-\varepsilon_1}}. \tag{15}$$

By (8) and (15), we obtain

$$0 \neq \left|\left(\frac{a_{\mu+1}}{a_\mu}\right)^{1/l} - \frac{x_\mu}{x_{\mu+1}}\right| < \frac{2k}{a_{\mu+1}x_{\mu+1}^l}. \tag{16}$$

Now we are going to apply a theorem of Baker [1] to obtain a lower bound for the left hand side of inequality (16). In the notation of this theorem, we put $a = la_{\mu+1}$, $b = la_\mu$, $m = 1$ and $n = l$. Observe that $1 \leqslant \mu_n \leqslant l$. By (15), we see that

$$\tfrac{7}{8}a \leqslant b < a \quad \text{and} \quad \left(\frac{a-b}{b}\right) \leqslant \frac{2c_9 \log k}{k^{1-\varepsilon_1}}$$

which, together with (10) and $(2\nu_l - 1) > 1/(l-1)$, implies that

$$\lambda \geqslant (lc_9 \log k)^{-2} k^{2-\tau-2\varepsilon_1} > k^{\nu_l^{-1}(2\nu_l-1)(1-4\varepsilon_1 l)}.$$

Further

$$2\mu_n(a+b) \leqslant 4l^2 k^\tau < k^{(1+\varepsilon_1)\nu_l^{-1}}.$$

Therefore $\varkappa$ is bounded by

$$\varkappa' = 1 + (1+12\varepsilon_1 l)(2\nu_l - 1)^{-1}.$$

Notice that $\varkappa' < l$ and $c^{-1} \leqslant 8l2^l a_{\mu+1}$. Hence we apply the above mentioned theorem of Baker to conclude that the left hand side of (16) exceeds

$$(c_{10} a_{\mu+1} x_{\mu+1}^{\varkappa'} 1)^{-1}. \tag{17}$$

Combining (16) and (17), we obtain

$$x_{\mu+1}^{l-\varkappa'} \leqslant c_{11} k.$$

Now we apply Lemma 1 to conclude that

$$(2-\tau_1-5\varepsilon_1)(l-\varkappa') \leqslant 1+\varepsilon_1. \tag{18}$$

By (3) and (17),

$$(2-\tau_1-5\varepsilon_1) \geqslant \frac{2l-3+u}{l-1}\left(1+\frac{\varepsilon}{10l}\right) \tag{19}$$

and

$$l-\varkappa' \geqslant \frac{l-1}{2l-3+u}\left(1-\frac{\varepsilon}{20l}\right). \tag{20}$$

By (18), (19) and (20), we see that $\varepsilon < 40\varepsilon_1 l$ which is a contradiction. This completes the proof of Theorem 1.

Proof of Corollary 1. Suppose that equation (1) with (2) and (4) is satisfied. In view of Lemma 6 of [9], we may assume that l is bounded by an effectively computable absolute constant. Let $l \geqslant 3$ be fixed. Now we apply Theorem 1 with $u = 4/(2l-1)$ and

$$2\varepsilon = \frac{4l^2-8l+7}{2(l-1)(2l^2-5l+4)} - \frac{2}{(2l-3)} = \frac{2l-5}{2(l-1)(2l-3)(2l^2-5l+4)}$$

to complete the proof of Corollary 1.

3. Proof of Theorem 2. Let $0 < \varepsilon < 1/4$. Suppose that equation (1) with (5) is satisfied. We assume that c_{12} is a sufficiently large number depending only on ε. Put $c_{13} = c_{12}^3$. We may assume that $k \geqslant c_{13}$. By (1) with $l = 2$,

$$m + d_i = A_i X_i^2, \quad 1 \leqslant i \leqslant t, \tag{21}$$

where A_i and X_i are positive integers such that $P(A_i) \leqslant k$ and A_i is square free. Further $A_1, \ldots, A_t$ are distinct, since $m > k^2$. Put $S_6 = \{A_1, \ldots, A_t\}$. Observe that

$$A_1 \ldots A_t \leqslant \prod_{p \leqslant k} p^{[k/p]+1} \leqslant (4k)^k.$$

Therefore there exists a subset S_7 of S_6 such that $|S_7| \geqslant \varepsilon k/2$ and

