Conspectus materiae tomi XLV, fasciculi 4

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange

Address of the Editorial Board and of the exchange

Die Adresse der Schriftleitung und des Austausches

Адрес редакции и книгообмена

ACTA ARITHMETICA

ul. Śniadeckich 8, 00-950 Warszawa

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires
The authors are requested to submit papers in two copies
Die Autoren sind gebeten um Zusendung von 2 Exemplaren jeder Arbeit
Рукописи статей редакция просит предлагать в двух экземплярах

ISBN 83-01-06274-6 ISSN 0065-1036

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

ACTA ARITHMETICA
XLV (1986)

A note on Selberg sieve

by

SAVERIO SALERNO (Salerno)

Introduction. Let $\mathcal{N}$ be a finite set of integers, $f(n)$ a non-negative integer function and $\mathcal{A}$ a sequence of integers. An important problem in number theory is to estimate $\sum_{n\in\mathcal{N}\cap\mathcal{A}} f(n)$ when $\mathcal{A}$ is, for example, the sequence of the primes or almost primes (we recall that a r-almost prime is an integer with at most r prime factors, counting the multiplicity). For this problem, the weighted sieve is often used. The idea works as follows: one introduces a weight $b(n)$, positive only if $n\in\mathcal{A}$ and not too large for these n. Then, if one is able to give a lower bound for $\sum_{n\in\mathcal{N}} f(n)b(n)$, one gets a lower bound for $\sum_{n\in\mathcal{N}\cap\mathcal{A}} f(n)$.

Hence, using Richert's logarithmic weights, we consider

$$(1)\qquad S=\sum_{n\in\mathcal{N}} f(n)\Big\{1-\sum_{\substack{p\mid n\\ p<t}}1-\lambda\sum_{\substack{p\mid n\\ p<y}}\Big(1-\frac{\log p}{\log y}\Big)\Big\}\Big(\sum_{\substack{\nu\mid n\\ \nu<z}}\lambda_\nu\Big)^2,\qquad t<z<y,$$

where $\{\lambda_\nu\}$ is a real sequence, $\lambda_\nu=0$ if $\mu(\nu)=0$ or if $\nu>\nu_0$ and λ_ν will be suitably chosen in the applications.

The basic reference on the subject is [2], in particular Chapter X. We point out that our choice of λ_ν, as well as our error term, are different from the usual ones. The possibility of working with different λ_ν is due to the use in the evaluation of (1) of a theorem of Bombieri [1] quoted in the sequel. The form of the error term is suggested by the bilinear form of the error introduced by Iwaniec.

Here, the bilinear form of the error depends on the fact that the weight of n is zero if n is not 1 or a prime. We hope to investigate the question of the bilinear form of the error of Selberg sieve in a forthcoming paper.

We point out that our method does not need any knowledge of the so-called "sieving limits"; this makes possible to avoid heavy computations. For a discussion of this question, we refer to [4].

I thank with pleasure the referee of the present paper for helpful remarks.

Definitions and statement of the results. We set

(2) $$\sum_{\substack{n\in\mathcal{A} \\ n\equiv 0 \pmod d}} f(n) = \frac{g(d)}{d}\sum_{n\in\mathcal{A}} f(n) + R_d$$

with $0 \leqslant g(d) < d$ and g multiplicative, that is $g(nm) = g(n)g(m)$ if $(n, m) = 1$; we define

$$g_1(d) = g(d)\frac{d}{d-g(d)};$$

moreover, let

(3) $$\zeta_r = \frac{\mu(r)r}{g_1(r)}\sum_{\nu}\frac{\lambda_{\nu r}g(\nu r)}{\nu r}$$

from which, by the Möbius inversion formula,

(4) $$\lambda_\nu = \frac{\mu(\nu)\nu}{g(\nu)}\sum_{r}\frac{\mu^2(r\nu)g_1(r\nu)}{r\nu}\zeta_{r\nu}.$$

Finally, let $\{a_d\}$ be a real sequence, $a_d = 0$ if $d > d_0$, $|a_d| \leqslant 1$. We shall use the following theorem of Bombieri ([1], Theorem 18) in a slightly modified form:

