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An Erdős–Kac theorem for integers without large prime factors

by

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Dedicated to Professor Erdős for his seventy fifth birthday

1. Introduction. Denote by $P(n)$ the largest prime factor of n if $n>1$ and set $P(1)=1$. For real numbers $x, y>1$ let $S(x, y)$ be the set of positive integers $n\leqslant x$ for which $P(n)\leqslant y$. As usual let $\Psi(x, y)$ denote the cardinality of $S(x, y)$. Also let $\alpha = \log x/\log y$.

In a recent paper [2] we derived a Turán–Kubilius inequality for $S(x, y)$. More specifically, for additive functions f we obtained a general upper bound for the variance of the sequence of values $f(n)$, where $n\in S(x, y)$. (Additive functions are those arithmetical functions f satisfying $f(mn)=f(m)+f(n)$ whenever $(m, n)=1$.) In the range

$$\exp\{(\log x)^{2/3}\}\leqslant y\leqslant x \tag{1.1}$$

the bound was uniform in f and y. In the special case when $f(n)=\nu(n)$, the number of prime factors of n, we were able to estimate the variance asymptotically. To be precise, in the range (1.1) we showed that as $x\to\infty$

$$\sum_{n\in S(x,y)}\{\nu(n)-\eta(x, y)\}^2 = \Psi(x, y)\,\Theta(x, y)\{1+o(1)\}. \tag{1.2}$$

Here

$$\eta(x, y) = \log\log y+\operatorname{li}(\alpha\xi) \tag{1.3}$$

and

$$\Theta(x, y) = \eta(x, y)-\alpha\xi/(\xi-1), \tag{1.4}$$

where $\xi=\xi(\alpha)$ is the unique positive solution of

$$e^{\xi}-1=\alpha\xi \tag{1.5}$$

and

$$\mathrm{li}(T) = \int_2^{\max(2,T)} dt/(\log t).$$

Interest in (1.2) derives from the fact that when α is much larger than $\log\log y$, we have $\Theta(x, y) = o(\eta(x, y))$. Classically, in probabilistic number theory, with regard to the distribution of $\nu(n)$ over all integers $n > 1$ or over special sequences, it always turns out that the mean and variance are asymptotically equal (see for instance Elliott [7], Vols. 1 and 2). So (1.2) is a natural example where this phenomenon fails. However, we conjectured in [2] that, as in the classical case, the analogue of the Erdős Kac theorem holds for $\nu(n)$, $n \in S(x, y)$, in the range (1.1).

The goal here is to prove Theorem A below which goes beyond our conjecture in two respects: (i) we establish a Gaussian limiting distribution by asymptotically estimating the moments and (ii) there is an improvement in the range (1.1) also.

For simplicity in certain arguments we shall prove Theorem A for $\Omega(n)$ (the number of prime factors of n counted with multiplicity) but the result holds as stated if $\Omega(n)$ is replaced by $\nu(n)$.

THEOREM A. *Let $\varepsilon > 0$ be arbitrarily small and*

$$(1.6) \qquad \exp\{(\log\log x)^{5/3+\varepsilon}\} < y \leqslant x.$$

Then for $k = 1, 2, 3, \ldots$

$$(1.7) \quad \lim_{x\to\infty} \frac{1}{\Psi(x, y)\,\Theta(x, y)^{k/2}} \sum_{n\in S(x,y)} \{\Omega(n)-\eta(x, y)\}^k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^k e^{-u^2/2}\,du.$$

Therefore as $x \to \infty$

$$(1.8) \qquad F_{x,y}(v) \overset{\mathrm{def}}{=} \frac{1}{\Psi(x, y)} \sum_{\substack{n\in S(x,y)\\ \Omega(n)-\eta(x,y) < v\sqrt{\Theta(x,y)}}} 1 \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-u^2/2}\,du.$$

The convergence in (1.7) *and* (1.8) *is uniform in y and v.*

Recently Hensley [8] has obtained a result like (1.8), more generally for certain additive functions but in a different range, namely

$$(\log x)^{2+\varepsilon} < y < \exp\{(\log x)^{\beta}\},$$

where $\beta < 1$. More recently Hildebrand [11] has established an analogue of (1.8) for $y \leqslant \exp\{(\log x)^{1/21}\}$ along with an estimate for the rate of convergence. Both Hensley [9] and Hildebrand [11] are able to discuss the local distribution of Ω as well, which we are unable to do at present. That is they can estimate

$$\Omega_k(x, y) = \sum_{\substack{n\in S(x,y)\\ \Omega(n)=k}} 1$$

asymptotically in their respective ranges for values k near the mean. But their results do not involve moments.

In a recent paper [1] we established an analogue of (1.8) for integers $n \leqslant x$ having all their prime factors $\geqslant y$. There we were able to discuss the local distribution as well. That earlier paper motivated us to study the distribution of $\Omega(n)$ in $S(x, y)$ but owing to technical difficulties we restricted attention first to the study of variance [2]; but this has now led us to Theorem A.

The method of this paper is based upon the asymptotic analysis of

$$(1.9) \qquad \Psi_z(x, y) = \sum_{n\in S(x,y)} z^{\Omega(n)}$$

for real values z close to 1. We employ a recent technique of Hildebrand [10] to estimate $\Psi_z(x, y)$ in the range (1.6). In addition we need to study the behaviour of certain functions satisfying a Volterra difference-differential equation. For this we employ ideas due to de Bruijn [4], [5] and certain improvements on these that were already utilized by us [2]. We interpret (1.9) suitably in terms of the bilateral Laplace transform of $F_{x,y}(v)$ and this yields (1.7) and (1.8).

To prove a result like (1.8) one would normally attempt to estimate the Fourier transform of $F_{x,y}(v)$ and appeal to quantitative Fourier inversion (see [7], Vol. 1, p. 69). Such an approach presented difficulties mainly in connection with the Volterra kernels which turned out to be complex valued. It is for this reason we chose to use the Laplace transform, in which case the Volterra kernels are positive (see § 4). The Laplace transform approach had the advantage of yielding estimates for the moments but the main drawback is that in passing from (1.7) to (1.8) the estimates for the rate of convergence are weak. That is why these error terms are not stated in Theorem A but they will be treated later during the course of the proof. At any rate the idea of 'quantitative Laplace inversion' discussed in Section 8 seems new and interesting for its own sake; it may have many other applications in situations where the Fourier transform does not succumb to a direct treatment.

All notation introduced so far will be retained. The $\ll$ and O symbols are equivalent and will be used interchangeably as is convenient. Implicit constants are absolute unless indicated otherwise, usually by subscripts. By c we shall mean a generic absolute constant and so its value need not be the same when used in different contexts.

The letter p will always denote a prime number. We shall use the Prime Number Theorem in its strong form

$$\pi(t) - \mathrm{li}(t) \ll tR(t),$$

where $\pi(t) = \sum_{p \leqslant t} 1$ and

$$R(t) = \exp\{-c(\log t)^{3/5}(\log\log t)^{-1/5}\}. \tag{1.10}$$

Throughout we assume x and y to be sufficiently large to avoid any confusion in various expressions involving them. Finally, z shall always satisfy $1/2 \leqslant z \leqslant 3/2$ so that all estimates will be uniform in z.

2. Preliminary estimates for $\Psi_z(x, y)$. The sum $\Psi(x, y)$ has been studied in detail (see [5], [6] and [10]). As in the case of Ψ, we have more generally for Ψ_z, the recurrence

$$\Psi_z(x, y) = \Psi_z(x, y^h) - z \sum_{y < p \leqslant y^h} \Psi_z(x/p, p). \tag{2.1}$$

Selberg [12] has shown that

$$\sum_{n \leqslant x} z^{\Omega(n)} = A(z)\, x(\log x)^{z-1}\left\{1 + O\left(\frac{1}{\log x}\right)\right\} \tag{2.2}$$

and the sum in (2.2) can be identified with $\Psi_z(x, y)$ when $y \geqslant x$. Here $A(z)$ is non-vanishing and continuously differentiable in z.