$$A_i \leqslant k(\log k)^{1-(\varepsilon/2)}, \quad A_i \in S_7. \tag{22}$$

Here we have used that the product of ν distinct elements of S_7 is greater than or equal to $\nu!$. Further we see from (5), (21) and (22) that

$$X_i > k^{1/4}, \quad A_i \in S_7. \tag{23}$$

Denote by S_8 the set of all $A_i \in S_7$ with $A_i \leqslant 3k$. In this paragraph, we show that the inequality $|S_8| > \varepsilon k/4$ implies that k is bounded by an effectively computable number depending only on ε. We assume that

$$|S_8| > \varepsilon k/4.$$

Let $b_1, \ldots, b_s$ be all the integers between $c_{12}^{-2}k$ and $3k$ such that every proper divisor of b_i is less than or equal to $c_{12}^{-2}k$. If $b_i > c_{12}^{-1}k$, then every prime divisor of b_i exceeds c_{12}. Then, by taking c_{12} sufficiently large, we see from sieve argument that

$$s \leqslant c_{12}^{-1}k + \varepsilon k/32 < \varepsilon k/16.$$

Denote by S_9 the set of all $A_i \in S_8$ such that $A_i \geqslant c_{12}^{-2}k$. Observe that every element of S_9 is divisible by at least one b_i. Further notice that

$$|S_9| \geqslant |S_8| - c_{12}^{-2}k > \varepsilon k/8$$

if c_{12} is sufficiently large. Let c_{12} be chosen suitably. If every b_i appears in at most two elements of S_9, then

$$|S_9| \leqslant 2s < \varepsilon k/8$$

and this is a contradiction. Thus we may assume that there exists b_ν with $1 \leqslant \nu \leqslant s$ such that three distinct elements of S_9 are divisible by b_ν. By permuting the suffixes of $d_1, \ldots, d_t$, there is no loss of generality in assuming that A_1, A_2 and A_3 are elements of S_9 and they are divisible by b_ν. Put

$$B_i = b_\nu^{-1} A_i, \quad 1 \leqslant i \leqslant 3,$$

and

$$R = b_\nu^{-1}(d_2 - d_1), \quad R' = b_\nu^{-1}(d_3 - d_1).$$

Observe that B_1, B_2, B_3 and R, R' are integers of absolute values at most $3c_{12}^2$. By (21),

$$B_2 B_3 (X_2 X_3)^2 = (B_1 X_1^2 + R)(B_1 X_1^2 + R').$$

Now we apply a theorem of Baker [2] and (23) to conclude that k is bounded by an effectively computable number depending only on ε.

Thus we may assume that $|S_8| \leqslant \varepsilon k/4$. Denote by S_{10} the complement of S_8 in S_7. Then

$$|S_{10}| \geqslant \varepsilon k/4. \tag{24}$$

Let A_i, A_j, A_μ and A_ν be elements of S_{10} such that

$$A_i A_j = A_\mu A_\nu. \tag{25}$$

Put

$$\Delta = (m+d_i)(m+d_j) - (m+d_\mu)(m+d_\nu). \tag{26}$$

If $\Delta \neq 0$, then (21) and (25) imply that

$$|\Delta| \geqslant \left((A_i A_j)(A_i X_i^2)(A_j X_j^2)\right)^{1/2} > 3km, \tag{27}$$

since $A_i, A_j \in S_{10}$. On the other hand, we see from (26) and $m > k^2$ that

$$|\Delta| \leqslant 2km + k^2 < (2k+1)m. \tag{28}$$

We see that (27) and (28) are inconsistent. Hence $\Delta = 0$. Then

$$d_i + d_j = d_\mu + d_\nu, \quad d_i d_j = d_\mu d_\nu,$$

since $m > k^2$. This implies that $d_i = d_\mu$ or $d_i = d_\nu$. Therefore $i = \mu$ or $i = \nu$.

Thus we have shown that there is no non-trivial relation (25) among the elements of S_{10}. Now we apply Lemma 4 of Erdős [4] and (22) to conclude that

$$|S_{10}| \leqslant 3k(\log k)^{-\varepsilon/2}. \tag{29}$$

By (24) and (29), we see that k is bounded by an effectively computable number depending only on ε. This completes the proof of Theorem 2.

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