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Improvement on Davenport's iterative method and new results in additive number theory III

by

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1. Introduction. In this paper, the results in [9] and [10] (Parts I and II in this series of papers) are improved to the following:

Theorem 1. *Every sufficiently large integer that is* $\not\equiv 0$, 14 *or* 15 (mod 16) *is the sum of at most* 13 *fourth powers.*

This improves the corresponding result $G^*(4) \leqslant 14$ of Davenport [1].

Theorem 2. $G(5) \leqslant 21$, $G(6) \leqslant 32$, $G(7) \leqslant 45$, $G(8) \leqslant 62$, $G(9) \leqslant 82$, $G(10) \leqslant 102$, $G(11) \leqslant 118$.

Also (in Waring–Goldbach problem), we have (cf. § 2 in Ch. 12 of [6])

Theorem 3. $H(6) \leqslant 33$, $H(7) \leqslant 47$, $H(8) \leqslant 63$, $H(9) \leqslant 83$, $H(10) \leqslant 107$; *and* $h(6) \leqslant 17$, $h(7) \leqslant 24$, $h(8) \leqslant 32$, $h(9) \leqslant 42$, $h(10) \leqslant 54$.

The author understands that R. C. Vaughan has obtained similar bounds together with his result in [13].(¹) However, differences in the proofs are expected. While the starting point is the same as in [9], further improvement arises from the direct use of Hardy–Littlewood method at the iterative steps. It is acknowledged that for this purpose, Vaughan's result in [12] (see (1.2) below) fits in well for the major arcs. When $3 \leqslant k \leqslant 11$, and α_{s-1} is close to 1, this is more economical in estimating α_s in $U_s^{(k)}(N) > N^{\alpha_s - \varepsilon}$. (In essence, we use Weyl's inequality for the $(k-1)$-st power instead of the kth power.) With g denoting the approximating function for f (on major arcs), Hua's result in [5] (which utilises Weil's result in Algebraic Geometry) with a partial summation leads to

$$f-g \ll q^{(1/2)+\varepsilon}(1+P^k|\beta|). \tag{1.1}$$

The factor $q^{(1/2)+\varepsilon}$ in (1.1) can be replaced by $q^{(3/4)+\varepsilon}$ using the results in [4]. Theorem 2 in [12] is the more precise result

$$f-g \ll q^{(1/2)+\varepsilon}(1+P^k|\beta|)^{1/2}. \tag{1.2}$$

(¹) Copies of [12] and [13] were made available to the author during early January 1985.

In the author's proof, the integration of $F(\alpha)$ over the major arcs is considered rather than that of $f_s(\alpha)$ (see (2.1), (2.3) and (2.4)). Nevertheless, for $k=4$, the proof given here requires (1.2) also. Some arguments in [10] (regarding the admissibility of exponents) also play a key role for $k=4$. While (1.1) may be required for $k=5$, for $k \geqslant 6$, it is sufficient to use it with $q^{(3/4)+\varepsilon}$ in place of $q^{(1/2)+\varepsilon}$. Also, for $k=5$, one can use Vaughan's result (1.2) replacing $q^{(1/2)+\varepsilon}$ with $q^{(3/4)+\varepsilon}$, and estimate the integrals of the error terms over the major arcs separately by using (with the standard notations) $\sum_q \sum_a q^{-4-(2/5)-\delta_0} |S(a, q)|^4 \ll 1$. Thus, overall, the proof given here does not require Weil's result for $k \geqslant 5$ (see § 15A). For uniformity of proofs, we do use (1.2) here for $k=5$. For $k=4$, (1.2) is used on the major arcs for f_4 and f_5 respectively in estimating α_5 and α_6.

Some results in [6] for exponential sums with integral-valued polynomials are modified for our use in this paper.

With Davenport's bounds for α_r (for suitable r), generally, we can take $\delta_i = 1/2^{k-2}$ for $i \geqslant r$ in estimating α_s with $s \geqslant r+1$ (till α_s gets very close to 1, the impossibility of this choice for δ_i then being seen from the estimates over the major arcs).

2. Some preliminary results. Let (as in [9]) ε denote an arbitrarily small positive number, and δ_0 a small positive constant. All constants implied in the '$\ll$' notation will depend at most on k and ε. Inequalities depending on other parameters (like in Lemmas 2.3 and 2.4) will be so indicated. We recall (from [9]) that with $\lambda_i = (k-1+\delta_i)/k$, $P_i = P^{\lambda_i}$,

$$f = \sum_{P<x<2P} e(\alpha x^k), \quad f_i = \sum_{P_i<x<2P_i} e(\alpha x^k). \tag{2.1}$$

Let

$$0 < \delta \leqslant 1/2^{k-2}, \tag{2.2}$$

and (uniformly)

$$\begin{gathered} F(\alpha) = \sum_{0<t<P^\delta} \sum_{P<x<2P} e(\Delta_t(x^k)\alpha), \\ F_r(\alpha) = \sum_t \sum_{t_1} \dots \sum_{t_r} \sum_x e(\Delta_{t,t_1,\dots,t_r}(x^k)\alpha). \end{gathered} \tag{2.3}$$

With $\{\lambda_1, \dots, \lambda_s\}$ forming admissible exponents, we use Hardy–Littlewood method (with $\delta_s = \delta$) directly to estimate

$$S = \int_0^1 |f|^2 |f_s f_{s-1} \dots f_1|^2 \, d\alpha. \tag{2.4}$$

When Davenport's estimate for α_r is used, we replace $(f_1 \dots f_r)$ by $U(\alpha)$, where $U(\alpha) = \sum_{u_i} e(\alpha u_i)$, the u_i's being distinct integers of the form $(x_1^k + \dots + x_r^k)$. The proofs then will have the obvious modifications as in § 9 in [9]. As explained in [9] (in the proof of the Fundamental Lemma), we have

$$S \ll P^{1+\varepsilon}(P_s \dots P_1) + I, \tag{2.5}$$

where (cf. (2.3))

$$I = \int_0^1 F(\alpha) |f_s \dots f_1|^2 \, d\alpha. \tag{2.6}$$

LEMMA 2.1. *If* $|\alpha - a/q| \leqslant q^{-2}$, *then*

$$F(\alpha) \ll P^{1+\delta+\varepsilon} \{P^{-1} + q^{-1} + P^{-k+1-\delta} q\}^{1/2^{k-2}}. \tag{2.7}$$

Proof. The proof is the same as that of Weyl's inequality (for $(k-1)$-st power). We start with (using Cauchy's inequality)

$$\begin{aligned} F(\alpha) &\ll (P^\delta)^{1/2} \Big\{ \sum_{t<P^\delta} \sum_x \sum_y e\big((\Delta_t(x^k) - \Delta_t(y^k))\alpha\big) \Big\}^{1/2} \\ &\ll (P^\delta)^{1/2} \{P^{1+\delta} + |F_1(\alpha)|\}^{1/2}, \end{aligned}$$

where $F_1(\alpha)$ is defined by (2.3) with $0 < t_1 < P$.

