is asymptotic to

$$2k^{1/2}\{\tfrac{3}{2}L(1-\theta^2)+\tfrac{1}{3}L(1-\theta^3)-L(1-\theta)-\tfrac{1}{6}L(1-\theta^6)\}^{1/2},$$

with $\theta = 2\cos 4\pi/9$. However, the theorem cannot be applied to (P13) because the terms of the series there do not have positive coefficients. This makes it possible for cancellation to occur between the terms, and the identity shows that this does in fact occur to a rather surprising extent.

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A generalization of Hasse's generalization of the Syracuse algorithm

by

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1. Introduction. In 1978, H. Möller [6] discussed an algorithm due to Hasse: Let m and d be relatively prime positive integers, $d \geqslant 2$; R_d is a complete set of residues mod d, not including a representative of the multiples of d; $N_d = \{n \in Z \mid d \nmid n\}$. Then $H\colon N_d \to N_d$ is defined by

$$H(x) = \frac{mx-r}{d^{\alpha}}, \tag{1.1}$$

where $mx - r = d^{\alpha}M$, $\alpha \geqslant 1$, $d \nmid M$, $r \in R_d$. (It is assumed that $r \in R_d \Rightarrow m \nmid r$, to ensure H is well-defined.)

Möller conjectured that the sequence of iterates $(H^k(n))_{k\geqslant 0}$ is periodic for all $n \in N_d$ if and only if $m < d^{d/(d-1)}$ and that the set of pure periods is finite for each choice of m, d and R_d. (See Terras [7], [8], Everett [3], Crandall [2] for the special case $d = 2$, $R_d = \{-1\}$, $m = 3$ known as the Syracuse algorithm, and Heppner [4] for the general case.)

Closely related to H is the mapping $T\colon Z \to Z$ defined by

$$T(x) = \begin{cases} (mx-r)/d & \text{if } d \nmid x, \text{ where } mx \equiv r \pmod d,\ r \in R_d, \\ x/d & \text{if } d \mid x. \end{cases} \tag{1.2}$$

In fact $H^k(n) = T^{\sigma_k}(n)$, where (using Möller's notation)

$$H^k(n) = \frac{mH^{k-1}(n) - r_{k-1}}{d^{\alpha_k}} \quad \text{and} \quad \sigma_k = \sum_{i=0}^{k} \alpha_i.$$

In the present paper a more symmetric mapping which generalizes T is studied. Let $d, m_1, \ldots, m_d$ be positive integers, $d \geqslant 2$, $\gcd(m_i, d) = 1$ for $i = 1, \ldots, d$; $R_d = \{x_1, \ldots, x_d\}$ is a complete set of residues mod d; $r_i \in R_d$ is defined for $i = 1, \ldots, d$ by $m_i x_i \equiv r_i \pmod d$. Then $T\colon Z \to Z$ is given by

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if} \quad x \equiv x_i \pmod d. \tag{1.3}$$

(This definition can be restated in terms of the integer part symbol:

$$T(x) = \left[\frac{m_i x}{d}\right] + \beta_i \quad \text{if} \quad x \equiv x_i \pmod d, \tag{1.4}$$

where $\beta_i = (\bar{r}_i - r_i)/d$ and $\bar{r}_i$ is the least non-negative residue of $r_i \pmod d$.)

Thus if $m_1 = \ldots = m_{d-1} = m$, $m_d = 1$, $x_d = 0$, (1.3) reduces to (1.2).

We are interested in the behaviour of the sequence of iterates $(T^K(n))_{K\geqslant 0}$, $n \in Z$. We remark that non-periodicity, unboundedness and $\lim_{K\to\infty} |T^K(n)| = \infty$ are all equivalent here.

Numerical evidence supports the following conjectures, the first three corresponding to Möller's conjectures:

Conjectures. (i) If $m_1 \ldots m_d < d^d$ the sequence $(T^K(n))_{K\geqslant 0}$ is periodic for all $n \in Z$.

(ii) If $m_1 \ldots m_d > d^d$, the sequence $(T^K(n))_{K\geqslant 0}$ is unbounded for almost all n.

(iii) For each choice of $d, m_1, \ldots, m_d, R_d$, the number of pure periods is finite.