(5) $$\sum_{n\in\mathcal{A}} f(n)\Big(\sum_{\substack{d\mid n \\ d<y}} a_d\Big)\Big(\sum_{\substack{\nu\mid n \\ \nu<z}}\lambda_\nu\Big)^2 = G\sum_{n\in\mathcal{A}} f(n) + O\Big(\sum_{d<y}\sum_{m<z^2} a_d b_m R_{[d,m]}\Big)$$

where

(6) $$G = \sum_{m<z}\sum_{\substack{d<y \\ (m,d)=1}}\frac{\mu^2(m)g_1(m)}{m}\frac{a_d g(d)}{d}\Big(\sum_{r\mid d}\mu(r)\zeta_{rm}\Big)^2$$

and

(7) $$b_m = \sum_{\substack{\nu_1,\nu_2<z \\ [\nu_1,\nu_2]=m}}\lambda_{\nu_1}\lambda_{\nu_2}.$$

The asymptotic term in (5) is exactly that of Theorem 18 of [1], except for the presence of $f(n)$, but this introduces no real difference because it suffices to consider the set $\mathcal{A}'$ formed by the same elements of $\mathcal{A}$, but where every n is repeated $f(n)$ times. Following the proof of the theorem of Bombieri quoted above, the error term is

$$\sum_{d<y}\sum_{\nu_1,\nu_2<z} a_d\lambda_{\nu_1}\lambda_{\nu_2}R_{[d,\nu_1,\nu_2]}$$

and from this our expression of the error term (5), (7) is clear.

In this work, we shall choose $\lambda_1 = 1$ and

(8) $$\zeta_\nu = \begin{cases}\zeta_1 & \text{if } \nu \leqslant z_1,\ \mu^2(\nu) = 1,\\ \zeta_2 & \text{if } z_1 < \nu < z,\ \mu^2(\nu) = 1,\\ 0 & \text{otherwise}\end{cases}$$

and we set

(9) $$X \doteq \sum_{n\in\mathcal{A}} f(n),\quad y = X^\alpha,\quad z = X^\beta,\quad z_1 = X^{\varrho\beta},$$

$$t = (\log X)^{C_1},\quad \tau = \zeta_2/\zeta_1$$

with $\beta < \alpha$ and $\varrho, \tau < 1$, $\delta = \max_{n\in\mathcal{A}}\frac{\log n}{\log y}$.

We denote by C_i, ε_i, $i = 1, 2, \ldots$, some constants.

Finally, we introduce the following conditions:

(A_1) $$g(p) \ll p^{1-\varepsilon_1} \quad \text{for some } \varepsilon_1 > 0,$$

(A_2) $$\sum_{p<x}\frac{g(p)-k}{p}\log p = O(1), \quad \text{with } k\in N^+,$$

(A_3) $$\sum_{z\leqslant p<y}\sum_{\substack{n\in\mathcal{A} \\ n\equiv 0 \pmod{p^2}}} f(n) \ll \frac{X}{z}+y, \quad 2\leqslant z<y.$$

We express (A_2) saying that the sieve is k-dimensional; we point out that the condition (A_2) could be weakened without affecting our results.

Then, we shall prove the following

THEOREM. *Assume* (A_1), (A_2); *then*

(10) $$S = GX + O\Big(\sum_{p<y}\sum_{\substack{m<z^2 \\ p\nmid m}} c_m R_{pm}\Big) + O\Big(\sum_{m<z^2} c_m R_m\Big)$$

where

(11) $$G \sim C(\log X)^{-k}\Big\{(\varrho^k+\tau^2-\tau^2\varrho^k)\Big[1-\lambda k\Big(\log\frac{\alpha}{\beta}\Big)+ + \sum_{i=1}^{k}\frac{1}{i} - 1 + \frac{1}{k+1}\frac{\beta}{\alpha}\Big] + \lambda k F(k,\varrho,\tau)\Big\}$$

with

(12) $$|c_m| \leqslant m^\varepsilon,$$

$$(11')\qquad F(k,\varrho,\tau)=2\tau(1-\tau)\varrho^k\Big(\sum_{i=1}^{k}\frac{1}{i}+\int_0^1\frac{t^k}{1/\varrho-t}\,dt-\log\frac{1}{1-\varrho}\Big)+$$
$$+\varrho^k(1-\tau)^2\Big((1-\varrho)\frac{\beta}{\alpha}-\log\frac{1}{\varrho}\Big)-\frac{k}{k+1}\frac{\beta}{\alpha}\varrho^k(1-\varrho)(1-\tau^2),$$

$$(11'')\qquad C=(\beta^k k!)^{-1}\big(\tau+\varrho^k(1-\tau)\big)^{-1}\prod_p\Big(1-\frac{g(p)}{p}\Big)^{-1}\Big(1-\frac{1}{p}\Big)^k.$$