Generally, we have

$$F_r(\alpha) \ll (P^{r+\delta})^{1/2} \{P^{r+1+\delta} + |F_{r+1}(\alpha)|\}^{1/2} \quad (1 \leqslant r \leqslant k-3).$$

(Indeed, these inequalities are implied in the proof of the Fundamental Lemma in [9].)

At the final step (as in the proof of Weyl's inequality, where the only difference is that we take $\delta = 1$), by actually carrying out the summation over x, and using $tt_1 \dots t_{k-2} \ll P^{k-2+\delta}$, we have

$$F_{k-2}(\alpha) \ll P^\varepsilon \sum_{0 \leqslant m \leqslant (k!)P^{k-2+\delta}} \min(P, \|\alpha m\|^{-1}).$$

The sum on the right-side is divided into $\ll \{(P^{k-2+\delta}/q) + 1\}$ blocks, and we get (in the standard way)

$$F_{k-2}(\alpha) \ll P^\varepsilon \{(P^{k-2+\delta}/q) + 1\} \{P + q \log q\}.$$

Putting these estimates together, we get (2.7).

We have

$$\Delta_t(x^k) = A_{k-1} x^{k-1} + A_{k-2} x^{k-2} + \dots + A_1 x + A_0, \tag{2.8}$$

where

$$A_{k-1} = kt, \quad A_{k-2} = \binom{k}{2} t^2, \quad \dots, \quad A_1 = kt^{k-1}, \quad A_0 = t^k, \tag{2.9}$$

so that, $\Delta_t(x^k)$ is (for given t) a polynomial of degree $(k-1)$ in x with positive integral coefficients $A_{k-1}, \ldots, A_0$.

For every given t with $0 < t < P^\delta$, let

$$h_t(\alpha) = \sum_{P < x < 2P} e(\Delta_t(x^k)\alpha) \tag{2.10}$$

(so that, from (2.3), $F(\alpha) = \sum_{0 < t < P^\delta} h_t(\alpha)$);

$$S_v(t) = \sum_{x=1}^{v} e\left(\frac{a}{q}(\Delta_t(x^k))\right) \quad (1 \leqslant v \leqslant q), \tag{2.11}$$

and

$$S(a, q, t) = S_q(t). \tag{2.12}$$

Since $(A_{k-1}, \ldots, A_2, q) \leqslant kt$ and $(a, q) = 1$, the next lemma follows from Theorem 2 in [6] (Ch. 1, § 8) (using $t < P^\delta$).

LEMMA 2.2.

$$\sum_{1 \leqslant x \leqslant P} e\left(\frac{a}{q}(\Delta_t(x^k))\right) - (P/q)S(a, q, t) \ll q^{1-1/(k-1)+\varepsilon} t^{1/(k-1)} \tag{2.13}$$

$$\ll q^{1-1/(k-1)+\varepsilon} P^{\delta/(k-1)} \quad \textit{(uniformly in } t\textit{)};$$

and, for $1 \leqslant v \leqslant q$,

$$S_v(t) \ll q^{1-1/(k-1)+\varepsilon} t^{1/(k-1)} \tag{2.14}$$

$$\ll q^{1-1/(k-1)+\varepsilon} P^{\delta/(k-1)} \quad \textit{(uniformly in } t\textit{)}.$$

(Note that the lemma in this form is ineffective if $q < t$. The result then, can be modified. The effect of the factor $t^{1/(k-1)}$ on the estimates over the major arcs can be reduced or dispensed with by some additional arguments. But, these will not be required for our purposes.)

LEMMA 2.3. *Let* $1 \leqslant l \leqslant k-1$, *and with* $A_l = \binom{k}{k-l} t^{k-l}$ (cf. (2.9)),

$$\psi(x) = \psi(x, t) = e(\beta A_l (qx)^l).$$

Then, if $q \leqslant P^{1-\varepsilon}$, $|\beta| \ll q^{-1} P^{-(l-1)-(k-l)\delta-\varepsilon}$, *and* $0 \leqslant x \leqslant (2P/q)$, *we have*

$$|\psi^{(r)}(x)| \leqslant C_1(k, r, \varepsilon) P^{-r\varepsilon} \quad \textit{(where } C_1 \textit{ is a suitable constant)}.$$

Proof. Let $y = (2\pi\beta A_l)^{1/l}(qx)$, so that $\psi(x) = e^{i(y^l)}$. Hence, by Lemma 7.6 in [6],

$$|\psi^{(r)}(x)| = |\psi(x) G_r(y) \{(2\pi\beta A_l)^{1/l} q\}^r|, \tag{2.15}$$

where $G_r(y)$ is a polynomial of degree $(l-1)r$ in y.

Now (from the hypotheses of the lemma),

$$(|\beta| A_l)^{1/l} q \ll q^{1-(1/l)} P^{-\{(l-1)+(k-l)\delta+\varepsilon\}/l} t^{(k-l)/l}$$

$$\ll P^{(1-\varepsilon)(1-1/l)-1+(1-\varepsilon)/l} \quad (\text{using } q \leqslant P^{1-\varepsilon} \text{ and } t < P^\delta)$$

$$\ll P^{-\varepsilon};$$

and

$$\{(|\beta| A_l)^{1/l} q\}^l x^{l-1} \ll |\beta| A_l q^l \left(\frac{2P}{q}\right)^{l-1} \ll q|\beta| t^{k-l} P^{l-1}$$

$$\ll P^{-\varepsilon} \quad (\text{using } t < P^\delta, \text{ and the bound for } q\beta).$$

Thus, from (2.15) (with C_2 denoting a suitable constant),

$$|\psi^{(r)}(x)| \leqslant C_2(k, r, \varepsilon)[1 + \{(|\beta| A_l)^{1/l}(qx)\}^{(l-1)r}] \{(|\beta| A_l)^{1/l} q\}^r \leqslant C_1(k, r, \varepsilon) P^{-r\varepsilon}.$$

LEMMA 2.4. *Let* $w_l(x) = A_l x^l + \ldots + A_0$ $(1 \leqslant l \leqslant k-1)$ (*so that,* $w_{k-1}(x) = \Delta_t(x^k)$), *and* $\varphi_l(x) = \varphi_l(x, t) = e(\beta w_l(qx))$.

Then, if

$$q \leqslant P^{1-\varepsilon}, \quad |\beta| \ll q^{-1} P^{-(l-1)-(k-l)\delta-\varepsilon} \quad \textit{and} \quad 0 \leqslant x \leqslant (2P/q), \tag{2.16}$$

we have (with a suitable constant C_3*)*

$$|\varphi_l^{(r)}(x)| \leqslant C_3(k, r, \varepsilon) P^{-r\varepsilon}.$$

Proof. This is proved by induction on l. When $l = 1$, we have

$$\varphi_1(x) = e(\beta(A_1 qx + A_0)),$$

so that

$$|\varphi_1^{(r)}(x)| = |(2\pi\beta A_1 q)^r \varphi_1(x)|. \tag{2.17}$$

Since $A_1 \ll t^{k-1} \ll P^{(k-1)\delta}$, and $q|\beta| \ll P^{-(k-1)\delta-\varepsilon}$, we see from (2.17) that the result is true for $l = 1$.