If $\sqrt{x} < y \leqslant x$ we let $y^h = x$ in (2.1). Then by (2.1), (2.2) and the Prime Number Theorem

$$\Psi_z(x, y) = A(z)\, x(\log x)^{z-1}\left(1 - z\int_y^x \frac{\left(1 - \frac{\log t}{\log x}\right)^{z-1} dt}{t\log t}\right)\left\{1 + O\left(\frac{1}{\log x}\right)\right\} \tag{2.3}$$
$$+ O\left(\sum_{y \leqslant p \leqslant x} \frac{x(\log(2x/p))^{z-2}}{p}\right).$$

In (2.3) the substitution $t = x^{1/s}$ yields

$$1 - z\int_y^x \frac{\left(1 - \frac{\log t}{\log x}\right)^{z-1}}{t\log t}\, dt = 1 - z\int_1^\alpha \frac{(s-1)^{z-1}}{s^z}\, ds. \tag{2.4}$$

With regard to the last term on the right-hand side of (2.3) the decomposition of $[y, x]$ into intervals $[x/(n+1), x/n]$, $n = 1, 2, \ldots$, shows that it is

$$\ll \frac{x}{\log x} \sum_{n < x/y+1} \frac{(\log 2n)^{z-2}}{n} = x(\log x)^{z-(3/2)} \sum_{n < x/y+1} \frac{(\log 2n)^{z-2}}{n(\log x)^{z-(1/2)}} \tag{2.5}$$
$$\ll x(\log x)^{z-(3/2)} \sum_{n=1}^{\infty} \frac{1}{n(\log 2n)^{3/2}} \ll x(\log x)^{z-(3/2)}.$$

If we denote by $\varrho_z(\alpha)$ the quantity in (2.4) for $1 \leqslant \alpha \leqslant 2$ then by (2.3) and (2.5)

$$\Psi_z(x, y) = A(z)\, x(\log x)^{z-1}\varrho_z(\alpha)\{1 + O(1/\sqrt{\log x})\} \quad \text{for} \quad 1 \leqslant \alpha \leqslant 2 \tag{2.6}$$

which is actually best possible when $z = 1/2$.

With the initial estimate (2.6) and the recurrence (2.1) we can show by induction on $[\alpha]$ that

$$\Psi_z(x, y) = A(z)\, x(\log x)^{z-1}\varrho_z(\alpha)\big(1 + O_\alpha(1/\sqrt{\log x})\big), \quad \alpha > 1 \tag{2.7}$$

where

$$\varrho_z(\alpha) = 1 - z\int_1^\alpha \frac{\varrho_z(s-1)(s-1)^{z-1}}{s^z}\, ds \quad \text{for} \quad \alpha \geqslant 2. \tag{2.8}$$

We omit the details of this derivation, since it will be a repetition of the well known procedure for $\Psi(x, y)$ (see [6]) but only mention that as in (2.3) the Prime Number Theorem will enable conversion of the sum in (2.1) into an integral.

Note that by setting $\varrho_z(\alpha) = 1$ for $0 < \alpha < 1$ we make $\varrho_z(\alpha)$ continuous for $\alpha > 0$, and extend (2.7) to $\alpha > 0$. In view of (2.6) it makes sense to define $\varrho_z(\alpha) = 0$ for $\alpha < 0$. This will make (2.8) valid for $\alpha > 0$.

3. The functions ϱ_z and ϱ_z^*. The advantage in comparing $\Psi_z(x, y)$ with $x(\log x)^{z-1}$ is that we immediately deduce $\varrho_z(\alpha)$ to be a decreasing function of α, because $z > 0$. But then there are other advantages in comparing Ψ_z with $x(\log y)^{z-1}$. So, for this purpose we define

$$\varrho_z^*(\alpha) = \varrho_z(\alpha)\alpha^{z-1}. \tag{3.1}$$

Then (2.8) becomes

$$\alpha^{1-z}\varrho_z^*(\alpha) = 1 - z\int_1^\alpha \frac{\varrho_z^*(s-1)}{s^z}\, ds \quad \text{for} \quad \alpha > 0. \tag{3.2}$$

From (3.2) and the initial conditions on $\varrho_z(\alpha)$ it follows that

$$\varrho_z^*(\alpha) = \frac{z}{\alpha}\int_{\alpha-1}^\alpha \varrho_z^*(t)\, dt, \quad \forall \alpha \neq 0 \tag{3.3}$$

and therefore by (3.1) we have

$$\varrho_z(\alpha) = \frac{z}{\alpha^z}\int_{\alpha-1}^\alpha \varrho_z(t)\, t^{z-1}\, dt. \tag{3.4}$$

Since ϱ_z is decreasing we see from (3.4) that

$$0 < \varrho_z(\alpha) \leqslant \frac{z\varrho_z(\alpha-1)}{\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{|z-1|}. \tag{3.5}$$

Therefore by iteration of (3.5) we get

$$\varrho_z(\alpha) \ll e^{-\alpha\log\alpha + O(\alpha)}. \tag{3.6}$$

From (3.1) and (3.6) it follows that

$$\varrho_z^*(\alpha) \ll e^{-\alpha\log\alpha + O(\alpha)} \tag{3.7}$$

as well.

In view of (3.6) it is of interest to check how far (2.6) yields an asymptotic estimate for Ψ_z as $\alpha \to \infty$ with x. But first we need to investigate the behaviour of $\varrho_z^*(\alpha)$ more closely, as $\alpha \to \infty$. This will involve the saddle point method and the study of a certain Volterra equation; these will be the contents of the next three sections.

4. A Volterra equation. The upper bound (3.7) is far too simple for our purpose. We shall obtain a fairly sharp asymptotic estimate for $\varrho_z^*(\alpha)$ by employing a method originally due to de Bruijn [4] and certain refinements on this method discussed by us [2].

Consider the Volterra equation

$$\lambda(\alpha) = \int_0^1 k(\alpha, t)\lambda(\alpha - t)\,dt, \tag{4.1}$$

where $k > 0$ for all $\alpha > 0$ and $0 < t < 1$. Also let k be *normalised*, in the sense that

$$\int_0^1 k(\alpha, t)\,dt = 1, \qquad \forall\alpha.$$

We say that the kernel k is *stabilising* if for every continuous λ

$$\lim_{\alpha\to\infty} \lambda(\alpha) = l$$

exists and is finite. The following lemma of de Bruijn [4] provides a sufficient condition for k to be stabilising:

LEMMA 1. *Let β be a positive constant $< 1/4$ and Φ a continuous function for $\alpha > 1$ that satisfies*

$$\Phi(\alpha) > 0, \qquad \sum_{n=1}^{\infty} \mu_n = \infty,$$

where

$$\mu_n = \min_{n\leqslant\alpha\leqslant n+2} \Phi(\alpha).$$

Then a sufficient condition for k to be stabilising is that

$$\int_E k(\alpha, t)\,dt > \Phi(\alpha)$$

holds for all measurable subsets E of $[0, 1]$ with measure $> \beta$. In this case we have

$$|\lambda(\alpha) - l| \leqslant l \prod_{2n+a<\alpha} \max\left\{\frac{3}{4(1-\beta)},\ 1-\mu_{2n+a}\right\}, \tag{4.2}$$

where $a = 0$ or 1 according as $[\alpha]$ is even or odd.

Suppose that $F(\alpha)$ is any continuous function satisfying

$$F(\alpha) = \frac{z}{\alpha}\int_{\alpha-1}^{\alpha} F(t)\,dt. \tag{4.3}$$

Let us set

$$\lambda(\alpha) = F(\alpha)/\varrho_z^*(\alpha) \tag{4.4}$$

and

$$k(\alpha, t) = k_z(\alpha, t) = \big(z\varrho_z^*(\alpha - t)\big)/\big(\alpha\varrho_z^*(\alpha)\big). \tag{4.5}$$

Clearly from (4.5) and (3.3) we see that k_z is positive and normalised. In addition on comparing (4.3) with (3.3) it follows that $\lambda(\alpha)$ in (4.4) satisfies (4.1).

We know that $\varrho_z(\alpha - t) > \varrho_z(\alpha)$ and therefore by (3.1)

$$\varrho_z^*(\alpha - t) > \frac{(\alpha - t)^{z-1}}{\alpha^{z-1}}\varrho_z^*(\alpha) \geqslant \left(1 - \frac{c}{\alpha}\right)\varrho_z^*(\alpha) \quad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{4.6}$$

In particular (4.6) yields

$$k_z(\alpha, t) \gg 1/\alpha \tag{4.7}$$

and so

$$\Phi(\alpha) \gg 1/(n+2). \tag{4.8}$$

Thus from Lemma 1 we deduce that there exists l such that

$$\frac{F(\alpha)}{\varrho_z^*(\alpha)} - l \to 0 \quad \text{as} \quad \alpha \to \infty. \tag{4.9}$$

Notice that from (4.8) and (4.2) of Lemma 1 we have an estimate on the rate of convergence in (4.9). But we can actually do better. We shall construct a special F satisfying (4.3) whose asymptotic behaviour can be estimated by the saddle point method. We can then use the well known technique of adjoint equations to get the value of l and therefore an asymptotic estimate for $\varrho_z^*(\alpha)$. This in turn leads to a superior quantitative version of (4.9).