(iv) If the sequence $(T^K(n))_{K\geqslant 0}$ is unbounded, it is uniformly distributed mod d^a for each $a \geqslant 1$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{K \leqslant N \mid T^K(n) \equiv j \pmod{d^a}\} = \frac{1}{d^a} \quad \text{for} \quad j = 0, \ldots, d^a - 1. \tag{1.5}$$

Conjecture (iv) has the following easily-proved consequences for Möller's algorithm: If the sequence $(H^k(n))_{k\geqslant 0}$ is unbounded then

$$\text{(a)} \quad \lim_{k\to\infty} \frac{\sigma_k}{k} = \frac{d}{d-1}, \tag{1.6}$$

$$\text{(b)} \quad \lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{k \leqslant N \mid H^k(n) \equiv s \pmod{d^{a+1}}\} = \frac{1}{(d-1)d^a} \quad \text{if } a \geqslant 0,\ d \nmid s, \tag{1.7}$$

$$\text{(c)} \quad \lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{k \leqslant N \mid a_k = j\} = \frac{d-1}{d^j} \quad \text{if } j \geqslant 1. \tag{1.8}$$

Our main result (Theorem 1) is that conjecture (iv) (with $a = 1$) implies conjecture (i).

In Section 3 we elaborate on the remark that definition (1.3) extends to that of a mapping $T\colon Z_d \to Z_d$ of the d-adic completion of Z. (Here $x \equiv x_i \pmod d$ means $(x - x_i)/d \in Z_d$.) We prove that T is measure-preserving and strongly-mixing with respect to Haar measure on Z_d. (See Billingsley [1] and Kuipers–Niederreiter [5] for background to these terms.) The ergodic theorem, applied to the congruence class $\{x \in Z_d \mid x \equiv x_i \pmod{d^a}\}$, then shows that conjecture (iv) holds for almost all d-adic integers n (Theorem 3).

2. Asymptotic results. For each $n \in Z$, $n \equiv x_i \pmod d$, we let $m(n) = m_i$, $r(n) = r_i$, $x(n) = x_i$, where $m_i x_i \equiv r_i \pmod d$, $r_i \in R_d$. Then definition (1.3) becomes

$$T(x) = \frac{m(x)x - r(x)}{d} \quad \text{if} \quad x \in Z. \tag{2.1}$$

We also let $m_K(n) = m(T^K(n))$, $r_K(n) = r(T^K(n))$, $x_K(n) = x(T^K(n))$, $K \geqslant 0$. Then the following results are easy exercises in induction:

Lemma 1. *If $K \geqslant 1$, we have*

$$\text{(a)} \quad T^K(n) = \frac{m_0(n) \ldots m_{K-1}(n)}{d^K}\left(n - \sum_{i=0}^{K-1} \frac{r_i(n) d^i}{m_0(n) \ldots m_i(n)}\right); \tag{2.2}$$

(b) *if $T^i(n) \neq 0$ for all $i \geqslant 0$, then*

$$T^K(n) = \frac{m_0(n) \ldots m_{K-1}(n)}{d^K}\, n \prod_{i=0}^{K-1}\left(1 - \frac{r_i(n)}{m_i(n) T^i(n)}\right). \tag{2.3}$$

Remark. With Möller, we point out that $T^K(n) \in Z$ implies the following interesting d-adically convergent expansion of n:

$$n = \sum_{i=0}^{\infty} \frac{r_i(n) d^i}{m_0(n) \ldots m_i(n)}. \tag{2.4}$$

Lemma 2. *Suppose the sequence $(T^K(n))_{K\geqslant 0}$ is unbounded. Then*

(a) *$T^K(n)$ has the same sign for all large K,*

$$\text{(b)} \quad \varliminf_{K\to\infty} \frac{1}{K} \sum_{i=0}^{K-1} \log m_i(n) \geqslant \log d, \tag{2.5}$$

$$\text{(c)} \quad |T^K(n)|^{1/K} \sim (m_0(n) \ldots m_{K-1}(n))^{1/K}/d \quad \textit{as} \quad K \to \infty. \tag{2.6}$$

Proof. Assume that $(T^K(n))_{K\geqslant 0}$ is unbounded. Then $\lim_{K\to\infty} |T^K(n)| = \infty$ and hence $T^K(n) \neq 0$ if $K \geqslant K_0$. Replacing n by $T^{K_0}(n)$ we can assume that $T^K(n) \neq 0$ for all $K \geqslant 0$; then (2.3) applies. As $1 - \dfrac{r_i(n)}{m_i(n) T^i(n)} > 0$ if $i \geqslant i_0$ it follows that $T^i(n)$ and $T^{i_0}(n)$ have the same sign if $i > i_0$, giving (a).