With $\psi(x)$ as in Lemma 2.3, we have (for $l \geqslant 2$)

$$\varphi_l(x) = \psi(x)\varphi_{l-1}(x). \tag{2.18}$$

By inductive assumption for $l-1$, we have

$$|\varphi_{l-1}^{(r)}(x)| \leqslant C_3(k, r, \varepsilon) P^{-r\varepsilon}$$

if

$$q \leqslant P^{1-\varepsilon}, \quad |\beta| \ll q^{-1} P^{-(l-2)-(k-l+1)\delta-\varepsilon} \quad \text{and} \quad 0 \leqslant x \leqslant (2P/q). \tag{2.19}$$

Since $\delta < 1$,

$$P^{-(l-1)-(k-l)\delta} < P^{-(l-2)-(k-l+1)\delta};$$

so that, if β is subject to (2.16), it also satisfies (2.19). Result now follows from Lemma 2.3 since (from (2.18)),

$$\varphi_l^{(r)}(x) = \psi^{(r)}(x)\varphi_{l-1}(x) + \binom{r}{1}\psi^{(r-1)}(x)\varphi'_{l-1}(x) + \ldots + \psi(x)\varphi_{l-1}^{(r)}(x).$$

Corollary 2.4. *Let (with $\varphi_l(x, t)$ as in Lemma 2.4)*

$$\varphi(x) = \varphi_{k-1}(x, t) = e\big(\beta\Delta_t((qx)^k)\big). \tag{2.20}$$

Then, if

$$q \leqslant P^{1-\varepsilon}, \quad |\beta| \leqslant q^{-1}P^{-(k-2)-\delta-\varepsilon}, \quad \text{and} \quad 0 \leqslant x \leqslant (2P/q), \tag{2.21}$$

we have

$$|\varphi^{(r)}(x)| \leqslant C_3(k, r, \varepsilon)P^{-r\varepsilon}. \tag{2.22}$$

Proof. In Lemma 2.4, we take $l = k-1$, so that $w_{k-1}(qx) = \Delta_t((qx)^k)$.

Lemma 2.5. *Let $\alpha = (a/q)+\beta$ and*

$$h_t^*(\alpha) = q^{-1}S(a, q, t)J(\beta) \quad (\text{cf. } (2.12)) \tag{2.23}$$

where

$$J(\beta) = \int_P^{2P} e\big(\Delta_t(x^k)\beta\big)dx. \tag{2.24}$$

Then, if $q \leqslant P^{1-\varepsilon}$ and $|\beta| \leqslant q^{-1}P^{-(k-2)-\delta-\varepsilon}$, we have (cf. (2.10))

$$\begin{aligned} h_t(\alpha) - h_t^*(\alpha) &\ll q^{1-1/(k-1)+\varepsilon}t^{1/(k-1)} \\ &\ll q^{1-1/(k-1)+\varepsilon}P^{\delta/(k-1)} \quad (\textit{uniformly in } t). \end{aligned} \tag{2.25}$$

Proof. Starting with Euler's summation formula, this is proved in the same way as Lemma 7.11 in [6] with some modifications. We start with the identity

$$h_t(\alpha) = \sum_{v=1}^{q} e\left(\frac{a}{q}\Delta_t(v^k)\right)A_v,$$

where (cf. (2.20))

$$A_v = \sum_{\substack{j \\ (P/q) < j+(v/q) < (2P/q)}} \varphi(j+(v/q)).$$

In using Lemma 7.2 in [6], the limits a, b are taken to be (P/q) and $(2P/q)$ (in place of 0 and P/q). As in [6], we take $l = [1/\varepsilon]+1$, so that,

$$\varphi^{(l)}(x) \ll P^{-1} \quad \text{and} \quad q\int_{P/q}^{2P/q} \varphi^{(l)}(x)dx \ll 1.$$

Furthermore, we use the inequality (2.14) (in place of the inequality $S_v \ll q^{1-(1/k)+\varepsilon}$ used in [6]). With these changes, and Corollary 2.4, the arguments are precisely the same as in [6].

Lemma 2.6. *With $J(\beta)$ defined as in (2.24),*

$$J(\beta) \ll \min(P, P^{2-k}t^{-1}|\beta|^{-1}). \tag{2.26}$$

Proof. (For every given t),

$$\frac{d}{dx}(\Delta_t(x^k)) = (k-1)A_{k-1}x^{k-2} + \ldots + A_1,$$

which is positive, monotonic increasing (for $P < x < 2P$), and $\geqslant tP^{k-2}$ (cf. (2.9)). Hence, the result follows in a standard way by the second Mean-value Theorem. (See for example Lemma 4.2 in [11].)

Lemma 2.7. *For every given t (with $h_t^*(\alpha)$ defined by (2.23) and $\alpha = (a/q)+\beta$), we have*

$$h_t^*(\alpha) \ll q^{-1/(k-1)+\varepsilon}t^{1/(k-1)}\min(P, P^{2-k}t^{-1}|\beta|^{-1}). \tag{2.27}$$

Proof. From (2.14),

$$S(a, q, t) \ll q^{1-1/(k-1)+\varepsilon}t^{1/(k-1)}.$$

Hence, result follows from Lemma 2.6.

3. Estimation of I. We estimate I (cf. (2.6)) as follows:

With δ satisfying (2.2), let

$$Q = P^{k-2+\delta+\delta_0}, \tag{3.1}$$

and, divide the unit interval

$$Q^{-1} < \alpha < 1+Q^{-1} \tag{3.2}$$

into basic intervals

$$\mathfrak{M}_{a,q}\colon |\alpha - a/q| \leqslant (qQ)^{-1} \quad \text{for} \quad 1 \leqslant q < P^{k\delta-\delta_0}, \tag{3.3}$$

with $\mathfrak{M}$ denoting their union. Also, let m denote the supplementary intervals in (3.2).

Lemma 3.1. *Let $k \geqslant 4$. Then, subject to (2.2), on m*

$$F(\alpha) \ll P^{1+\delta_0} \quad (\text{cf. } (2.3)). \tag{3.4}$$

Proof. On m with $q \geqslant P^{1-\delta_0}$, result follows from Lemma 2.1. Let $\alpha \in m$ with

$$P^{k\delta-\delta_0} \leqslant q \leqslant P^{1-\delta_0}. \tag{3.5}$$

From Lemmas 2.5 and 2.7 (cf. (2.25) and (2.27)), it follows from (3.5) that (for $0 < t < P^{\delta}$)

$$h_t(\alpha) \ll P^{1-\delta+\delta_0} + P^{1-(1/(k-1))(1-\delta)+\delta_0} \ll P^{1-\delta+\delta_0} (\text{using } \delta \leqslant 1/2^{k-2} \text{ and } k \geqslant 4).$$

Hence,

$$F(\alpha) = \sum_{t < P^{\delta}} h_t(\alpha) \ll P^{1+\delta_0},$$

proving the lemma.

LEMMA 3.2. *With $\lambda_1, \ldots, \lambda_s$ and α_s defined as before, let*

$$\int_0^1 |f_s \ldots f_1|^2 \, d\alpha \ll (P_s \ldots P_1) P^{\delta_0}. \tag{3.6}$$

Then

$$\int_{m} F(\alpha) |f_s \ldots f_1|^2 \, d\alpha \ll P^{1+3\delta_0} P^{k\lambda_s \alpha_s}.$$

(Constants multiplying δ_0 are not indicated in all the inequalities.)