5. Saddle point method. We construct the special F by considering a Laplace integral

$$F(\alpha) = \int_W e^{-\alpha t} p(t)\, dt, \tag{5.1}$$

where p will be determined using (4.3) and W will be chosen in such a way that the integrand vanishes rapidly along the ends of W. From (5.1) and (4.3) we are led to

$$F(\alpha) = z \int_W \frac{e^{-\alpha t}}{\alpha}\left(\frac{e^t-1}{t}\right) p(t)\, dt. \tag{5.2}$$

On the other hand integration-by-parts of (5.1) yields

$$F(\alpha) = \frac{e^{-\alpha t} p(t)}{-\alpha}\Big|_{\mathrm{end}\,W} + \int_W \frac{e^{-\alpha t}}{\alpha} p'(t)\, dt = \int_W \frac{e^{-\alpha t}}{\alpha} p'(t)\, dt \tag{5.3}$$

provided the evaluations at the ends of W vanish. On comparing (5.2) and (5.3) we require p to satisfy

$$p'(t) = z\left(\frac{e^t-1}{t}\right) p(t). \tag{5.4}$$

In view of this we choose

$$F(\alpha) = \frac{1}{2\pi i}\int_W \exp\left\{-\alpha s + z\int_0^s \frac{e^t-1}{t}\, dt\right\} ds, \tag{5.5}$$

where W is the contour starting at $-\pi i+\infty$ running parallel to the real axis upto $-\pi i$, then the line segment $[-\pi i, \pi i]$, and finally the half line πi to $\pi i+\infty$.

Next, let

$$g_\alpha(s) = g_{\alpha,z}(s) = \alpha s - z\int_0^s \frac{e^t-1}{t}\, dt. \tag{5.6}$$

Then the saddle points are given by $\frac{d}{ds}\{g_\alpha(s)\} = 0$. The additional convenience of W is that there is exactly one such saddle point $\xi_z = \xi_z(\alpha)$ "inside W", namely one that satisfies

$$z(e^{\xi_z}-1) = \alpha\xi_z. \tag{5.7}$$

It is easily seen that

$$\xi = \xi_1 = \log\alpha + \log\log\alpha + \frac{\log\log\alpha}{\log\alpha} + O\left(\frac{\log\log\alpha}{\log^2\alpha}\right).$$

Note from (5.7) (and (1.5)) that

$$\xi_z(\alpha) = \xi_1(\alpha/z). \tag{5.8}$$

Therefore

$$\xi_z = \log\alpha + \log\log\alpha - \log z - \frac{\log z}{\log\alpha} + \frac{\log\log\alpha}{\log\alpha} + O\left(\frac{\log\log\alpha}{\log^2\alpha}\right). \tag{5.9}$$

In the case $z = 1$ de Bruijn estimated $F(\alpha)$ asymptotically and in [2] we stated a quantitative version of his result. Since the saddle point analysis remains the same for $1/2 \leqslant z \leqslant 3/2$ we now give an estimate without proof:

$$F(\alpha) = \frac{1}{\sqrt{2\pi |g''_\alpha(\xi_z)|}} \exp\left\{-\alpha\xi_z + z\int_0^{\xi_z} \frac{e^t-1}{t}\, dt\right\}\{1+\varphi(\alpha)\}. \tag{5.10}$$

In (5.10) the derivative is with respect to s and

$$-g''_\alpha(\xi_z) \sim \alpha \quad \text{as} \quad \alpha \to \infty$$

because of (5.6) and (5.9). As in ([2], eqns. (1.12) and (3.8)), $\varphi(\alpha)$ satisfies

(i) $\varphi(\alpha) \ll \alpha^{-1}$ and (ii) $\varphi(\alpha) - \varphi(\alpha-u) \ll_n u(\alpha\xi_z^n)^{-1}$, $\forall n$, and $u \ll 1$

but here we shall only make use of (i). At any rate from (4.9) and (5.10) we have

$$\varrho_z^*(\alpha) = \frac{l(z)(1+o(1))}{\sqrt{2\pi |g''_\alpha(\xi_z)|}} \exp\left\{-\alpha\xi_z + z\int_0^{\xi_z} \frac{e^t-1}{t}\, dt\right\}. \tag{5.11}$$

In the next section we shall evaluate both $l(z)$ and the $o(1)$ in (5.11).

6. The adjoint equation. We begin by rewriting (4.3) as

$$\{\alpha F(\alpha)\}' = z\{F(\alpha) - F(\alpha-1)\}. \tag{6.1}$$

The *adjoint* of the equation is

$$\alpha H'(\alpha-1) = z\{H(\alpha) - H(\alpha-1)\}. \tag{6.2}$$

For any pair of functions F, H satisfying (6.1) and (6.2), the expression

$$\langle F, H\rangle = z\int_{\alpha-1}^{\alpha} F(u) H(u)\, du - \alpha F(\alpha) H(\alpha-1) \tag{6.3}$$

is constant. To verify this note that

$$\{z\int_{\alpha-1}^{\alpha} F(u) H(u)\, du\}' = z\{F(\alpha) H(\alpha) - F(\alpha-1) H(\alpha-1)\}$$

whereas by (6.1) and (6.2)

$$\begin{aligned}\{\alpha F(\alpha)H(\alpha-1)\}' &= \{\alpha F(\alpha)\}' H(\alpha-1)+\alpha F(\alpha)H'(\alpha-1)\\ &= z\{F(\alpha)-F(\alpha-1)\}H(\alpha-1)+zF(\alpha)\{H(\alpha)-H(\alpha-1)\}\\ &= z\{F(\alpha)H(\alpha)-F(\alpha-1)H(\alpha-1)\}\end{aligned}$$

and so $\langle F, H\rangle$ in (6.3) does not depend on α.

As in the case of (4.3) we now obtain by the use of Laplace integrals

$$H(\alpha) = \int_{-\infty}^{\infty} \exp\left\{\alpha s + s - z\int_0^s \frac{e^t-1}{t}\,dt\right\}\frac{ds}{s}, \qquad \alpha > -1 \tag{6.4}$$

as a special solution to (6.2). In (6.4) we interpret

$$\int_{-\infty}^{\infty} = \lim_{\delta\to 0^+}\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}$$

as the Cauchy principal value. The saddle point for H is $\xi^* = \xi_z^*(\alpha)$ which satisfies

$$(\alpha+1)\xi^* = z(e^{\xi^*}-1)-1. \tag{6.5}$$

Since the principal contribution from the right-hand side of (6.5) is due to ze^{ξ^*} it follows by comparison with (5.7) that

$$\xi^* = \xi_z(\alpha+1)+O\left(\frac{1}{\alpha\log\alpha}\right). \tag{6.6}$$

Analogous to (5.10) the saddle point method yields

$$H(\alpha) = \sqrt{\frac{2\pi}{g''_{\alpha+1}(\xi^*)}}\exp\left\{(\alpha+1)\xi^* - z\int_0^{\xi^*}\frac{e^t-1}{t}\,dt\right\}\frac{\{1+O(\alpha^{-1})\}}{\xi^*}, \tag{6.7}$$

where g is as in (5.6). In view of (6.6) estimate (6.7) remains unchanged if ξ^* is replaced by $\xi_z(\alpha+1)$.