Next, without loss of generality, we can assume that $T^K(n)$ has constant sign for $K \geq 0$. Then, because the integers $T^K(n)$ are distinct, from (2.3) we have

$$1 \leq |T^K(n)| \leq \frac{m_0(n)\ldots m_{K-1}(n)}{d^K}|n|\prod_{i=0}^{K-1}\left(1+\frac{|r_i(n)|}{m_i(n)|T^i(n)|}\right)$$
$$\leq \frac{m_0(n)\ldots m_{K-1}(n)}{d^K}|n|\prod_{i=0}^{K-1}\left(1+\frac{R}{i+1}\right),$$

where $|r| \leq R$ if $r \in R_d$.

Hence, taking logarithms,

$$0 \leq \sum_{i=0}^{K-1}\log m_i(n) - K\log d + \log|n| + \sum_{i=0}^{K-1}\frac{R}{i+1}$$

and

$$\log d \leq \frac{1}{K}\sum_{i=0}^{K-1}\log m_i(n) + O\left(\frac{\log K}{K}\right) \quad \text{as} \quad K\to\infty. \tag{2.7}$$

Then (b) follows.

The proof also shows that

$$\log|T^K(n)| = \sum_{i=0}^{K-1}\log m_i(n) - K\log d + O(\log K) \quad \text{as} \quad K\to\infty, \tag{2.8}$$

and this gives (c).

As a corollary to Lemma 2 we have

THEOREM 1. *Suppose that the sequence $(T^K(n))_{K\geq 0}$ is unbounded and is uniformly distributed* mod d. *Then*

(a) $m_1 \ldots m_d > d^d$,

(b) $\lim\limits_{K\to\infty}|T^K(n)|^{1/K} = \dfrac{(m_1\ldots m_d)^{1/d}}{d}$.

Proof. With $\sum'$ denoting summation restricted by the condition $T^i(n) \equiv x_j \pmod d$, we have

$$\frac{1}{K}\sum_{i=0}^{K-1}\log m_i(n) = \frac{1}{K}\sum_{j=1}^{d}\sum_{i=0}^{K-1}{}'\log m_i(n) = \sum_{j=1}^{d}\log m_j\frac{1}{K}\sum_{i=0}^{K-1}{}'1 \tag{2.9}$$
$$\to \frac{1}{d}\sum_{j=1}^{d}\log m_j \quad \text{as } K\to\infty,$$

if the sequence $(T^K(n))_{K\geq 0}$ is uniformly distributed mod d.

Then by Lemma 1, if the sequence $(T^K(n))_{K\geq 0}$ is also unbounded, we have

$$\frac{1}{d}\sum_{j=1}^{d}\log m_j \geq \log d,$$

and hence $m_1\ldots m_d \geq d^d$ and consequently $m_1 \ldots m_d > d^d$, thus proving (a).

(b) follows from (2.9) and (2.6) (c).

THEOREM 2. *Let* $t_i(n) = \dfrac{r_i(n)}{m_0(n)\ldots m_i(n)}$. *Then if the series* $\sum\limits_{i=0}^{\infty} t_i(n)d^i$ *converges in the usual sense to a real number* s, *we have*

$$\lim_{K\to\infty}|T^K(n)| = \infty \quad \textit{if} \quad s \neq n.$$

Proof. From (2.2) we have

$$T^K(n) = \frac{m_0(n)\ldots m_{K-1}(n)}{d^K}\left(n - \sum_{i=0}^{K-1}t_i(n)d^i\right).$$

Now if $\sum\limits_{i=0}^{\infty}t_i(n)d^i$ converges to s, we have $\lim\limits_{i\to\infty}t_i(n)d^i = 0$. Hence

$$\lim_{i\to\infty}\frac{d^i}{m_0(n)\ldots m_i(n)} = 0$$

if $r_i(n) \neq 0$ for infinitely many i, as $r_i(n) \in Z$. However $r_i(n) = 0$ for $i \geq i_0$ implies $n = \sum\limits_{i=0}^{i_0-1}t_i(n)d^i$ by (2.4) and hence $n = s$.

