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Concerning a conjecture of R. L. Graham

by

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Graham [1] has conjectured that if $a_1 < \ldots < a_n$ are positive integers then

$$\max_{1 \leqslant i,j \leqslant n} \frac{a_i}{(a_i, a_j)} \geqslant n.$$

Many special cases of this result are known (see Wong [3]). Simpson [2] has shown that no counter example can contain a prime in the sequence.

It is the purpose of this note to show

THEOREM. *If* $a_1 < \ldots < a_n$ *and* $\frac{a_i}{(a_i, a_j)} < n$ *for all* i, j *then no* a_i *can be a power of a prime.*

Proof. Suppose that g.c.d. $(a_1, \ldots, a_n) = 1$, $a_k = p^m$ for some prime p, $1 \leqslant k \leqslant n$ and

$$\frac{a_i}{(a_i, a_j)} < n, \quad 1 \leqslant i, j \leqslant n. \tag{1}$$

Then the numbers $a_1, \ldots, a_n$ are all of the form

$$s_{ij} = ip^{m-j+1}, \quad 1 \leqslant i \leqslant n-1, \quad 1 \leqslant j \leqslant m+1.$$

Form the $(n-1) \times (m+1)$ matrix $S = (s_{ij})$.

Our proof will be complete if we can find a matrix T, a permutation of S, so that at most one distinct element from each row of T can lie in our sequence. Let σ denote the following function defined on $1, \ldots, n-1$.

$$\sigma(i) = pi \quad \text{if} \quad 1 \leqslant i \leqslant \left[\frac{n-1}{p}\right],$$

$$\sigma(i) = \frac{i+1}{p^\alpha}, \quad \text{where} \quad p^\alpha \| i+1, \quad \text{if} \quad \left[\frac{n-1}{p}\right] < i < n-1$$

$$= \frac{\left[\frac{n-1}{p}\right]+1}{p^\alpha} \quad \text{where} \quad p^\alpha \| \left[\frac{n-1}{p}\right]+1.$$

It is easily seen that σ is in fact a permutation of $\{1, \ldots, n-1\}$. Put $T_{ij} = (\sigma^{(j-1)}(i))\,p^{m+1-j}$; then $T = (T_{ij})$ is a permutation of S. Fix l; then at most one number of the form T_{lj} can be in our sequence.

Put

$$k_i = \sigma^{i-1}(l).$$

Lemma. *If $i > j$ then either*

$$k_i\, p^{m+1-i} = k_j\, p^{m+1-j}$$

or

$$\frac{k_j\, p^{m-j+1}}{(k_j\, p^{m-j+1},\, k_i\, p^{m-i+1})} \geqslant n.$$

Proof. By the definition of σ, we may assume without loss of generality

$$k_j > \left[\frac{n-1}{p}\right] \quad \text{and} \quad (k_i, p) = 1.$$

Now

$$\frac{k_j\, p^{m-j+1}}{(k_j\, p^{m-j+1},\, k_i\, p^{m-i+1})} = \frac{k_j\, p^{i-j}}{(k_j\, p^{i-j},\, k_i)} = \frac{k_j\, p^{i-j}}{(k_i,\, k_j)}.$$

We distinguish two cases:

(i) If $k_r = n-1$ for some $j \leqslant r < i$.

(ii) Not (i).

We estimate (k_i, k_j).

(i) $k_j = n - r_1$ where $1 \leqslant r_1 \leqslant r-j+1$,

$$k_i = \frac{\left[\frac{n-1}{p}\right]+s}{p^{\alpha}} \quad \text{where } 1 \leqslant s \leqslant i-r.$$

Then

$$\begin{aligned}(k_i, k_j) &\leqslant (p^{\alpha+1} k_i,\, k_j)\\ &= \left(ps + p\left[\frac{n-1}{p}\right],\, n-r_1\right)\\ &= (n+p^s-a,\, n-r_1) \quad \text{some } 1 \leqslant a \leqslant p\\ &\leqslant ps-a+r_1 \leqslant p(i-r)+(r-j)\\ &\leqslant p(i-j).\end{aligned}$$

Therefore

$$\frac{k_j}{(k_i,\, k_j)}\, p^{i-j} > \frac{(n-(i-j))\, p^{i-j}}{p(i-j)} \geqslant n-1$$

since $(R_i, p) = 1$ for all p, $i-j$ except $p = i-j = 2$, by considering the derivative of $(n-x)\,p^{x-1}/x$. For the case $p = i-j = 2$ it is easily verified that $\dfrac{n_j\, p^{i-j}}{(k_i,\, k_j)} \geqslant n$.

(ii) $k_i = \dfrac{k_j+s}{p^{\alpha}}$ where $1 \leqslant s \leqslant i-j$.

Then

$$(k_i, k_j) \leqslant (p^{\alpha} k_i,\, k_j) \leqslant i-j.$$

Hence

$$\frac{k_j\, p^{i-j}}{(k_i,\, k_j)} \geqslant \frac{k_j\, p^{i-j}}{i-j} \geqslant k_j\, p > n-1.$$

This completes the proof.

Added in proof. Recently M. Szegedy has settled the question of Graham for all sufficiently large n: M. Szegedy, *The solution of Graham's common divisor problem*, Hungarian Academy of Sciences. Preprint 27/1985.

References

[1] R. L. Graham, *Problem 5749*, Amer. Math. Monthly 77 (1970), p. 775.

[2] R. J. Simpson, *On a conjecture of R. L. Graham*, Acta Arith. 40 (1982), pp. 209–211.

[3] W. W.-C. Wong, *On a number theoretic conjecture of Graham*, Ph.D. thesis, 1981, Columbia.

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