Analogous to our earlier estimate in the case $z = 1$ (see [2], eqn. (3.14)) we now have

$$\xi_z(\alpha)-\xi_z(\alpha-u) = \frac{u\xi_z}{\alpha(\xi_z-1)}+O\left(\frac{u}{\alpha^2}\right), \qquad u \ll 1. \tag{6.8}$$

From (5.6) and (6.8) it follows (as in [2] eqn. (3.15)) that

$$g_\alpha(\xi_z(\alpha)) - g_{\alpha-u}(\xi_z(\alpha-u)) = u\xi_z + O(u/\alpha), \qquad u \ll 1. \tag{6.9}$$

On comparing (6.7) and (5.10) we see that

$$F(\alpha)H(\alpha) \sim e^{\xi_z}/(\alpha\xi_z) \sim 1/z \quad \text{as} \quad \alpha\to\infty \tag{6.10}$$

because of (6.6), (6.8) and (5.7). By similar reasoning

$$\alpha F(\alpha)H(\alpha-1) \sim 1/\xi_z \quad \text{as} \quad \alpha\to\infty. \tag{6.11}$$

Plugging (6.10) and (6.11) into (6.3) and letting $\alpha\to\infty$ we deduce that

$$\langle F, H\rangle = 1. \tag{6.12}$$

Next we compute $\langle \varrho_z^*, H\rangle$. The initial conditions on ϱ_z^* are

$$\varrho_z^*(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha < 0,\\ \alpha^{z-1} & \text{if} \quad 0 < \alpha < 1. \end{cases} \tag{6.13}$$

In the expression

$$\langle \varrho_z^*, H\rangle = z\int_{\alpha-1}^{\alpha} \varrho_z^*(u)H(u)\,du - \alpha\varrho_z^*(\alpha)H(\alpha-1) \tag{6.14}$$

we let $\alpha\to 0^+$. Since $H(u)$ is continuous for $u > -1$, it is bounded for $u\in[0, 1]$. So (6.13) shows that the integral in (6.14) is

$$\ll z\int_0^{\alpha}\varrho_z^*(u)\,du = \alpha^z \to 0 \quad \text{as} \quad \alpha\to 0^+.$$

Therefore

$$\langle \varrho_z^*, H\rangle = \lim_{\alpha\to 0^+}\{-\alpha\varrho_z^*(\alpha)H(\alpha-1)\}. \tag{6.15}$$

In (6.4), for the outer integral, the Cauchy principal value of $\int_{-1}^{\infty}$ is bounded. Thus (6.15) yields

$$\begin{aligned}\langle \varrho_z^*, H\rangle &= \lim_{\alpha\to 0^+} -\alpha\varrho_z^*(\alpha)\int_{-\infty}^{-1}\exp\left\{\alpha s - z\int_0^s\frac{e^t-1}{t}\,dt\right\}\frac{ds}{s}\\ &= \lim_{\alpha\to 0^+}\alpha\varrho_z^*(\alpha)\int_1^{\infty}\exp\left\{-\alpha s - z\int_0^{-s}\frac{e^t-1}{t}\,dt\right\}\frac{ds}{s}\\ &= \lim_{\alpha\to 0^+}\alpha\varrho_z^*(\alpha)\int_1^{\infty}e^{-\alpha s}\varphi_z(s)s^{z-1}\,ds,\end{aligned} \tag{6.16}$$

where

$$\varphi_z(s) = s^{-z}\exp\left\{-z\int_0^{-s}\frac{e^t-1}{t}\,dt\right\} = \varphi_1(s)^z.$$

It is known (see [5], p. 30) that $\varphi_1(s)\to e^{\gamma}$ as $s\to\infty$, where γ is Euler's constant. So $\varphi_z(s)\to e^{\gamma z}$. Finally, setting $u = \alpha s$ in (6.16) and using (6.13), we get

$$\langle \varrho_z^*, H\rangle = \lim_{\alpha\to 0^+}\int_{\alpha}^{\infty} e^{-u}u^{z-1}\varphi_z(u/\alpha)\,du = e^{\gamma z}\Gamma(z). \tag{6.17}$$

On comparing (6.17) with (6.12) we deduce that in (5.11)

$$l(z) = e^{\gamma z}\Gamma(z) \tag{6.18}$$

and so we have an asymptotic estimate for $\varrho_z^*(\alpha)$, as $\alpha\to\infty$.

Actually the method yields more; we can estimate the convergence in (4.9) quite accurately. As in the case of $z=1$ (see [2], Lemma 3), (5.7), (5.9), (5.11) and (6.9) now combine to give

$$\varrho_z^*(\alpha-u) = \varrho_z^*(\alpha)\left(\frac{\alpha\log\alpha}{z}\right)^u\{1+o(1)\}, \quad \text{for} \quad 0\le u\le 1. \tag{6.19}$$

In particular the Volterra kernel k in (4.5) satisfies

$$k(\alpha, u) \sim (\log\alpha)^u(\alpha/z)^{u-1}, \quad 0\le u\le 1,\ \alpha\to\infty.$$

This improvement over (4.7) yields a better lower bound for $\Phi(\alpha)$ than (4.8), namely,

$$\Phi(\alpha) \gg_\varepsilon n^{(-3/4)-\varepsilon}.$$

So by (4.2) of Lemma 1 and (6.18) we get as $\alpha\to\infty$

$$\varrho_z^*(\alpha) = e^{\gamma z}\Gamma(z)F(\alpha)\{1+O_\varepsilon(\exp\{-\alpha^{(1/4)-\varepsilon}\})\}. \tag{6.20}$$

Thus the $o(1)$ in (5.11) and (6.19) may be replaced by $O(\alpha^{-1})$.(*)

7. Asymptotic estimate for $\Psi_z(x, y)$. We shall now employ a recent method of Hildebrand [10] to establish

THEOREM 1. *Let $1/2\le z\le 3/2$ and (1.6) hold. Then*

$$\Psi_z(x, y) = A(z)x(\log y)^{z-1}\varrho_z^*(\alpha)\left\{1+O_\varepsilon\left(\frac{\log\alpha}{\log y}\right)+O\left(\frac{1}{\sqrt{\log x}}\right)\right\}.$$

Proof. To begin, we define Δ_z by

$$\Psi_z(x, y) = A(z)x(\log y)^{z-1}\varrho_z^*(\alpha)\{1+\Delta_z(y, \alpha)\}. \tag{7.1}$$

In Section 2 it was shown that

$$\Delta_z(y, \alpha) = O\left(\frac{1}{\sqrt{\log x}}\right) \quad \text{for} \quad x\ge 2 \text{ and } y\ge\sqrt{x}. \tag{7.2}$$

For the sake of convenience let

$$\Delta_z^*(y, \alpha) = \sup_{1/2\le\alpha'\le\alpha}|\Delta_z(y, \alpha')|, \qquad \Delta_z^{**}(y, \alpha) = \sup_{\log 2/\log y\le\alpha'\le\alpha}|\Delta_z(y, \alpha')|. \tag{7.3}$$

(*) In a recent paper, D. Hensley (J. London Math. Soc. (2) 33 (1986), pp. 395–406) has proved a result similar to (5.11) but by a different method, involving convolutions of $\varrho_1(\alpha)$.

In (7.3) the lower bound $\log 2/\log y$ for α is to ensure $x\ge 2$. Clearly

$$\Delta_z^{**}(y, \alpha) = \Delta_z^*(y, \alpha)+O(1) \quad \text{for} \quad \alpha\ge 1/2. \tag{7.4}$$

In view of (7.2) we shall assume in the course of the proof that $\alpha>2$.

Next, consider the sum

$$\sum_{n\in S(x,y)} z^{\Omega(n)}\log n = \sum_{n\in S(x,y)} z^{\Omega(n)}\sum_{p^m|n}\log p = \sum_{\substack{p\le y\\ p^m\le x}} z^m\Psi_z(x/p^m, y)\log p. \tag{7.5}$$

By partial summation

$$\sum_{n\in S(x,y)} z^{\Omega(n)}\log n = \Psi_z(x, y)\log x - \int_1^x\frac{\Psi_z(t, y)}{t}dt. \tag{7.6}$$

Note that (7.5) and (7.6) combine to give

$$\Psi_z(x, y)\log x = \int_1^x\frac{\Psi_z(t, y)}{t}dt + \sum_{\substack{p\le y\\ p^m\le x}} z^m\Psi_z(x/p^m, y)\log p. \tag{7.7}$$