Remarks. 1. If $n > 0$ and $r \in R_d \Rightarrow r \leq 0$, the condition $s \neq n$ is satisfied trivially.

2. It is tempting to conjecture that the converse of Theorem 2 holds, i.e.

(a) If $\sum\limits_{i=0}^{\infty}t_i(n)d^i$ is divergent, then $(T^K(n))_{K\geq 0}$ is periodic.

(b) If $\sum\limits_{i=0}^{\infty}t_i(n)d^i = n$ then $(T^K(n))_{K\geq 0}$ is periodic.

(If inequality (2.5) of Lemma 2 could be strengthened to ">", then (a) would follow by the ith root test for convergence.)

3. The converse of Theorem 2 holds if the sequence $(T^K(n))_{K\geq 0}$ is uniformly distributed mod d. For, as observed in the proof of Theorem 1, we have

$$\lim_{K\to\infty}\frac{1}{K}\sum_{i=0}^{K-1}\log m_i(n) = \frac{1}{d}\sum_{j=1}^{d}\log m_j > \log d.$$

Hence the series $\sum_{i=0}^{\infty} t_i(n)d^i$ is absolutely convergent to s, say. To prove that $n \neq s$, we first note that it suffices to assume that $T^K(n) \neq 0$ for all $K \geqslant 0$. Equations (2.2) and (2.3) then give

$$(2.10) \qquad n - \sum_{i=0}^{K-1} t_i(n)d^i = n \prod_{i=0}^{K-1} \left(1 - \frac{r_i(n)}{m_i(n)T^i(n)}\right).$$

However the infinite product $\prod_{i=0}^{\infty} \left(1 - \frac{r^i(n)}{m_i(n)T^i(n)}\right)$ is absolutely convergent, as by Theorem 1(b), the series $\sum_{i=0}^{\infty} 1/T^i(n)$ is absolutely convergent. Hence $n - s \neq 0$.

3. Measure-theoretic results. From now on we regard T as a mapping $T: Z_d \to Z_d$ of the ring of d-adic integers. With the Haar measure on Z_d normalised so that $\mu(Z_d) = 1$, the congruence class $B(y, d^a) = \{x \in Z_d | x \equiv y \pmod{d^a}\}$ has measure $1/d^a$.

Lemma 3. *T is measure-preserving. (i.e. $T^{-1}(A)$ is measurable for each measurable subset A of Z_d and $\mu(T^{-1}(A)) = \mu(A)$.)*

Proof. It suffices to verify the result for a congruence class. Then it is easily verified that

$$T^{-1}(B(y, d^a)) = \bigcup_{i=1}^{d} B\left(\frac{r_i + dy}{m_i}, d^{a+1}\right),$$

a disjoint union of d congruence classes, each of measure $1/d^{a+1}$.

Lemma 4. *With $x_i(n)$ defined as in Section 2, and $x_{i_0}, \ldots, x_{i_{K-1}} \in R_d$, we define*

$$(3.1) \qquad \Delta_{x_{i_0}, \ldots, x_{i_{K-1}}} = \{n \in Z_d | \ x_0(n) = x_{i_0}, \ldots, x_{K-1}(n) = x_{i_{K-1}}\}.$$

Then

$$(3.2) \qquad \Delta_{x_{i_0}, \ldots, x_{i_{K-1}}} = B\left(\frac{r_{i_0}}{m_{i_0}} + \ldots + \frac{r_{i_{K-1}} d^{K-1}}{m_{i_0} \ldots m_{i_{K-1}}}, d^K\right).$$

Proof. From (2.2), as $T^K(n) \in Z_d$, we have for $n \in Z_d$,

$$(3.3) \quad n \equiv \sum_{i=0}^{K-1} t_i(n)d^i \pmod{d^K}, \quad \text{where} \quad t_i(n) = \frac{r_i(n)}{m_0(n) \ldots m_i(n)}.$$

Then if $n \in \Delta_{x_{i_0}, \ldots, x_{i_{K-1}}}$, we have

$$x_0(n) = x_{i_0}, \quad \ldots, \quad x_{K-1}(n) = x_{i_{K-1}}$$

and hence

$$m_0(n) = m_{i_0}, \ldots, m_{K-1}(n) = m_{i_{K-1}} \text{ and } r_0(n) = r_{i_0}, \ldots, r_{K-1}(n) = r_{i_{K-1}}.$$

Hence from (3.3)

$$n \in B\left(\frac{r_{i_0}}{m_{i_0}} + \ldots + \frac{r_{i_{K-1}} d^{K-1}}{m_{i_0} \ldots m_{i_{K-1}}}, d^K\right).$$

The converse is also straightforward.