Proof. The proof follows directly from (3.6) and Lemma 3.1 on noting that

$$P_s \ldots P_1 \ll P^{k\lambda_s \alpha_s + \delta_0}.$$

For $k \geqslant 5$, we estimate the integral over $\mathfrak{M}$ as in the next lemma. For $k = 4$, some adjustments will be used.

LEMMA 3.3. *With the f_i's as occurring in I, suppose that on $\mathfrak{M}$ (for some r with $1 \leqslant r \leqslant k$)*

$$f_i \ll q^{-1/k} P_i \quad \textit{for} \quad s - r + 1 \leqslant i \leqslant s. \tag{3.7}$$

Then

$$\int_{\mathfrak{M}} F(\alpha) |f_s \ldots f_1|^2 \, d\alpha \ll P^{\delta/(k-1)} P^{2-k+\varepsilon} (P^{2k\lambda_s \alpha_s + \delta_0}) \max(1, P^{\mu}),$$

where

$$\mu = k \{2 - (2r/k) - 1/(k-1)\} \delta. \tag{3.8}$$

Proof. From Lemmas 2.5 and 2.7, on $\mathfrak{M}$ (using $t < P^{\delta}$)

$$h_t(\alpha) \ll q^{-1/(k-1)+\varepsilon} P^{\delta/(k-1)} \{\min(P, P^{2-k} t^{-1} |\beta|^{-1}) + q\}. \tag{3.9}$$

Also, from (3.7),

$$|f_s \ldots f_1|^2 \ll (P_s \ldots P_1)^2 q^{-(2r/k)} \ll P^{2k\lambda_s \alpha_s + \delta_0} q^{-(2r/k)}. \tag{3.10}$$

Hence, from (3.9),

$$\begin{aligned} \int_{\mathfrak{M}} F(\alpha) |f_s \ldots f_1|^2 \, d\alpha &= \sum_{0 < t < P^{\delta}} \int_{\mathfrak{M}} h_t(\alpha) |f_s \ldots f_1|^2 \, d\alpha \\ &\ll (P^{2k\lambda_s \alpha_s + \delta_0}) P^{\delta/(k-1)} \times \\ &\quad \times \sum_{0 < t < P^{\delta}} \Big\{ \sum_{q \leqslant P^{k\delta - \delta_0}} \sum_{a} q^{-\left(\frac{1}{k-1} + \frac{2r}{k}\right)} I(a, q) \Big\} \end{aligned} \tag{3.11}$$

(cf. (3.3)), where

$$\begin{aligned} I(a, q) &= \int_{\mathfrak{M}_{a,q}} \{\min(P, P^{2-k} t^{-1} |\beta|^{-1}) + q\} \, d\beta \\ &\ll P^{2-k+\varepsilon} t^{-1} + q (qQ)^{-1} \\ &\ll P^{2-k+\varepsilon} t^{-1} + P^{2-k-\delta-\delta_0} \quad \text{(cf. (3.1))}. \end{aligned} \tag{3.12}$$

Furthermore,

$$\sum_{0 < t < P^{\delta}} (P^{2-k+\varepsilon} t^{-1} + P^{2-k-\delta-\delta_0}) \ll P^{2-k+2\varepsilon}, \tag{3.13}$$

and

$$\sum_{q \leqslant P^{k\delta - \delta_0}} \sum_{a} q^{-\left(\frac{1}{k-1} + \frac{2r}{k}\right)} \ll \max(1, P^{\mu}) \quad \text{(cf. (3.8))}. \tag{3.14}$$

The lemma now follows from (3.11), (3.12), (3.13) and (3.14).

LEMMA 3.4. *Let r be chosen with the f_i's satisfying (3.7). Suppose that (with α_s and δ defined as before) either*

(a) *with $0 \leqslant r \leqslant k-1$,*

$$k - 1 > (k - 1 + \delta) \alpha_s + (2k - 2r - 1) \delta; \tag{3.15}$$

or

(b) *(with $r \geqslant k$)*

$$k - 1 > (k - 1 + \delta) \alpha_s + \delta/(k-1). \tag{3.16}$$

Then (with S defined by (2.4))

$$S \ll P^{1+\delta_0} (P_s \ldots P_1), \tag{3.17}$$

and $U_{s+1}^{(k)}(N) > N^{\alpha_{s+1} - \delta_0}$, where

$$\alpha_{s+1} = (1/k) + (k - 1 + \delta) \alpha_s / k. \tag{3.18}$$

Proof. (A) In case (a), from Lemma 3.3 (with $P^{\mu} \geqslant 1$),

$$\begin{aligned} \int_{\mathfrak{M}} F(\alpha) |f_s \ldots f_1|^2 \, d\alpha &\ll P^{\delta/(k-1)} (P^{(2-k+\varepsilon)}) P^{(2k\lambda_s \alpha_s + \delta_0)} (P^{\mu}) \\ &\ll (P^{1+\delta_0} P^{k\lambda_s \alpha_s}) P^{\{\delta/(k-1) + (1-k+\varepsilon) + k\lambda_s \alpha_s + \delta_0 + \mu\}}. \end{aligned}$$

The second factor (with $k\lambda_s = k - 1 + \delta$) is

$$\ll \{P^{-(k-1) + 2\delta_0}\} P^{(k-1+\delta)\alpha_s + \mu + \delta/(k-1)},$$

and, this is easily verified to be $\ll 1$ by using (3.8) and (3.15). Hence, the integral over $\mathfrak{M}$ is $\ll P^{1+\delta_0} (P^{k\lambda_s \alpha_s})$.

(B) In case (b), the integral over $\mathfrak{M}$ is

$$\begin{aligned} &\ll (P^{1+\delta_0}) (P^{k\lambda_s \alpha_s}) P^{-(k-1) + 2\delta_0} \{P^{(k-1+\delta)\alpha_s + \delta/(k-1)}\} \\ &< P^{1+\delta_0} P^{k\lambda_s \alpha_s} \quad \text{(from (3.16))}. \end{aligned}$$

Thus, from Lemma 3.2 (and (2.4), (2.5), (2.6)), $S \ll P^{1+\delta_0} P^{k\lambda_s \alpha_s}$. Result now follows in a standard way (with $P^{k\lambda_s \alpha_s} \ll P_s \ldots P_1$).

4. The case $k = 4$.

LEMMA 4.1. *Let*

(4.1) $$\delta_1 = 1, \quad \delta_2 = 3/7, \quad \delta_3 = 27/103.$$

Then, with

$$\lambda_3^{(3)} = (3+\delta_3)/4, \quad \lambda_2^{(3)} = \lambda_1^{(3)} = (3+\delta_2)\lambda_3^{(3)}/4,$$

$\{\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}, 1\}$ *form admissible exponents, and* $U_4^{(4)}(N) > N^{\alpha_4 - \varepsilon}$, *where*

(4.2) $$\alpha_4 = (\lambda_1^{(3)} + \lambda_2^{(3)} + \lambda_3^{(3)} + 1)/4 = 331/412.$$

Proof. These are precisely the same bounds obtained by Davenport (for the δ's and α_3, α_4) in [1], with the difference that the arguments used there do not establish the admissibility of exponents. For this, we use precisely the same arguments as in the proof of Lemmas 4.2 and 5.1 in [10]. The differences in the details are only computational. (Note that the result $S_{3,6} \ll P_3 P_2^{7/2+\varepsilon}$ in Lemma 3.3 of [10] holds for $k = 4$ also with δ_2 as in (4.1).)