In order to avoid values of ϱ_z^* at arguments very close to zero we observe that the contribution from $p^m\ge x/2$ and $t\le 2$ in (7.7) is

$$\ll 1+\sum_{\substack{p\le y\\ x/2\le p^m\le x}} z^m\log p \ll \sqrt{x} \tag{7.8}$$

since $m>2$. So, upon dividing both sides of (7.7) by $A(z)x(\log y)^{z-1}\times\varrho_z^*(\alpha)\log x$, we get

$$\begin{aligned} 1+\Delta_z(y, \alpha) = {}& \frac{1}{\varrho_z^*(\alpha)x\log x}\int_2^x\varrho_z^*\left(\frac{\log t}{\log y}\right)\left\{1+\Delta_z\left(y, \frac{\log t}{\log y}\right)\right\}dt \\ &+O(x^{-1/2}\varrho_z^*(\alpha)^{-1})+\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{\substack{p\le y\\ p^m\le x/2}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right) \\ &\times\left\{1+\Delta_z\left(y, \alpha-\frac{m\log p}{\log y}\right)\right\} \end{aligned} \tag{7.9}$$

because of (7.1) and (7.8). Now we introduce a function $\beta_z(\alpha)$ using the decomposition

$$\begin{aligned} 1 = \frac{z}{\alpha\varrho_z^*(\alpha)}\int_{\alpha-1}^{\alpha}\varrho_z^*(t)\,dt &= \frac{z}{\alpha\varrho_z^*(\alpha)}\int_{\alpha-1/2}^{\alpha}\varrho_z^*(t)\,dt+\frac{z}{\alpha\varrho_z^*(\alpha)}\int_{\alpha-1}^{\alpha-1/2}\varrho_z^*(t)\,dt \\ &= \beta_z(\alpha)+(1-\beta_z(\alpha)), \text{ respectively.} \end{aligned} \tag{7.10}$$

With this (7.9) can be rewritten as

$$\Delta_z(y,\alpha) = \frac{1}{\varrho_z^*(\alpha)\,x\log x}\int_2^x \varrho_z^*\left(\frac{\log t}{\log y}\right)\left\{1+\Delta_z\left(y,\frac{\log t}{\log y}\right)\right\}dt + O\left(x^{-1/2}\varrho_z^*(\alpha)^{-1}\right) + \left[\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{\substack{p\le y\\ p^m\le x/2}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right)\times\left\{1+\Delta_z\left(y,\alpha-\frac{m\log p}{\log y}\right)\right\}-\beta_z(\alpha)-(1-\beta_z(\alpha))\right]. \tag{7.11}$$

In (7.11) we decompose the expression inside [ ] as

$$\begin{aligned}&\left\{\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{\substack{p\le y\\ y<p^m\le x/2}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right)\times\left(1+\Delta_z\left(y,\alpha-\frac{m\log p}{\log y}\right)\right)\right\}\\ &+\left\{\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{p^m\le\sqrt{y}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right)-\beta_z(\alpha)\right\}\\ &+\left\{\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{\sqrt{y}<p^m\le y}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right)-(1-\beta_z(\alpha))\right\}\\ &+\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{p^m\le y}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right)\Delta_z\left(y,\alpha-\frac{m\log p}{\log y}\right).\end{aligned} \tag{7.12}$$

Let

$$R_1 = \frac{1}{\varrho_z^*(\alpha)\,x\log x}\int_1^x \varrho_z^*\left(\frac{\log t}{\log y}\right)dt. \tag{7.13}$$

Also, it is convenient to denote the expressions in (7.12) within { } by R_2, R_3 and R_4 respectively, where for R_2 the summation is taken without $(1+\Delta_z)$ factor. With regard to the last summation in (7.12) note that it is

$$< \frac{\Delta_z^*(y,\alpha)}{\varrho_z^*(\alpha)\log x}\sum_{p^m\le\sqrt{y}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right) + \frac{\Delta_z^*(y,\alpha-\frac12)}{\varrho_z^*(\alpha)\log x}\sum_{\sqrt{y}<p^m\le y}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right) \tag{7.14}$$

because of (7.3). At this point it is useful to note the inequalities

$$\begin{aligned}&\text{(i)}\quad \varrho_z^*(\alpha-u)/\varrho_z^*(\alpha)\ll e^{2u\log\alpha},\quad 0<u<\alpha-1,\\ &\text{(ii)}\quad 1/\varrho_z^*(\alpha)\ll x^{1/4},\quad \alpha<y^{1/8}\end{aligned} \tag{7.15}$$

which follow from (5.11) and (6.19). Therefore by (7.3) and (7.11)–(7.15) we arrive at

$$|\Delta_z(y,\alpha)| < \left(1+\Delta_z^{**}(y,\alpha)\right)\sum_{i=1}^4 R_i + \beta_z(\alpha)\Delta_z^*(y,\alpha) + (1-\beta_z(\alpha))\Delta_z^*(y,\alpha-\tfrac12)+O(x^{-1/4}) \tag{7.16}$$

provided $\alpha<y^{1/8}$. We shall bound $\Delta_z(y,\alpha)$ suitably using (7.16). This involves bounds for the R_i which we obtain next.

Put $t=y^s$. Then from (7.13) and (7.15) we get

$$\begin{aligned}R_1 &= \frac{1}{\varrho_z^*(\alpha)\,x\log x}\int_0^\alpha \varrho_z^*(s)\,y^s\log y\,ds = \frac{1}{\alpha\varrho_z^*(\alpha)}\int_0^\alpha\frac{\varrho_z^*(\alpha-s)}{y^s}ds\\ &\ll \frac{1}{\alpha\varrho_z^*(\alpha)\sqrt{x}}+\frac{1}{\alpha}\int_0^{\alpha-1}e^{-s(\log y-2\log\alpha)}ds \ll \frac{1}{\alpha\log y}=\frac{1}{\log x},\end{aligned} \tag{7.17}$$

for $\alpha<y^{1/8}$.

With regard to R_2 note that

$$\frac{1}{\varrho_z^*(\alpha)\log x}\sum_{\substack{p\le y\\ y<p^m\le x/2}}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right) = \frac{1}{\varrho_z^*(\alpha)\log x}\Big(\sum_{\substack{p\le y\\ y<p^m\le x/y}}+\sum_{\substack{p\le y\\ x/y<p^m\le x/2}}\Big).$$

With $u=2\log\alpha/\log y$ we see from (7.15) that the above expression is

$$\ll \frac{1}{\alpha}\sum_{\substack{p\le y\\ p^m>y}}\frac{1}{p^{m(1-u)}}+(\log x)^z x^{1/4}\sum_{\substack{p\le y\\ x/y<p^m\le x/2}}\left(\frac{z}{p}\right)^m$$

so long as $\alpha<y^{1/8}$. Since $m>2$ this implies that

$$R_2 \ll \frac{1}{\alpha\log y}=\frac{1}{\log x}\quad\text{for}\quad \alpha<y^{1/8}. \tag{7.18}$$

To bound R_3 consider

$$S_\theta = \frac{1}{\varrho_z^*(\alpha)\log x}\sum_{p^m\le y^\theta}\frac{z^m\log p}{p^m}\varrho_z^*\left(\alpha-\frac{m\log p}{\log y}\right) \tag{7.19}$$

for $0<\theta<1$. By the Prime Number Theorem

$$M_z(t) \overset{\text{def}}{=} \frac{1}{y^t}\sum_{p^m\le y^t} z^m\log p = z+O(R(y^t)), \tag{7.20}$$

where R is as in (1.10). On comparing (7.19) and (7.20) we get

$$(7.21)\qquad S_\theta = \frac{1}{\varrho_z^*(\alpha)\log x}\int_0^\theta y^{-t}\varrho_z^*(\alpha-t)\,d\{y^t M_z(t)\}$$
$$= \frac{M_z(\theta)\varrho_z^*(\alpha-\theta)}{\varrho_z^*(\alpha)\log x} - \frac{1}{\varrho_z^*(\alpha)\log x}\int_0^\theta M_z(t)\,y^t\frac{d}{dt}\{y^{-t}\varrho_z^*(\alpha-t)\}\,dt$$
$$= M^* + R^*,$$

where M^* is the main term and R^* the remainder term obtained by inserting (7.20) into the above expression for S_θ. That is

$$M^* = \frac{z\varrho_z^*(\alpha-\theta)}{\varrho_z^*(\alpha)\log x} + \frac{z}{\varrho_z^*(\alpha)\log x}\int_0^\theta \{\varrho_z^*(\alpha-t)' + \varrho_z^*(\alpha-t)\log y\}\,dt,$$

where ‘ ′ ’ denotes $\frac{d}{dt}$. Clearly

$$(7.22)\qquad M^* = \frac{z}{\log x} + \frac{z}{\alpha\varrho_z^*(\alpha)}\int_0^\theta \varrho_z^*(\alpha-t)\,dt = \frac{z}{\log x} + \beta_z(\alpha, \theta)$$

say.