We also obtain the following

COROLLARY. *Assume* (A_1), (A_2), (A_3) *and suppose that* y *and* z *are chosen in such a way that the error terms of* (10) *are* $O(GX)$. *Then*

$$(13)\qquad \sum_{\substack{n\leqslant X\\ n=P_r}} f(n)\geqslant\frac{X}{(\log X)^k}$$

provided

$$(14)\qquad r\geqslant\Big[\frac{\delta}{\alpha}+k\Big(\log\frac{\alpha}{\beta}+\sum_{i=1}^{k}\frac{1}{i}-1+\frac{\beta}{\alpha}\,\frac{1}{k+1}\Big)-\frac{kF(k,\varrho,\tau)}{\varrho^k+\tau^2-\tau^2\varrho^k}+\varepsilon\Big]$$

for some $\varepsilon>0$ (we denote by $[x]$ the integer part of x).

With the usual choice of ζ_ν, that is $\zeta_1=\zeta_2$, we should get for the order of the almost-primes r the value given by (14) with $F(k,\varrho,\tau)=0$. Since we can choose ϱ_k, τ_k in order to have $F(k,\varrho_k,\tau_k)>0$, we obtain an improvement, which is appreciable when the dimension of the sieve k is relatively large.

Finally, we apply our results to the almost-primes represented by the product of k linear polynomials. We obtain improvements for $k\geqslant 4$.

Proofs.

Proof of the theorem. Let us now denote by G_1 the G defined by (5) with the following choise of the a_d:

1) $$a_d=\begin{cases}1 & \text{if}\quad d=1,\\ 0 & \text{if}\quad d>1.\end{cases}$$

Similarly, G_2 and G_3 are the G's corresponding to the choices

2) $$a_d=\begin{cases}1 & \text{if}\quad d=p<y,\ p\text{ prime},\ y>z,\\ 0 & \text{otherwise};\end{cases}$$

3) $$a_d=\begin{cases}\dfrac{\log p}{\log y} & \text{if}\quad d=p<y,\ p\text{ prime},\\ 0 & \text{otherwise},\end{cases}$$

respectively. We have by (A_1), (A_2)

$$(15)\qquad \sum_{n<x}\frac{\mu^2(n)g_1(n)}{n}=C_2(\log x)^k+O\big((\log x)^{k-1}\big)$$

with

$$(16)\qquad C_2=(k!)^{-1}\prod_p\Big(1-\frac{g(p)}{p}\Big)^{-1}\Big(1-\frac{1}{p}\Big)^k>0.$$

A proof of this can be found in [2], Chapter 5, no 3, formula (3.12).

We easily obtain

$$(17)\qquad G_1\sim C_2\zeta_1^2\log^k z_1+C_2\zeta_2^2(\log^k z-\log^k z_1)$$
$$=C_2\zeta_1^2(\log X)^k\{\varrho^k+\tau^2-\tau^2\varrho^k\}\beta^k.$$

Moreover

$$(18)\qquad \Big(\sum_{r|p}\mu(r)\zeta_{rm}\Big)^2=(\zeta_{pm}-\zeta_m)^2$$
$$=\begin{cases}0 & \text{if}\quad pm<z_1\text{ and}\\ & \text{if}\quad z_1<pm<z\text{ and }z_1<m<z,\\ (\zeta_2-\zeta_1)^2 & \text{if}\quad m<z_1\text{ and }z_1<pm<z,\\ \zeta_1^2 & \text{if}\quad m<z_1\text{ and }pm>z,\\ \zeta_2^2 & \text{if}\quad z_1<m<z\text{ and }pm>z.\end{cases}$$