I. Estimation of α_5. Let

(4.3) $$\delta_4 = 0.234,$$

and (with δ_i $(1 \leqslant i \leqslant 3)$ as in (4.1)),

(4.4) $$\lambda_4 = \lambda_4^{(4)} = (3+\delta_4)/4, \quad \lambda_i = \lambda_i^{(4)} = (3+\delta_i)\lambda_{i+1}^{(4)}/4 \quad (1 \leqslant i \leqslant 3),$$

and define f, f_i $(1 \leqslant i \leqslant 4)$ as before.

With $\delta = \delta_4$, let the $\mathfrak{M}_{a,q}$'s be defined as in (3.3) with

(4.5) $$1 \leqslant q < P^{4\delta_4 - \delta_0}.$$

LEMMA 4.2. *On* $\mathfrak{M}$,

(a) $f_i \ll q^{-\nu_2} P_2$ $(i = 1, 2)$ (cf. $f_1 = f_2$), (b) $f_3 \ll q^{-\nu_3} P_3$, (c) $f_4 \ll q^{-\nu_4} P_4$,

where

(4.6) $$\nu_2 = 0.103805, \quad \nu_3 = 0.204442, \quad \nu_4 = 1/4.$$

Proof. From (1.2) (with g_i denoting the approximating function to f_i),

(4.7) $$f_i \ll g_i + q^{1/2+\varepsilon}(1 + P_i^4 |\beta|)^{1/2}.$$

With the estimate $g_i \ll q^{-1/4} P_i$, it is easily verified that

(4.8) $$f_i \ll q^{-1/4} P_i + q^{1/2+\varepsilon} \quad (1 \leqslant i \leqslant 4).$$

Now, by computing $\lambda_i (1 \leqslant i \leqslant 4)$, and using (4.5), (a), (b), and (c) are verifed from (4.8).

LEMMA 4.3. (With S as in (2.4).) $S \ll P^{1+\delta_0} P^{(4\lambda_4 \alpha_4 + \delta_0)}$ *and* $U_5^{(4)}(N) > N^{\alpha_5 - \delta_0}$, *where*

(4.9) $$\alpha_5 = (1/4) + \lambda_4^{(4)} \alpha_4 > 0.8995472.$$

Proof. We estimate S (cf. (2.4)) as in the general case. Since $\delta_4 < 1/4$, the integral over m (arguing as in the general case) is

(4.10) $$\ll P^{1+\delta_0} P^{(4\lambda_4 \alpha_4 + \delta_0)} \quad \text{(cf. Lemmas 3.1 and 3.2).}$$

For the estimate of the integral over $\mathfrak{M}$, we use Lemma 3.3 replacing μ by μ', where

(4.11) $$\mu' = 4(2 - 2\nu - 1/3)\delta_4$$

with

(4.12) $$\nu = 2\nu_2 + \nu_3 + \nu_4 = 0.662052.$$

It is now verified from (4.2), (4.3), (4.4), (4.6), (4.11) and (4.12) (and Lemma 3.3) that the integral over $\mathfrak{M}$ is also bounded by the estimate in (4.10). Result now follows with α_5 as in (4.9).

II. Estimation of α_6. Let

(4.13) $$\delta_5 = 0.1985,$$

and (with δ_i $(1 \leqslant i \leqslant 4)$ as above),

$$\lambda_5 = \lambda_5^{(5)} = (3+\delta_5)/4, \quad \lambda_i = \lambda_i^{(5)} = (3+\delta_i)\lambda_{i+1}^{(5)}/4 \quad (1 \leqslant i \leqslant 4).$$

Let the $\mathfrak{M}_{a,q}$'s be defined with

(4.14) $$1 \leqslant q < P^{4\delta_5 - \delta_0}.$$

LEMMA 4.4. *On* $\mathfrak{M}$,

$$\text{(a) } f_i \ll q^{-\nu_2'} P_2 \ (i = 1, 2), \quad \text{(b) } f_3 \ll q^{-\nu_3'} P_3,$$
$$\text{(c) } f_4 \ll q^{-\nu_4'} P_4, \quad \text{(d) } f_5 \ll q^{-\nu_5'} P_5,$$

where

(4.15) $$\nu_2' = 0.069167, \quad \nu_3' = 0.164025, \quad \nu_4' = \nu_5' = 1/4.$$

Proof. This is verified from (4.8), (4.14) and (4.15) by computing λ_i $(1 \leqslant i \leqslant 5)$.

LEMMA 4.5.

(4.16) $$S = \int_0^1 |f|^2 |f_5 \ldots f_1|^2 \, d\alpha \ll P^{1+\delta_0} P^{(4\lambda_5 \alpha_5 + \delta_0)},$$

and

$$U_6^{(4)}(N) > N^{\alpha_6 - \delta_0},$$

where

$$\alpha_6 = (1/4)+\lambda_5^{(5)}\alpha_5 > 1-(1/32)+(5/10^4). \tag{4.17}$$

Proof. Same as Lemma 4.3, where in place of (4.11) and (4.12), we use

$$\mu' = 4(2-2\nu'-1/3)\delta_5 \quad \text{with} \quad \nu' = 2\nu_2'+\nu_3'+\nu_4'+\nu_5' = 0.802359.$$

Proof of Theorem 1. We follow the same proof as in § 11 of [10] by considering $\mathfrak{M}_{a,q}$'s with $|\alpha-a/q| \leqslant q^{-1}P^{-3-\delta_0}$ for $1 \leqslant q \leqslant P^{1/2}$. For the treatment of the minor arcs m, we use (4.17) and Weyl's inequality for an additional 4-th power. Note that with admissibility of exponents, the treatment of the singular series is simplified considerably (as there is no need to impose congruence conditions on the summands in defining the exponential sums). It now follows in a standard way that

$$G^*(4) \leqslant 2(6)+1 = 13. \tag{4.18}$$

5. The case $k = 5$. We start with Lemma 7.1 in [10] taking δ_i $(1 \leqslant i \leqslant 5)$ as in (3.3) (in [10]). Then $\{\lambda_1^{(5)}, \ldots, \lambda_5^{(5)}, 1\}$ form admissible exponents, and (cf. (7.3) in [10])

$$0.823065 < \alpha_6 < 0.823069. \tag{5.1}$$

In the estimates of α_7, α_8 and α_9 (corresponding to (4.8)), we now use

$$f_i \ll q^{-1/5}P_i+q^{1/2+\varepsilon}. \tag{5.2}$$

We can successively take $\delta_i = 1/8$ for $6 \leqslant i \leqslant 8$.