As for R^* it follows from (1.10) that

$$(7.23)\qquad R^* \ll \frac{1}{\log x}\left(1+\alpha\log^2(\alpha+1)\,R(y^\theta)\right) \ll_\varepsilon \frac{1}{\log x}$$

so long as (1.6) holds. Now take $\theta = 1/2$ and observe that in (7.22)

$$\beta_z = (\alpha, 1/2) = \beta_z(\alpha)$$

as in (7.10). Thus we deduce from (7.12) and (7.19)–(7.23) that if (1.6) holds then

$$(7.24)\qquad R_3 = S_{1/2} - \beta_z(\alpha) \ll_\varepsilon \frac{1}{\log x}.$$

Lastly, with regard to R_4, we look at the sum complementary to the one in (7.19), namely $y^\theta < p^m < y$, $\theta = 1/2$, and obtain by similar reasoning

$$(7.25)\qquad R_4 \ll_\varepsilon \frac{1}{\log x}$$

for α as in (1.6).

To complete the proof it remains only to employ the bounds (7.17), (7.18), (7.24) and (7.25) for the R_i in (7.16). For α as in (1.6) each R_i $\ll_\varepsilon (\log x)^{-1}$ and so

$$(7.26)\qquad |\Delta_z(y, \alpha)| < \Delta_z^*(y, \alpha)\beta_z(\alpha) + \Delta_z^*(y, \alpha-\tfrac{1}{2})(1-\beta_z(\alpha)) + O_\varepsilon\left(\frac{1+\Delta_z^{**}(y, \alpha)}{\log x}\right).$$

In view of (7.4) we may replace Δ_z^{**} by Δ_z^* in (7.26). In addition, since Δ_z^* is (for fixed y) an increasing function of α (see (7.3)), (7.26) will remain valid if $|\Delta_z(y, \alpha)|$ is replaced by $\Delta_z^*(y, \alpha)$. With these changes (7.26) implies that

$$(7.27)\qquad \Delta_z^*(y, \alpha)\{1-\beta_z(\alpha)+O_\varepsilon(1/\log x)\} \leqslant \Delta_z^*(y, \alpha-\tfrac{1}{2})(1-\beta_z(\alpha)) + O_\varepsilon(1/\log x).$$

From (6.19) and (7.10) it follows that $\beta_z(\alpha)\to 0$ as $\alpha\to\infty$. Hence from (7.27) we obtain

$$(7.28)\qquad \Delta_z^*(y, \alpha) \leqslant \Delta_z^*(y, \alpha-\tfrac{1}{2})\left\{1+O_\varepsilon\left(\frac{1}{\alpha\log y}\right)\right\} + O_\varepsilon\left(\frac{1}{\alpha\log y}\right), \qquad \alpha > \alpha_0.$$

By iterating (7.28) we arrive at

$$(7.29)\qquad \Delta_z^*(y, \alpha) \ll_\varepsilon \Delta_z^*(y, \alpha_0) + (\log\alpha/\log y),$$

for $\alpha \geqslant \alpha_0$ and α as in (1.6). For $\alpha < \alpha_0$ it was shown in Section 2 that $\Delta_z(y, \alpha) \ll (\log x)^{-1/2}$. So Theorem 1 follows from (7.29) and (1.6).

8. Quantitative Laplace inversion. Suppose F_j is a sequence of probability distributions whose Fourier transforms $\hat{F}_j$ converge as $j\to\infty$ to the Fourier transform $\hat{\varphi}$ of a probability distribution φ. Then, as is well known, the F_j converge weakly to φ as $j\to\infty$. If φ satisfies certain smoothness conditions then it is possible to obtain a quantitive version of this phenomenon. An example of such a result (for a proof see Elliott [7], Vol. 1, p. 69) is

Lemma 2 (Quantitative Fourier inversion). *Suppose F and φ are probability distributions whose Fourier transforms $\hat{F}$ and $\hat{\varphi}$ are close, in the sense that there exists ε (small) and U (large) such that*

$$\int_{-U}^{U}\left|\frac{F(u)-\varphi(u)}{u}\right|du < \varepsilon.$$

Suppose further that $|\varphi'(v)| \leqslant B$ for $-\infty < v < \infty$. Then

$$\sup_v |F(v)-\varphi(v)| \leqslant \varepsilon + B/U.$$

In view of our interest in moments it is natural to consider an analogue of the above lemma with the Fourier transform replaced by the bilateral Laplace transform. We now state a result for the special case where φ is the Gaussian distribution $G(v)$ in (1.8).

Theorem 2. *Let F be a probability distribution. Suppose there exists $t > 10$ such that*

$$(8.1)\qquad \int_{-\infty}^{\infty} e^{|uv|}\,dF(v) \ll e^{u^2/2} \quad \textit{for} \quad -t \leqslant u \leqslant t.$$

Assume further that there is t^*, positive and removed from zero, such that

$$(8.2)\qquad \left|\int_{-\infty}^{\infty} e^{uv}\,dF(v) - \int_{-\infty}^{\infty} e^{uv}\,dG(v)\right| \leqslant e^{u^2/2}\delta \quad \textit{for} \quad -t^* \leqslant u \leqslant 0.$$

Then for $k = 0, 1, 2, 3, \ldots$, we have

$$|\int_{-\infty}^{\infty} v^k dF(v) - \int_{-\infty}^{\infty} v^k dG(v)| \ll_k \mu(\delta, t) \tag{8.3}$$

where $\mu(\delta, t) \to 0$ as $\delta \to 0$, uniformly in t. Furthermore

$$\sup_v |F(v) - G(v)| \ll \frac{1}{\sqrt{t}} + 8\sqrt{\frac{\log|\log\delta|}{|\log\delta|}}. \tag{8.4}$$

Remarks. (i) In practice t will be large and δ small. If t^* cannot be chosen to be away from zero, then, naturally, the implicit constants in (8.3) and (8.4) will depend on t^*.

(ii) In (8.1) it does not matter whether we use $e^{|uv|}$ or e^{uv} because for real x

$$e^x < e^{|x|} < e^x + e^{-x}.$$

(iii) In contrast to Lemma 2 note that in Theorem 2 we only assume that the bilateral Laplace transforms are close on one side of the origin and that too only for values $-u < t^*$. However, with t taken large in (8.1) $dF(v)$ will be small as $|v| \to \infty$.

(iv) Motivated by applications to Probabilistic Number Theory a general qualitative version of Theorem 2 was already established by us ([3], Lemma 1) by not assuming the limiting distribution to be Gaussian. In view of that earlier result and Lemma 2 it is of interest to extend Theorem 2, by only imposing smoothness conditions on φ. The proof of Theorem 2 which follows can certainly be adapted to a more general situation and will be considered elsewhere.

(v) Note that in (8.3) we do not require $t \to \infty$ in order to make $\mu(\delta, t) \to 0$. In fact it was already noticed in [3] that the moments can be made to converge by just assuming the boundedness of $1/t^*$.

(vi) It is desirable to improve the quality of the error term in (8.4). While the precision of Lemma 2 may not be attainable by the moment method, we make note that (8.4) can be improved; we shall take this up on a later occasion.

Proof of Theorem 2. We begin by observing that for $|u| \leq t$

$$\int_{|v|>T} dF(v) \leq e^{-|u|T} \int_{-\infty}^{\infty} e^{|uv|} dF(v) \ll e^{-uT + (u^2/2)}$$

follows from (8.1). With $u = \min(t, T)$ the above inequality yields

$$\int_{|v|>T} dF(v) \ll e^{-T\min(t,T)/2}. \tag{8.5}$$

By the Cauchy–Schwarz inequality, (8.1) and (8.5)

$$\int_{|v|>T} e^{|uv|} dF(v) \leq \Big(\int_{-\infty}^{\infty} e^{|2uv|} dF(v)\Big)^{1/2} \Big(\int_{|v|>T} dF(v)\Big)^{1/2} \ll e^{u^2} e^{-T\min(t,T)/4} \quad \text{for} \quad |u| \leq t/2. \tag{8.6}$$

Since

$$|v|^k \leq k! \frac{e^{|av|}}{a^k},$$

holds for all $a > 0$, once again (8.1) shows that

$$\int_{-\infty}^{\infty} |v|^k dF(v) \ll \frac{k!\, e^{a^2/2}}{a^k} \quad \text{for} \quad 0 < a \leq t. \tag{8.7}$$

Therefore by (8.7)

$$\int_{|v|>T} |v|^k dF(v) \leq \frac{1}{T} \int_{-\infty}^{\infty} |v|^{k+1} dF(v) \ll \frac{(k+1)!\, e^{a^2/2}}{T\cdot a^{k+1}} \tag{8.8}$$

where a will be specified later.