Hence

$$(19)\qquad G_2=\sum_{m<z_1}\frac{\mu^2(m)g_1(m)}{m}\Big\{\sum_{\substack{z_1/m<p<z/m\\ p\nmid m}}(\zeta_2-\zeta_1)^2\frac{g(p)}{p}+\sum_{\substack{z/m<p<y\\ p\nmid m}}\zeta_1^2\frac{g(p)}{p}\Big\}+$$
$$+\sum_{z_1<m<z}\frac{\mu^2(m)g_1(m)}{m}\sum_{\substack{z/m<p<y\\ p\nmid m}}\zeta_2^2\frac{g(p)}{p}.$$

The condition $p\nmid m$ can be eliminated without affecting the asymptotic value of G_2 because we have, using (15),

$$\sum_{p<z}\frac{g(p)g_1(p)}{p^2}\sum_{z/p^2<t<z/p}\frac{\mu^2(t)g_1(t)}{t}$$
$$\ll(\log z)^{k-1}\sum_{p<z}\frac{g(p)g_1(p)\log p}{p^2}\ll(\log z)^{k-1}.$$

Also, we obtain by partial summation, using (A_2)

$$(20)\qquad \sum_{p<x}\frac{g(p)}{p}=k\log(1+\log x)+C_3+O\Big(\frac{1}{1+\log x}\Big).$$

Then, we deduce

$$G_2 = (\zeta_2-\zeta_1)^2 k \sum_{m<z_1} \frac{\mu^2(m)g_1(m)}{m} \log \frac{1+\log(z/m)}{1+\log(z_1/m)} + \tag{21}$$

$$+k\zeta_1^2 \sum_{m<z_1} \frac{\mu^2(m)g_1(m)}{m} \log \frac{\log y}{1+\log(z/m)} +$$

$$+k\zeta_2^2 \sum_{z_1<m<z} \frac{\mu^2(m)g_1(m)}{m} \log \frac{\log y}{1+\log(z/m)} +$$

$$+O\Big(\sum_{m<z} \frac{\mu^2(m)g_1(m)}{m} \cdot \frac{1}{1+\log(z/m)}\Big).$$

For the error term of (21), we remark that, by (15),

$$\sum_{m<z} \frac{\mu^2(m)g_1(m)}{m} \cdot \frac{1}{1+\log(z/m)}$$

$$\ll \frac{1}{\log\log z} \sum_{m<z/\log z} \frac{\mu^2(m)g_1(m)}{m} + \sum_{z/\log z<m<z} \frac{\mu^2(m)g_1(m)}{m}$$

$$\ll \frac{(\log z)^k}{\log\log z} + (\log\log z)(\log z)^{k-1}.$$

To estimate the main term of (21), we use (15) and partial summation. We have indeed, for $y > x$

$$\sum_{m<x} \frac{\mu^2(m)g_1(m)}{m} \log\big(1+\log(y/m)\big) \tag{22}$$

$$= C_2(\log x)^k \log\big(1+\log(y/x)\big) + C_2 \int_1^x (\log t)^k \frac{dt}{t\log(y/t)}$$

$$= C_2(\log x)^k \Big\{\log\big(1+\log(y/x)\big) + \int_0^1 \frac{t^k}{(\log y/\log x)-t}\, dt\Big\} + O\big((\log x)^{k-1}\big)$$

and also

$$\sum_{m<x} \frac{\mu^2(m)g_1(m)}{m} \log\big(1+\log(x/m)\big) = C_2 \int_1^x (\log t)^k \frac{dt}{t\big(1+\log(x/t)\big)} \tag{23}$$

$$= C_2(\log x)^k \Big\{\log\log x + \sum_{i=1}^{k} \frac{(-1)^i}{i}\binom{k}{i}\Big\} + O\big((\log x)^{k-1}\big).$$

So (21) becomes, recalling (9),

$$G_2 \sim C_2 k\zeta_1^2 (\log X)^k \varrho^k \beta^k \Big\{2\tau(1-\tau)\log\frac{1}{1-\varrho} + (1-\tau)^2 \log\frac{1}{\varrho} + \tag{24}$$

$$+\Big(1-\tau^2+\frac{\tau^2}{\varrho^k}\Big)\log\frac{\alpha}{\beta} - 2\tau(1-\tau)\int_0^1 \frac{t^k}{(1/\varrho)-t}\, dt -$$

$$-\Big[(1-\tau)^2+\frac{\tau^2}{\varrho^k}\Big] \sum_{i=1}^{k} \frac{(-1)^i}{i}\binom{k}{i}\Big\}.$$