In estimating α_7, α_8 and α_9, we use $f_i \ll q^{-1/5}P_i$ (as easily verified from (5.2)) respectively with $3 \leqslant i \leqslant 6$; $4 \leqslant i \leqslant 7$; and $5 \leqslant i \leqslant 8$. (3.15) is satisfied with $k = 5$, $\delta = 1/8$ and $r = 4$ in each of these cases. (Actually, it is sufficient to take $r = 3$ in estimating α_7.) (α_{10} may also be estimated with δ_9 slightly less than 1/8.)

Thus, starting with (5.1), and using

$$\alpha_{i+1} = 1/5+\alpha_i\{4+(1/8)\}/5 = 1/5+(33/40)\alpha_i \quad (6 \leqslant i \leqslant 8),$$

we have

LEMMA 5.1.

$$S = \int_0^1 |f|^2|f_8\ldots f_1|^2\,d\alpha \ll P^{1+\delta_0}(P_8\ldots P_1), \tag{5.3}$$

and

$$\alpha_9 > 0.963288 > 1-(3/80)+(7/10^4). \tag{5.4}$$

Proof that $G(5) \leqslant 21$. We follow precisely the same proof in § 11 of [10] introducing three more 5-th powers to deal with m. With Weyl's inequality, the saving contributed on m by these three 5-th powers is $N^{-(3/80)+\delta_0}$.

Hence, from (5.4), it follows that

$$G(5) \leqslant 2(9)+3 = 21. \tag{5.5}$$

6. The case $k = 6$. The next two lemmas will be used for $k \geqslant 6$.

LEMMA 6.1. *Let*

$$\lambda = \{(k-1)+(1/2^{k-2})\}/k. \tag{6.1}$$

Then, for $k \geqslant 6$,

$$\lambda^k > k\{(3/4+(1/k)\}/2^{k-2}. \tag{6.2}$$

Proof. For $k = 6$, this can be verified numerically. Also, $\lambda^k > (1-1/k)^k$, and $(1-1/k)^k$ is an increasing function of k, while the right-side of (6.2) is a decreasing function. Hence, result follows with verification for the case $k = 7$.

LEMMA 6.2. *Let* $1 \leqslant q \leqslant P^{(k/2^{k-2})}$. *Then, with* λ *as in* (6.1), *we have*

$$P^{(\lambda^r)} > q^{3/4+(1/k)+\varepsilon} \quad \textit{for} \quad 1 \leqslant r \leqslant k. \tag{6.3}$$

Proof. The proof follows directly from (6.3).

NOTE. If (1.2) is used, we can replace 1/2 by 3/4 in (6.3). Corresponding to (5.2), we use $f_i \ll q^{-1/k}P_i+q^{(3/4)+\varepsilon}$, which, with (6.2) gives (on major arcs)

$$f_i \ll q^{-1/k}P_i \tag{6.4}$$

for all the f_i's considered in the proofs.

Let $U(\alpha) = \sum_{u_i} e(\alpha u_i)$, where the u_i's are distinct integers of the form $(\sum_{i=1}^{7} x_i^6)$ occurring in the estimate of α_7 in [6]. The next lemma follows from [6] (Ch. 9, Lemma 9.9).

LEMMA 6.3.

$$\int_0^1 |f|^2|U(\alpha)|^2\,d\alpha \ll P^{1+\varepsilon}U(0), \tag{6.5}$$

and

$$U_8^{(6)}(N) > N^{\alpha_8},$$

where

$$\alpha_8 = 0.8283548. \tag{6.6}$$

LEMMA 6.4. *With* f, f_i*'s and* $U(\alpha)$ *defined (iteratively) as before,*

$$S = \int_0^1 |f|^2|f_{14}f_{13}\ldots f_8|^2|U(\alpha)|^2\,d\alpha \ll P^{1+\delta_0}(P_{14}\ldots P_8)U(0), \tag{6.7}$$

and

$$U_{15}^{(6)}(N) > N^{\alpha_{15}},$$

where

(6.8) $$\alpha_{15} > 0.9941155 > 1-(2/192)+(4/10^3).$$

Proof. We take $\delta_i = 1/16$ for $8 \leqslant i \leqslant 14$ (and use (6.4) with $k = 6$). At the first step (of estimating α_9), we take $\delta = 1/16$ and $r = 1$ in (3.15). (Actually, it is sufficient to take $r = 0$, and use trivial estimate for f_8 over the major arcs.) Then r is taken one larger at each of the successive steps (of estimating $\alpha_{10}, \alpha_{11}, \ldots$). When $r = 6$, we use (3.16). The rest of the proofs are simple computational verifications (with (3.15) and (3.16)). Using

(6.9) $$\alpha_{i+1} = 1/6+\alpha_i\{5+(1/16)\}/6 = 1/6+(27/32)\alpha_i \quad (8 \leqslant i \leqslant 14),$$

α_{15} is computed to satisfy (6.8).

Proof that $G(6) \leqslant 32$. We introduce two additional 6-th powers (for dealing with the minor arcs m) and use (6.8). We follow the proof of Lemma 4.8 in [9] defining $\mathfrak{M}_{a,q}$'s with $1 \leqslant q \leqslant P^{3/16}$. The saving contributed by the two additional 6-th powers over m (using Weyl's inequality) is $N^{-(2/192)+\delta_0}$. The result

(6.10) $$G(6) \leqslant 2(15)+2 = 32$$

now follows in a standard way from (6.8).

7. The case $k = 7$. For $k \geqslant 7$, conditions (3.15) and (3.16) are verified as in the case $k = 6$ (with enough to spare in the estimates) with $\delta = 1/2^{k-2}$. So, we indicate only the main results in these cases.

With $U(\alpha) = \sum_{u_i} e(\alpha u_i)$, where the u_i's are distinct integers of the form $(\sum_{i=1}^{10} x_i^7)$ occurring in the estimate of α_{10} (cf. Lemma 9.10 in [6]), and starting with

(7.1) $$\alpha_{11} = 0.85627,$$

we compute the α's with

(7.2) $$\alpha_{i+1} = 1/7+\alpha_i\{6+(1/32)\}/7 = 1/7+(193/224)\alpha_i \quad (11 \leqslant i \leqslant 21).$$

We have

LEMMA 7.1.

(7.3) $$S = \int_0^1 |f|^2 |f_{21}\ldots f_{11}|^2 |U(\alpha)|^2 d\alpha \ll P^{1+\delta_0}(P_{21}\ldots P_{11})U(0),$$

and

$$U_{22}^{(7)}(N) > N^{\alpha_{22}},$$

where

(7.4) $$\alpha_{22} > 0.998068 > 1-(1/448)+(2/10^4).$$

We now introduce an additional 7-th power, and follow the proof of Lemma 4.8 in [9]. It is verified from (7.4) that there is sufficient saving over m (with Weyl's inequality for the additional 7-th power); so that

(7.5) $$G(7) \leqslant 2(22)+1 = 45.$$

8. The case $k = 8$. Let $U(\alpha) = \sum_{u_i} e(\alpha u_i)$, where the u_i's are distinct integers of the form $(\sum_{i=1}^{13} x_i^8)$ occurring in the estimate of α_{13} (cf. Lemma 9.11 in [6]). Starting with

(8.1) $$\alpha_{14} = 0.8714775,$$

we compute the α's with

(8.2) $$\alpha_{i+1} = 1/8+\alpha_i\{7+(1/64)\}/8 = 1/8+(449/512)\alpha_i \quad (14 \leqslant i \leqslant 29)$$

to get the following:

LEMMA 8.1.