Lemma 5. *Let A and B be congruence classes in Z_d, $A = B(a, d^a)$, $B = B(b, d^\beta)$. Then, if $K \geqslant \beta$ we have*

$$(3.4) \qquad \text{(i)} \quad T^{-K}(A) \cap B = \bigcup \Delta_{x_0(b), \ldots, x_{\beta-1}(b), c_\beta, \ldots, c_{K-1}, x_0(a), \ldots, x_{a-1}(a)},$$

where $c_\beta, \ldots, c_{K-1} \in R_d$, a disjoint union of $d^{K-\beta}$ congruence classes $\pmod{d^{K+a}}$,

$$(3.5) \qquad \text{(ii)} \quad \mu(T^{-K}(A) \cap B) = \mu(A)\mu(B).$$

Proof. From (2.2), with n replaced by $T^K(n)$, we have

$$(3.6) \qquad T^K(n) \equiv m_0(n) \ldots m_{K-1}(n) \sum_{i=0}^{a-1} t_{K+i}(n)d^i \pmod{d^a}.$$

Also

$$(3.7) \qquad a \equiv \sum_{i=0}^{a-1} t_i(a)d^i \pmod{d^a},$$

$$(3.8) \qquad b \equiv \sum_{i=0}^{\beta-1} t_i(b)d^i \pmod{d^\beta},$$

$$(3.9) \qquad n \equiv \sum_{i=0}^{\beta-1} t_i(n)d^i \pmod{d^\beta}.$$

Then $n \in T^{-K}(A) \cap B \Rightarrow T^K(n) \in A$ and $n \in B$

$\Rightarrow$ r.h.s. of (3.6) and (3.7) are congruent $\pmod{d^a}$

and r.h.s. of (3.8) and (3.9) are congruent $\pmod{d^\beta}$,

$$\Rightarrow x_K(n) = x_0(a), \ldots, x_{K+a-1}(n) = x_{a-1}(a);$$
$$x_0(n) = x_0(b), \ldots, x_{\beta-1}(n) = x_{\beta-1}(b).$$

Hence if $K \geqslant \beta$ we have $n \in \Delta_{x_0(b),\ldots,x_{\beta-1}(b),c_\beta,\ldots,c_{K-1},x_0(a),\ldots,x_{\alpha-1}(a)}$, where $c_\beta = x_\beta(n), \ldots, c_{K-1} = x_{K-1}(n)$. The converse implication is also straightforward.

(ii) follows as

$$\mu\big(T^{-K}(A)\cap B\big) = d^{K-\beta}\frac{1}{d^{K+\alpha}} = \frac{1}{d^{\alpha+\beta}} = \mu(A)\mu(B).$$

COROLLARY 1. *If A and B are measurable subsets of Z_d then*

(3.10) $$\lim_{K\to\infty}\mu\big(T^{-K}(A)\cap B\big) = \mu(A)\mu(B).$$

Proof. The result holds trivially from (ii) Lemma 5 if A and B are congruence classes. Then standard arguments about approximating measurable sets give the general result. (See Billingsley [1], p. 12.)

THEOREM 3. *For $j = 0, \ldots, d^\alpha-1$, we have*

(3.11) $$\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{K \leqslant N \mid T^K(n) \equiv j \pmod{d^\alpha}\} = \frac{1}{d^\alpha}$$

for almost all $n \in Z_d$.

Proof. By Corollary 1, T is strongly-mixing and hence ergodic (i.e. $T^{-1}(A) = A \Rightarrow \mu(A) = 0$ or 1). Hence the ergodic theorem applied to χ_A, the characteristic function of $A = B(j, d^\alpha)$, gives

$$\lim_{N\to\infty}\frac{1}{N}\sum_{K=0}^{N}\chi_A\big(T^K(n)\big) = \mu(A)$$

for almost all $n \in Z_d$, and this is just (3.11) restated.

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