Next, observe that

$$(e^{uv} - 1)^k = (uv)^k \{1 + O(|kuv|)\}$$

holds if $|kuv| < 1$. Therefore

$$\begin{aligned} &\int_{-T}^{T} v^k dF(v) - \int_{-T}^{T} v^k dG(v) \\ &= u^{-k}\Big(\int_{-T}^{T} (e^{uv} - 1)^k dF(v) - \int_{-T}^{T} (e^{uv} - 1)^k dG(v)\Big) \\ &\quad + O\Big(k|u| \Big\{\int_{-T}^{T} |v|^{k+1} dF(v) + \int_{-T}^{T} |v|^{k+1} dG(v)\Big\}\Big) \quad \text{for} \quad |kuT| \leq 1. \end{aligned} \tag{8.9}$$

Because T will be large we have $|ku| \leq t^*$ in (8.9). So with u in (8.9) chosen to be negative and close to zero we obtain from (8.2), (8.6) and (8.8)

$$\begin{aligned} &|\int_{-\infty}^{\infty} v^k dF(v) - \int_{-\infty}^{\infty} v^k dG(v)| \\ &\ll \frac{2^k \delta}{|u|^k} + \frac{k(k+1)!\, e^{a^2/2}}{a^k}\Big(|u| + \frac{1}{Ta}\Big) + e^{-T\min(t,T)/4} \end{aligned} \tag{8.10}$$

so long as $|kuT| < 1$. Now, given δ we let $-u = 2\exp\{-(\lambda/2)\log|\log\delta|\}$, where

(8.11) $$\lambda = \sqrt{|\log\delta|/\log|\log\delta|}\,.$$

Also let T be chosen to satisfy $T\log^2 T = -u^{-1}$ and assume that $k<\lambda$. With these choices we get from (8.10)

(8.12) $$\left|\int_{-\infty}^{\infty} v^k\,dF(v) - \int_{-\infty}^{\infty} v^k\,dG(v)\right| \ll \sqrt{\delta} + k(k+1)!\,|u|\,e^{a^2/2}/a^k.$$

Next, let $\zeta = i\omega$ where ω is real. The inequality

(8.13) $$\left|e^{\zeta} - \sum_{k=0}^{N} \zeta^k/k!\right| < |\zeta|^{N+1}/(N+1)!$$

can be easily established by induction on N. So by (8.7), (8.12) and (8.13) we have

(8.14) $$|\hat F(\omega) - \hat G(\omega)| = \left|\int_{-\infty}^{\infty} e^{\zeta v}\,dF(v) - \int_{-\infty}^{\infty} e^{\zeta v}\,dG(v)\right|$$
$$< \sum_{k=0}^{[\lambda]} \frac{|\omega|^k}{k!}\left|\int_{-\infty}^{\infty} v^k\,dF(v) - \int_{-\infty}^{\infty} v^k\,dG(v)\right| + \frac{|\omega|^{[\lambda]+1}\,e^{a^2/2}}{a^{[\lambda]+1}}$$
$$\ll \sqrt{\delta}\,e^{|\omega|} + \frac{|u|\,e^{a^2/2}\,\lambda^2\,|\omega|}{a} + \frac{|\omega|^{[\lambda]+1}\,e^{a^2/2}}{a^{\lambda}},$$

provided $|\omega| < a/2$.

We now choose

(8.15) $$a = \min(t,\ \sqrt{\lambda}/2).$$

Then from (8.14) and (8.15) we deduce that

(8.16) $$|\hat F(\omega) - \hat G(\omega)| \ll e^{-c\lambda} \quad \text{for} \quad |\omega| < \sqrt{a}.$$

On the other hand when ω is close to zero the trivial estimate

(8.17) $$|\hat F(\omega) - \hat G(\omega)| \ll |\omega|$$

holds. The idea is to use these upper bounds for $\hat F - \hat G$ along with Lemma 2 to get (8.4). More precisely if $U \leqslant \sqrt{a}$ then by (8.16) and (8.17)

(8.18) $$\int_{-U}^{U}\left|\frac{F(\omega)-G(\omega)}{\omega}\right| d\omega = \int_{-\varepsilon'}^{\varepsilon'} + \int_{\varepsilon'\leqslant|\omega|\leqslant U} = O(\varepsilon' + |\log\varepsilon'|\,e^{-c\lambda}\,U) \ll e^{-c\lambda/2}$$

by suitably choosing ε'. Thus with $\varepsilon = \exp\{-c\lambda/2\}$, $U = \sqrt{a}$ in Lemma 2 we deduce (8.4) from (8.18), (8.11) and (8.15).

Finally, with regard to (8.3), notice that it is a consequence of (8.12) and (8.15).

9. Proof of Theorem A. We begin by estimating the difference $\xi_z - \xi_1$. For convenience let

(9.1) $$h(t) = (e^t - 1)/t \quad \text{and} \quad j(t) = h^{-1}(t).$$

By (5.7) and (9.1)

(9.2) $$h(\xi_z) = \alpha/z$$

and so

(9.3) $$h'(\xi_z) = \big(\alpha(\xi_z - 1) + 1\big)/(z\xi_z).$$

Similarly

(9.4) $$h''(\xi') = O(\alpha) \quad \text{if} \quad |\xi' - \xi_1| \leqslant |\xi_z - \xi_1|.$$

In addition

(9.5) $$j'(t) = 1/h'\big(j(t)\big) \quad \text{and} \quad j''(\alpha') = O(\alpha^{-2}) \quad \text{if} \quad |\alpha' - \alpha| \leqslant |\alpha/z - \alpha|.$$

Since

$$\xi_z = j\left(\frac{\alpha}{z}\right) = j(\alpha) + \left(\frac{\alpha}{z} - \alpha\right) j'(\alpha) + O\left(\frac{\alpha}{z} - \alpha\right)^2 |j''(\alpha')|$$

(9.1), (9.2), (9.3) and (9.5) combine to give

(9.6) $$\xi_z - \xi_1 = \frac{(1-z)\,\xi_1}{\xi_1 - 1} + O\left((z-1)^2 + \frac{|z-1|}{\alpha}\right).$$

Next we compute $\varrho_z^*(\alpha)/\varrho_1^*(\alpha)$, as $z \to 1$. Note that by (5.6) and (9.1)

$$g''_{\alpha,z}(\xi_z) = -z h'(\xi_z)$$

whence by (9.3) and (9.6)

$$\sqrt{g''_{\alpha,1}(\xi_1)/g''_{\alpha,z}(\xi_z)} = 1 + O(|z-1|).$$

With regard to $l(z)$ it follows from (6.18) that

$$l(z)/l(1) = 1 + O(|z-1|).$$

Therefore by (5.10) and (6.20)

(9.7) $$\frac{\varrho_z^*(\alpha)}{\varrho_1^*(\alpha)}$$
$$= \exp\left\{-\alpha(\xi_z - \xi_1) + z\int_{\xi_1}^{\xi_z} h(t)\,dt + (z-1)\int_{0}^{\xi_1} h(t)\,dt\right\}\{1 + O(\alpha^{-1}) + O(|z-1|)\}.$$

From (9.2), (9.3), (9.4) and (9.6) we get

$$(9.8)\qquad z\int_{\xi_1}^{\xi_z} h(t)\,dt = z\int_{\xi_1}^{\xi_z} \{h(\xi_1)+(t-\xi_1)h'(\xi_1)+O((t-\xi_1)^2\alpha)\}\,dt$$

$$= z\alpha(\xi_z-\xi_1)+\frac{z\alpha(\xi_z-\xi_1)^2(\xi_1-1)}{2\xi_1}+O(|z-1|^3\alpha)+O((z-1)^2)$$

$$= \alpha(\xi_z-\xi_1)+(z-1)\alpha(\xi_z-\xi_1)+\frac{\alpha(\xi_z-\xi_1)^2(\xi_1-1)}{2\xi_1}$$

$$+O(|z-1|^3\alpha+(z-1)^2)$$

$$= \alpha(\xi_z-\xi_1)-\frac{(z-1)^2\alpha\xi_1}{2(\xi_1-1)}+O(|z-1|^3\alpha+|z-1|).$$