(8.3) $$S = \int_0^1 |f|^2 |f_{29}\ldots f_{14}|^2 |U(\alpha)|^2 d\alpha \ll P^{1+\delta_0}(P_{29}\ldots P_{14})U(0),$$

and

$$U_{30}^{(8)}(N) > N^{\alpha_{30}},$$

where

(8.4) $$\alpha_{30} > 0.998205 > 1-(2/1024)+(1/10^4).$$

Now, with (Weyl's inequality for) two additional 8-th powers, we get from (8.4)

(8.5) $$G(8) \leqslant 2(30)+2 = 62.$$

9. The case $k = 9$. Let u_i's denote distinct integers of the form $(\sum_{i=1}^{16} x_i^9)$ occurring in the estimate of α_{16}. With the estimates as in § 7 of [7], we can take

(9.1) $$\alpha_{17} = 0.882015.$$

Now, the α's are computed with

(9.2) $$\alpha_{i+1} = (1/9)+\alpha_i\{8+(1/128)\}/9 = (1/9)+(1025/1152)\alpha_i \quad (17 \leqslant i \leqslant 39).$$

We have

LEMMA 9.1.

(9.3) $$S = \int_0^1 |f|^2 |f_{39} \ldots f_{17}|^2 |U(\alpha)|^2 d\alpha \ll P^{1+\delta_0} (P_{39} \ldots P_{17}) U(0),$$

and

$$U_{40}^{(9)}(N) > N^{\alpha_{40}},$$

where

(9.4) $$\alpha_{40} > 0.9993 > 1-(2/2304)+(1/10^4).$$

Now, with (Weyl's inequality for) two additional 9-th powers, we get (from (9.4))

(9.5) $$G(9) \leqslant 2(40)+2 = 82.$$

10. The cases $k = 10$ and 11.

(A) $k = 10$. With the estimates as in (b) of § 7 in [7], we have

$$U_{20}^{(10)}(N) > N^{\alpha_{20}},$$

where

(10.1) $$\alpha_{20} = 0.89095.$$

Now, we compute the α's with

(10.2) $$\alpha_{i+1} = (1/10)+\alpha_i \{9+(1/256)\}/10 = (1/10)+(461/512)\alpha_i,$$

to get

(10.3) $$\alpha_{30} > 0.9643594,$$

(10.4) $$\alpha_{36} > 0.9828417,$$

and (as required in the estimate of $H(10)$)

(10.5) $$\alpha_{51} > 0.999553 > 1-(3/5120)+(1/10^4).$$

Condition (14) in [8] is satisfied with $s_1 = 30$, $s_2 = 36$, $\gamma_1 = \alpha_{30}$, and $\gamma_2 = \alpha_{36}$, so that

(10.6) $$G(10) \leqslant 2(36)+30 = 102.$$

(B) $k = 11$. As in (c) of § 7 in [7], we have $U_{24}^{(11)}(N) > N^{\alpha_{24}}$, where

(10.7) $$\alpha_{24} = 0.90774.$$

The α's are now computed with

(10.8) $$\alpha_{i+1} = (1/11)+\alpha_i \{10+(1/512)\}/11 = 1/11+(5121/5632)\alpha_i,$$

giving

(10.9) $$\alpha_{34} > 0.96556 \quad \text{and} \quad \alpha_{42} > 0.98495.$$

Condition (14) in [8] is satisfied with $s_1 = 34$, $s_2 = 42$, $\gamma_1 = \alpha_{34}$ and $\gamma_2 = \alpha_{42}$. Hence,

(10.10) $$G(11) \leqslant 2(42)+34 = 118.$$

11. Proof of Theorem 3. For $6 \leqslant k \leqslant 9$, Theorem 3 follows from Theorem 2 as indicated in § 17 of [9]. For $k = 10$, using (10.5) and Weyl's inequality for 3 additional 10-th powers (in place of the method in [8]), we get $G(10) \leqslant 2(51)+3 = 105$, and this implies that $H(10) \leqslant 107$.

12. A theorem on admissible exponents. The next theorem is a slight improvement for $4 \leqslant k \leqslant 8$ on the corresponding result of Davenport and Erdös in [3]. α_3 is computed with $\alpha = \alpha_2 = 2/k$, and $l = 2$ in Theorem 2 of [2]. But, the additional arguments as in Lemma 2.5 of [10] are required to establish the admissibility of exponents.

THEOREM 4. *Let $4 \leqslant k \leqslant 8$ and $\lambda = 3k/(3k+2)$. Then $\{1, \lambda, \lambda\}$ form admissible exponents.*

13. Precise bounds for $U_s^{(5)}(N)$ and $U_s^{(6)}(N)$. The bounds for $U_s^{(5)}(N)$ and $U_s^{(6)}(N)$ so far used in the proofs work in the estimates of both $G(k)$ and $H(k)$. However, the bounds can be made more precise by starting with Davenport's estimates for the α_i's (with small i's). This is made possible only because the major arcs are considered with small values of q, namely $q \leqslant P^{k/2^{k-2}}$. With the α's defined as before, we have the following:

THEOREM 5. (a) *For $k = 5$,*

(13.1) $$\alpha_4 = 569/845, \quad \alpha_5 = (7+33\alpha_4)/5(7+\alpha_4) = 6173/8105,$$

(13.2) $$\alpha_s = (8+33\alpha_{s-1})/40 \quad (6 \leqslant s \leqslant 9).$$

In particular,

(13.3) $$\alpha_8 = \frac{481782661}{518720000} \quad \textit{and} \quad \alpha_9 = \frac{20048587813}{20748800000}.$$

(b) *For $k = 6$,*

(13.4) $$\alpha_6 = 575117/787182, \quad \alpha_s = (15+81\alpha_{s-1})/6(15+\alpha_{s-1}) \quad (7 \leqslant s \leqslant 8),$$

(13.5) $$\alpha_s = (16+81\alpha_{s-1})/96 \quad (9 \leqslant s \leqslant 15).$$

Here,

(13.6) $$\alpha_8 = \frac{649358333}{781898958}.$$

Proof. Of these, (13.1) and (13.4) are Davenport's and are given in [2]. With the arguments already given in the paper, for $k=5$, we take $\delta_i = 1/8$ for $5 \leqslant i \leqslant 8$, and use $\alpha_s = 1/5+(33/40)\alpha_{s-1}$ leading to (13.2); for $k=6$, we take $\delta_i = 1/16$ for $8 \leqslant i \leqslant 14$, and use $\alpha_s = 1/6+(27/32)\alpha_{s-1}$, which establishes (13.5).

It should be noted here that in Davenport's results, x_5 can be kept explicit for $k=5$, and x_8 for $k=6$ (in estimating the number of integers of the forms $x_1^5+\ldots+x_5^5$, and $x_1^6+\ldots+x_8^6$). For $k=5$, it is also necessary to modify the arguments (for the major arcs) as indicated in the introduction. In this connection, Vaughan's result (1.2) may have to be used (where, it would be sufficient to work with $q^{(3/4)+\varepsilon}$ in place of $q^{(1/2)+\varepsilon}$). Also, the integrals (over the major arcs) of the error terms are to be estimated separately.