The calculation of G_3 can be performed in a similar manner, and so we give only a sketch of the computations. Using (A_2), we have

$$\frac{G_3(\log y)}{k} \sim \{(\zeta_2-\zeta_1)^2 \log(z/z_1) + \zeta_1^2 \log(y/z)\} \sum_{m<z_1} \frac{\mu^2(m)g_1(m)}{m} + \tag{25}$$

$$+\zeta_1^2 \sum_{m<z_1} \frac{\mu^2(m)g_1(m)}{m} \log m + \zeta_2^2 \Big\{\log(y/z) \sum_{z_1<m<z} \frac{\mu^2(m)g_1(m)}{m} +$$

$$+ \sum_{z_1<m<z} \frac{\mu^2(m)g_1(m)}{m} \log m\Big\}.$$

It follows from (15) and partial summation, recalling (9),

$$G_3 \sim C_2 k\zeta_1^2 \frac{\beta^k}{\alpha} (\log X)^k \Big\{\varrho^k (1-\tau)^2 \beta(1-\varrho) + \frac{k}{k+1}\beta\varrho^{k+1} + \tag{26}$$

$$+(\varrho^k+\tau^2-\tau^2\varrho^k)(\alpha-\beta) + \frac{k\tau^2\beta}{k+1}(1-\varrho^{k+1})\Big\}.$$

Now, the main term of (10) is obtained by (5) collecting together (17), (24) and (26) and observing that the contribution to the sum S of the term $\sum_{n\in\mathcal{A}} f(n)\big(\sum_{\substack{p|n\\ p<t}} 1\big)\big(\sum_{\substack{\nu|n\\ \nu<z}} \lambda_\nu\big)^2$ is negligible. In fact, if we denote by G_4 the G corresponding to the choice

$$a_d = \begin{cases} 1 & \text{if} \quad d = p < t,\ t < z, \\ 0 & \text{otherwise} \end{cases}$$

we obtain by computations similar to the previous ones

$$G_4 \ll \Big(\frac{\log t}{\log X}\Big)^k (\log X)^{-k}$$

which is negligible in view of (9).

For the error term, we have by (5)

$$\sum_{d<y}\sum_{m<z^2} a_d b_m R_{[d,m]} = \sum_{p<y}\sum_{m<z^2} a_p b_m R_{[p,m]}$$

$$= \sum_{p<y}\sum_{\substack{m<z^2\\ p\nmid m}} a_p b_m R_{pm} + O\Big(\sum_{m<z^2}\big(\sum_{p\mid m} a_p\big)|b_m|\,|R_m|\Big)$$

and in view of (7) it suffices to show that $|\lambda_v| \leqslant 1$. We have

$$\lambda_v = \frac{\mu(v)g_1(v)}{g(v)}\Big\{\zeta_1 \sum_{r<z_1/v}\frac{\mu^2(r)g_1(r)}{r} + \zeta_2 \sum_{z_1/v<r<z/v}\frac{\mu^2(r)g_1(r)}{r}\Big\}$$

$$= \mu(v)\sum_{d\mid v}\frac{g_1(d)}{d}\Big\{\zeta_1 \sum_{\substack{r<z_1/v\\ (r,v)=1}}\frac{\mu^2(r)g_1(r)}{r} + \zeta_2 \sum_{\substack{z_1/v<r<z/v\\ (r,v)=1}}\frac{\mu^2(r)g_1(r)}{r}\Big\}.$$

We deduce, using $\zeta_2 \leqslant \zeta_1$

$$(27)\qquad |\lambda_v| \leqslant \sum_{d\mid v}\frac{g_1(d)}{d}\Big\{\zeta_1 \sum_{r<z_1/v}\frac{\mu^2(r)g_1(r)}{r} + \zeta_2 \sum_{z_1/v<r<z/v}\frac{\mu^2(r)g_1(r)}{r}\Big\}$$

$$\leqslant \zeta_1 \sum_{r<z_1}\frac{\mu^2(r)g_1(r)}{r} + \zeta_2 \sum_{z_1<r<z}\frac{\mu^2(r)g_1(r)}{r} = \lambda_1 = 1.$$