On the other hand by change of variables

$$(9.9)\qquad (z-1)\int_0^{\xi_1} h(t)\,dt = (z-1)\,\mathrm{li}(\alpha\xi_1)+O(|z-1|\log\xi_1).$$

So by combining (9.7), (9.8) and (9.9) we arrive at

$$(9.10)\qquad \varrho_z^*(\alpha)/\varrho_1^*(\alpha)$$

$$= \exp\left\{\frac{-(z-1)^2\alpha\xi_1}{2(\xi_1-1)}+(z-1)\,\mathrm{li}(\alpha\xi_1)\right\}\{1+O(\alpha^{-1})+O(|z-1|\log\xi_1)\}.$$

The idea is to apply (9.10) to the bilateral Laplace transform of $F_{x,y}(v)$, namely

$$(9.11)\qquad \int_{-\infty}^{\infty} e^{uv}\,dF_{x,y}(v) = \frac{1}{\Psi(x,y)}\sum_{n\in S(x,y)} e^{u(\Omega(n)-\eta(x,y))/\sqrt{\Theta(x,y)}}$$

$$= e^{-u\eta(x,y)/\sqrt{\Theta(x,y)}}\,\frac{\Psi_z(x,y)}{\Psi_1(x,y)},$$

where

$$(9.12)\qquad z = e^{u/\sqrt{\Theta(x,y)}}.$$

By Theorem 1 the expression in (9.11) is

$$(9.13)\qquad e^{-u\eta(x,y)/\sqrt{\Theta(x,y)}}(\log y)^{z-1}\frac{\varrho_z^*(\alpha)}{\varrho_1^*(\alpha)}\left\{1+O_\varepsilon\left(\sqrt{\frac{\log 2\alpha}{\log y}}\right)+O(|z-1|)\right\}$$

when (1.6) holds. In (9.13) we utilised

$$A(z) = 1+O(|z-1|)$$

which readily follows from (2.2) and the differentiability of $A(z)$. With z as in (9.12)

$$(9.14)\qquad z-1 = \frac{u}{\sqrt{\Theta(x,y)}}+\frac{u^2}{2\Theta(x,y)}+O\left(\frac{|u|^3}{\Theta^{3/2}(x,y)}\right).$$

Hence from (9.10), (9.13), (9.14), (1.3) and (1.4) we deduce that the expression in (9.11) is

$$(9.15)\qquad \exp\left\{-\frac{u\eta(x,y)}{\sqrt{\Theta(x,y)}}+(z-1)\eta(x,y)-\frac{(z-1)^2\alpha\xi_1}{2(\xi_1-1)}\right\}$$

$$\times\left\{1+O(\alpha^{-1})+O(|z-1|\log\xi)+O_\varepsilon\left(\sqrt{\frac{\log 2\alpha}{\log y}}\right)\right\}$$

$$= \exp\left\{\frac{u^2\eta(x,y)}{2\Theta(x,y)}-\frac{u^2\alpha\xi_1}{2\Theta(x,y)(\xi_1-1)}\right\}$$

$$\times\left\{1+O(\alpha^{-1})+O\left(\frac{(1+|u|)^3\log^2\alpha}{\sqrt{\Theta(x,y)}}\right)+O_\varepsilon\left(\sqrt{\frac{\log\alpha}{\log y}}\right)\right\}$$

$$= e^{u^2/2}\left\{1+O(\alpha^{-1})+O\left(\frac{(1+|u|)^3\log^2\alpha}{\sqrt{\Theta(x,y)}}\right)+O_\varepsilon\left(\sqrt{\frac{\log\alpha}{\log y}}\right)\right\}$$

so long as (1.6) holds. This is to be compared with (8.2) of Theorem 2 when α is 'large'.

When α is 'small' and z close to 1 an estimate for $\varrho_z^*(\alpha)/\varrho_1^*(\alpha)$ more useful than (9.10) can be easily established by induction on $[\alpha]$. To be more precise, observe that (3.2) yields

$$(9.16)\qquad \varrho_z^*(\alpha)\alpha^{1-z}-\varrho_1^*(\alpha) = \int_1^{\alpha}\frac{\varrho_1^*(t-1)-\varrho_z^*(t-1)}{t}\,dt-\int_1^{\alpha}\varrho_z^*(t-1)\left(\frac{z}{t^z}-\frac{1}{t}\right)dt.$$

In (9.16)

$$(9.17)\qquad \frac{z}{t^z}-\frac{1}{t}\ll\frac{|z-1|\log t}{t}\quad\text{and}\quad \alpha^{1-z}-1\ll|z-1|\log\alpha,$$

provided $|z-1|\log\alpha$ is bounded, which will be case with z as in (9.12). Now (3.7), (9.16) and (9.17) imply that

$$(9.18)\qquad |\varrho_z^*(\alpha)-\varrho_1^*(\alpha)|\leqslant\log\alpha\cdot\sup_{1\leqslant\alpha'<\alpha-1}|\varrho_z^*(\alpha')-\varrho_1^*(\alpha')|+O(|z-1|\log\alpha).$$

From (9.18) the inequality

$$(9.19)\qquad |\varrho_z^*(\alpha)-\varrho_1^*(\alpha)|\ll|z-1|\,e^{O(\alpha\log\alpha)}$$

follows by iteration. In view of (5.11) we obtain from (9.19)

$$\varrho_z^*(\alpha)/\varrho_1^*(\alpha) = 1 + O(|z-1|\exp\{c\alpha\log\alpha\}). \tag{9.20}$$

Going back to (9.11) we see from Theorem 1, (9.14), (9.20), (1.3) and (1.4) that analogous to (9.15) we now have

$$\int_{-\infty}^{\infty} e^{uv}\,dF_{x,y}(v) \tag{9.21}$$

$$= \exp\left\{\frac{-u\eta(x,y)}{\sqrt{\Theta(x,y)}} + (z-1)\log\log y\right\}\left\{1+O_\varepsilon\left(\sqrt{\frac{\log 2\alpha}{\log y}}\right)\right\}\frac{\varrho_z^*(\alpha)A(z)}{\varrho_1^*(\alpha)}$$

$$= e^{u^2/2}\left\{1+O\left(\frac{|u|\exp\{c\alpha\log\alpha\}}{\sqrt{\Theta(x,y)}}\right)+O_\varepsilon\left(\sqrt{\frac{\log 2\alpha}{\log y}}\right)\right\}.$$

This is to be compared with (8.2) of Theorem 2 when α is 'small'.

The final step is to make the terms 'small' and 'large' precise for α. When α is less than

$$\frac{\log_{(3)}(x)}{4c\log_{(4)}(x)} \tag{9.22}$$

we use (9.21) whereas we utilise (9.15) when α is larger. Here by $\log_{(n)}(x)$ we mean the iterated logarithm $\log(\log_{(n-1)}(x))$, where $\log_{(1)}(x) = \log x$. We take

$$t = t^* = \sqrt[8]{\log_{(2)}(x)}. \tag{9.23}$$

Then from (9.15), (9.21), (9.22) and (9.23) it follows that (8.1) and (8.2) of Theorem 2 hold with $F_{x,y}$ in the place of F and with

$$\delta \ll_\varepsilon \frac{\log_{(4)}(x)}{\log_{(3)}(x)}. \tag{9.24}$$

Thus from (9.23), (9.24) and (8.4) of Theorem 2 we deduce that

$$|F_{x,y}(v) - G(v)| \ll_\varepsilon \sqrt[8]{\frac{\log_{(5)}(x)}{\log_{(4)}(x)}} \tag{9.25}$$

uniformly for all v and x, y satisfying (1.6). Similarly if (1.6) holds then by (8.3) of Theorem 2 (see also (8.12))

$$\left|\int_{-\infty}^{\infty} v^k\,dF_{x,y}(v) - \int_{-\infty}^{\infty} v^k\,dG(v)\right| \ll_\varepsilon (k+2)!\exp\left\{-c\sqrt{\log_{(4)}(x)\log_{(5)}(x)}\right\} \tag{9.26}$$

uniformly in y. With (9.25) and (9.26) Theorem A is proved.

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