14. Addendum. For $k=6$, Davenport's estimate of α_8 allows only x_8 to be explicit in $x_1^6+\ldots+x_8^6$. The methods in [9] and [10] (as in the case $k=5$) allow the choice of the parameters δ_i $(1 \leqslant i \leqslant 7)$ close to that of Davenport's, at the same time retaining the use of admissible exponents (so that, $x_1, \ldots, x_8$ are all explicit). These were used in the author's earlier estimate $G(6) \leqslant 34$.

15. Additional remarks.

(A) In the proof of $G(k) \leqslant k\{3 \log k+\log 108\}$ (in [7]), the author used (1.1) for the treatment of major arcs, and the same proofs were utilised in [8] also. However, it is now seen that it would be sufficient to use (1.1) with $q^{(3/4)+\varepsilon}$ replacing $q^{(1/2)+\varepsilon}$. For this (with the notations as in [7], with the major arcs defined as in [8], and with $h=[(k+1)/2]$), the integrals of E_i $(0 \leqslant i \leqslant k)$ where $E_0 = (f^2-g^2)(f_1\ldots f_k)^2$, and for $1 \leqslant i \leqslant k$, $E_i = g^2(g_1\ldots g_{i-1})^2(f_i^2-g_i^2)(f_{i+1}\ldots f_k)^2$ are estimated separately over the major arcs using

$$\sum_q \sum_a q^{-(2i+1)+(2i-3)/k}|S(a,q)|^{2i} \ll 1 \quad \text{for} \quad 2 \leqslant i \leqslant h,$$

and also

$$\sum_q \sum_a q^{-(2h+2+\delta_0)}|S(a,q)|^{2h+2} \ll 1.$$

Here, the results of Davenport and Heilbronn in [4] provide sufficiently good estimates over the major arcs for (f_i-g_i) with $1 \leqslant i \leqslant h$. For $h+1 \leqslant i \leqslant k$, a non-trivial estimate for (f_i-g_i) would suffice. This may be obtained by using Weyl's inequality for f_i over the major arcs with large q (and standard estimate for g_i).

(B) For $k=5$, the recurrence relation (13.2) may be established with $s=9$ also either by (i) removing the effect of the factor $t^{(1/k-1)}$ (arising from (2.14) by using a suitable summation technique over the major arcs; or (ii) by considering the integral of $f_9(\alpha)$ rather than that of $F(\alpha)$ over the major arcs.

(C) For $k=4$, more precise bounds for α_5 and α_6 can be obtained as follows: As mentioned in (B), remove the effect of the factor $t^{1/(k-1)}$, and integrate the error terms over the major arcs separately using the inequality

$$\sum_q \sum_a q^{-(4+1/4+\delta_0)}|S(a,q)|^4 \ll 1.$$

In addition to these, in estimating α_6, the major arcs may be split into two sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ (those with small and large q's respectively). $\mathfrak{M}_1$ is treated as in the paper. In treating $\mathfrak{M}_2$, f_2 is better estimated by using Weyl's inequality, rather than using the estimates for major arcs.

(D) The reference no [17] given in [9] to the author's paper is changed to Portugaliae Math. 42 (4) (1985), pp. 447–465.

Added in proofs: (1) The case $k=6$. The author has withheld a (previously planned) separate publication for the case $k=6$ as further improvements have been obtained since the announcement of the method, the latest being $G(6) \leqslant 31$. The bound for α_{15} given by (13.5) falls only slightly short for proving $G(6) \leqslant 2(15)+1 = 31$. This gap can be filled. One way of filling this gap is to iterate with Hardy–Littlewood method at the last step also in dealing with the minor arcs (without separately using Weyl's inequality for a 6-th power) to get $G(6) \leqslant 15+16 = 31$. Here, we do not estimate α_{16} as we already know that $\alpha_{16}=1$ (and that it cannot exceed 1), but obtain a better saving than what is provided by Weyl's inequality. With μ corresponding to α_{16} given by (13.5), and $\beta = (\alpha_{15}+\mu)/2$, the saving on the minor arcs (by a combination with Schwarz's inequality) would be $N^{-\beta} \ll N^{-1-\delta_0}$. In fact it would be sufficient to work with a number slightly less than μ, so that the major arcs need be considered with even smaller values of q (than the ones already considered with $q < P^{3/8}$). This would further simplify the treatment of major arcs. While putting these together requires an additional argument, a closer reflection will reveal that the crucial idea is the iterative use of Hardy–Littlewood method with $F(\alpha)$ (cf. (2.3)). This is explicitly and clearly stated in Part I ([9]). What makes the iteration successful is the fact that t is taken in a small interval (as compared to x). (2). In the footnote of page 1 of Part II ([10]), the equality $\delta_i = (1/8)$ has appeared incorrectly with 118 in place of (1/8).

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Editor's note. The results of Vaughan referred to in the introduction have already appeared in print, see R. C. Vaughan, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) 52 (1986), pp. 445–463, and *On Waring's problem for sixth powers*, J. London Math. Soc. (2) 33 (1986), pp. 227–236.

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Bounds for solutions of additive equations in an algebraic number field I

by

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1. Introduction. Let k be a rational integer $\geqslant 1$. Similar to Waring's problem, one can show by the Hardy–Littlewood's method that an equation

$$a_1 x_1^k + \ldots + a_s x_s^k = 0,$$

where $a_1, \ldots, a_s$ are given rational integers but not all of the same sign, has a nontrivial solution in nonnegative rational integers $x_1, \ldots, x_s$, provided only that $s \geqslant c_1(k)$. (See, e.g., H. Davenport [3]). Here we use $c(f, \ldots, g)$ to denote a positive constant depending on $f, \ldots, g$. As for a bound of these solutions, it was shown by J. Pitman [10] that if $s \geqslant c_2(k)$, then there exists a nontrivial solution in nonnegative integers such that

$$\max_i x_i < c_3(k) \max(1, |a_1|, \ldots, |a_s|)^{c_4(k)} \tag{1}$$

where c_2 and c_4 are explicit. Under suitable conditions and if s is very large, the estimation can be considerably improved. (See, B. J. Birch [2] and W. M. Schmidt [11], [12].) In particular, Schmidt proved that if $s \geqslant c_5(k, \varepsilon)$, the equation

$$a_1 x_1^k + \ldots + a_s x_s^k = b_1 y_1^k + \ldots + b_s y_s^k$$

with positive rational integer coefficients has a nontrivial solution in nonnegative rational integers $x_1, \ldots, x_s, y_1, \ldots, y_s$ such that

$$\max_{i,j}(x_i, y_j) \leqslant \max_{i,j}(a_i, b_j)^{1/k+\varepsilon}. \tag{2}$$

We use hereafter $\varepsilon, \varepsilon_1, \ldots$ to denote arbitrary preassigned positive numbers < 1. The number $1/k$ in (2) is best possible. Although the circle method is still used in the proof of (2), the treatment of the minor arcs is completely distinct from that in Waring's problem.

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