Now, the theorem is completely proved. ■

Proof of Corollary 1. Here, we use Richert's logarithmic weights. We define

$$(28)\qquad h(n) = 1 - \sum_{\substack{p\mid n\\ p<t}} 1 - \sum_{\substack{p\mid n\\ p<y}}\Big(1-\frac{\log p}{\log y}\Big).$$

First of all, we observe that the contribution to the sum S of the $n\in\mathcal{N}$ such that $h(n) > 0$ and there exists $p^2|n$ with $p<y$ is negligible. Indeed, $h(n)>0$ implies $p>t$ and so, using (A_3), this contribution is majorized by $y+X/t$.

Hence, we can assume that $p^2|n$ implies $p>y$. In this case

$$h(n) \leqslant 1 - \lambda\sum_{p\mid n}{}'\Big(1-\frac{\log p}{\log y}\Big)$$

where the dash indicates that the sum is performed counting the multiplicity. So $h(n)>0$ implies

$$\Omega(n) \leqslant \frac{1}{\lambda} + \frac{\log n}{\log y}, \qquad \text{where} \qquad \Omega(n) = \sum_{p\mid n}{}' 1.$$

Moreover $|\lambda_v| \leqslant 1$, $|a_d| \leqslant 1$. Hence, we obtain

$$(29)\qquad \sum_{\substack{n\in\mathcal{N}\\ \Omega(n)\leqslant\delta+(1/\lambda)}} f(n) \gg \sum_{n\in\mathcal{N}} f(n)h(n)\Big(\sum_{\substack{v\mid n\\ v<z}}\lambda_v\Big)^2.$$

Collecting together (29) and (10), our conclusion follows provided we choose λ in such a manner that G given by (10) is positive. The greatest suitable value of λ is given by (14). ■

Applications. We are concerned with the almost-primes represented by the polynomial $p_k(n) = (a_1 n + b_1)\dots(a_k n + b_k)$. In this case we define

$$\mathcal{N}_k = \{p_k(n) \mid 1 \leqslant n \leqslant x\}, \qquad R_d = \sum_{\substack{n\in\mathcal{N}_k\\ n\equiv 0 (\mathrm{mod}\, d)}} 1 - \frac{\varrho_k(d)}{d}x,$$

$$\varrho_k(d) = \#\{p_k(n) \equiv 0 (\mathrm{mod}\ d) \mid n\in \mathbf{Z}_d\}.$$

We introduce the following natural assumptions

(P_1^*) $\qquad p_k(n)$ has no fixed divisor,

(P_2^*) $\qquad \prod_{i=1}^{k} a_i \prod_{1\leqslant r<s\leqslant k}(a_r b_s - a_s b_r) \neq 0.$

Condition (P_2^*) ensures that no linear factor of $p_k(n)$ is a multiple of another. For further details, we refer to Theorem 10.5 of [2].

We use the following trivial estimate for the error term.

Lemma. *Assume* (P_1^*), (P_2^*); *then, for every* $\varepsilon>0$ *we have for some* $\delta>0$

$$(30)\qquad \sum_{d<x^{1-\varepsilon}} |R_d| \ll x^{1-\delta}. \ ■$$

Our result is the following:

Proposition. *Assume* (P_1^*), (P_2^*); *then we have*

$$(31)\qquad \#\{n\leqslant x \mid p_k(n) = P_{r(k)}\} \gg \frac{x}{(\log x)^k},$$

for suitable $r(k)$, *whose value is given in the table below for* $2\leqslant k\leqslant 6$

k	2	3	4	5	6
$r(k)$	6	10	14	18	23

Moreover, we have $r(k) < ck\log k$, *with* $c<1$, *for* k *large enough.*

Proof. We use the corollary, choosing $\alpha+2\beta = 1-\varepsilon$, $\delta = k = \dim \mathcal{N}_k$. In view of (30), the conclusion of the corollary holds. To obtain our thesis, we make the following choice of the parameters:

$$\alpha = \tfrac{1}{2}-\varepsilon, \qquad \beta = \tfrac{1}{4}-\varepsilon, \qquad \tau = \tfrac{1}{3}, \qquad \varrho = 4^{-k}. \ ■$$

Our proposition has to be compared with Theorem 10.5 and Corollary 10.11.2 of [2] where $r(k \sim k\log k$ for k large. We have improvements for $k\geqslant 4$.

References

[1] E. Bombieri, *Le grand crible dans la théorie analytique des nombres*, Astérisque 18 (1974), Soc. Math. de France.
[2] H. Halberstam and H. E. Richert, *Sieve methods*, Academic Press, London 1974.
[3] H. Iwaniec, *Almost-primes represented by quadratic polynomials*, Invent. Math. 47 (1978), pp. 171–188.
[4] — *Sieving limits*, Sem. Delange–Pisot–Poitou 1979/80, pp. 151–165, Birkhäuser, Boston–Basel–Stuttgart 1981.
[5] — *A new form of the error term in the linear sieve*, Acta Arith. 37 (1980), pp. 307–320.
[6] A. Selberg, *Sieve methods*, Proc. Sympos. Pure Math. 20 (1971), pp. 311–351.

ISTITUTO DI MATEMATICA
UNIVERSITÀ DI SALERNO
84 100 SALERNO, ITALY

Received on 24.2.1981
and in revised form on 12.6.1984 (1241)

ACTA ARITHMETICA
XLV (1986)

On the difference between perfect powers

by

JAN TURK (Apeldoorn)

1. Introduction. The problem whether there exists a function $\varphi\colon N \to N$ with $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$(0)\qquad |x^n - y^m| \geqslant \varphi(x^n) \quad \text{for all perfect powers } x^n \neq y^m$$

is still unsolved (see [5], p. 66). Actually, nothing beyond $|x^n - y^m| \geqslant 2$ for sufficiently large perfect powers $x^n \neq y^m$ ([14]) is known when there are no restrictions on the variables x, y, n, m other than the obvious ones ($x, y, n, m \in N$, $n \geqslant 2$, $m \geqslant 2$).

When two of the four variables are restricted the following results have been established (see Section 4.1 for more details).

Two restricted bases: for every $x, y \in N$ there exists a (large) number $c = c(x, y)$ such that (0) holds with $\varphi(t) = t(2\log t)^{-c}$.

One restricted base and one restricted exponent: for every $x, m \in N$ with $m \geqslant 2$ there exist (small) positive numbers $\varepsilon_i = \varepsilon_i(x, m)$, $i = 1, 2$, such that (0) holds with $\varphi(t) = \varepsilon_1 t^{\varepsilon_2}$.

Two restricted exponents: for every $n, m \in N$ with $n \geqslant 2$, $m \geqslant 2$ there exist (small) positive numbers $\varepsilon_i = \varepsilon_i(n, m)$, $i = 3, 4$, such that (0) holds with $\varphi(t) = \varepsilon_3 (\log t)^{\varepsilon_4}$.

It is the purpose of this paper to obtain functions φ for which (0) holds when only one of the variables is restricted. Our results are as follows (see Section 4.2 for more details).

One restricted base: for every $x \in N$ there exist (small) positive numbers $\delta_i = \delta_i(x)$, $i = 1, 2$, such that (0) holds with $\varphi(t) = \delta_1 t^{\delta_2}$.

One restricted exponent: for every $n \in N$ with $n \geqslant 2$ there exist (small) positive numbers $\delta_i = \delta_i(n)$, $i = 3, 4$, such that (0) holds with $\varphi(t) = \delta_3 \exp\big(\delta_4 (\log\log\log(t+16))^{1/2}\big)$.

The result for the case of one restricted base can actually be inferred from the detailed results on the case of one restricted base and one restricted exponent. The proof for the case of one restricted exponent depends on explicit bounds for the solutions m, x, y of the Diophantine equation $F(x) = ay^m$ (where $a \in Z$ and $F \in Z[X]$ are given) that we derive in Sections 2 and 3, thereby obtaining more explicit results than in [9] and [13].

The author thanks the referee for his thorough comments.