Thus

(b) $$T = (1-\lambda)\frac{\psi'(z^{-1})}{z}\frac{1}{n^{2-\lambda}} + O\left(\frac{\log n}{n^{3-\lambda}}\right).$$

Combining (a) and (b) we see that

(30) $$X' = \psi(z^{-1})\frac{1}{n^{1-\lambda}} + \frac{(1-\lambda)\psi'(z^{-1})}{z}\frac{1}{n^{2-\lambda}} + O\left(\frac{\log n}{n^{3-\lambda}}\right).$$

Similar (and easier) calculations give

(31) $$X'' = \psi(z^{-1})\frac{1}{n^{2-\lambda}} + O\left(\frac{1}{n^{3-\lambda}}\right),$$

(32) $$X''' = O\left(\frac{\log n}{n^{3-\lambda}}\right).$$

Relation (29) combined with relations (30), (31) and (32) gives the desired relation (28) and hence completes the proof of Theorem 1.

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On an extension of a theorem of S. Chowla

by

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1. Introduction. In [4] S. Chowla proved that if p is an odd prime, then the $(p-1)/2$ real numbers $\cot(2\pi a/p)$, $a = 1, 2, \ldots, (p-1)/2$ are linearly independent over the field Q of rational numbers. Other proofs were given by H. Hasse [5], R. Ayoub [1], [2] and T. Okada [8].

The purpose of this note is to show the following theorem, which is an extension of S. Chowla's theorem mentioned above.

THEOREM. *Let k and q be integers with $k > 0$ and $q > 2$. Let T be a set of $\varphi(q)/2$ representatives* mod q *such that the union $\{T, -T\}$ is a complete set of residues prime to q. Then the real numbers $D^{k-1}(\cot\pi z)|_{z=a/q}$, $a \in T$ are linearly independent over Q, where φ is the Euler totient function and $D = d/dz$.*

In the case $k = 2$, this corresponds to the result of H. Jager and H. W. Lenstra, Jr. [6].

2. Preliminary results. We put

$$F_k(z) = \begin{cases} \dfrac{k}{(-2\pi i)^k} D^{k-1}(\pi\cot\pi z) & \text{if } z \text{ is not an integer,} \\ 0 & \text{if } z \text{ is an integer and } k \text{ is odd,} \\ B_k & \text{if } z \text{ is an integer and } k \text{ is even,} \end{cases}$$

where B_k is the kth Bernoulli number. Then we have the following partial fraction decomposition of $F_k(z)$:

(1) $$F_k(z) = -\frac{k!}{(2\pi i)^k}\sum_{n=-\infty}^{\infty}{}' \frac{1}{(z+n)^k},$$

where the dash $'$ means that the term with $n = -z$ is omitted if z is an integer. (If $k = 1$, we interpret the sum as grouping the corresponding positive and negative terms together.)

Let ψ be an arithmetical function which is periodic $\bmod q$. Then we have from (1)

(2)
$$\sum_{n=-\infty}^{\infty}{}' \frac{\psi(n)}{n^k} = \sum_{m=0}^{q-1}\psi(m)\sum_{n=-\infty}^{\infty}{}'\frac{1}{(nq+m)^k} = \frac{1}{q^k}\sum_{m=0}^{q-1}\psi(m)\sum_{n=-\infty}^{\infty}{}'\frac{1}{(m/q+n)^k}$$
$$= -\frac{(2\pi i)^k}{k!\,q^k}\sum_{m=0}^{q-1}\psi(m)F_k\left(\frac{m}{q}\right).$$

If we put
$$\psi_*(m) = \frac{1}{q}\sum_{n=0}^{q-1}\psi(n)\,e\left(\frac{-mn}{q}\right),$$
where we write $e(w)$ for $e^{2\pi i w}$, then ψ_* is also periodic $\bmod q$ and the following inversion formula holds:
$$\psi(n) = \sum_{m=0}^{q-1}\psi_*(m)\,e\left(\frac{mn}{q}\right).$$
From this we get

(3)
$$\sum_{n=-\infty}^{\infty}{}'\frac{\psi(n)}{n^k} = \sum_{m=0}^{q-1}\psi_*(m)\sum_{n=-\infty}^{\infty}{}'\frac{e(mn/q)}{n^k} = -\frac{(2\pi i)^k}{k!}\sum_{m=0}^{q-1}\psi_*(m)P_k\left(\frac{m}{q}\right),$$

where

(4)
$$P_k(x) = \begin{cases}0 & \text{if } k=1 \text{ and } x \text{ is an integer,}\\ B_k(x-[x]) & \text{otherwise,}\end{cases}$$
$$= -\frac{k!}{(2\pi i)^k}\sum_{n=-\infty}^{\infty}{}'\frac{e(nx)}{n^k}$$

is the kth Bernoulli function, $B_k(x)$ denoting the kth Bernoulli polynomial (cf. [9], p. 16).

Letting ψ be the characteristic function of the set $\{nq+b \mid n=0, \pm 1, \pm 2, \ldots\}$ and noting that $\psi_*(m) = \frac{1}{q}e\left(-\frac{mb}{q}\right)$, we have from (1) and (3)

(5)
$$F_k\left(\frac{b}{q}\right) = -\frac{k!}{(2\pi i)^k}\sum_{n=-\infty}^{\infty}{}'\frac{1}{(b/q+n)^k} = -\frac{k!\,q^k}{(2\pi i)^k}\sum_{n=-\infty}^{\infty}{}'\frac{\psi(n)}{n^k}$$
$$= q^{k-1}\sum_{m=0}^{q-1}e\left(\frac{-mb}{q}\right)P_k\left(\frac{m}{q}\right).$$

Since $P_k(-x) = (-1)^k P_k(x)$, we have from (5)

(6)
$$F_k\left(\frac{-b}{q}\right) = (-1)^k F_k\left(\frac{b}{q}\right).$$

We say that ψ is even (resp. odd) if $\psi(-n) = \psi(n)$ (resp. $\psi(-n) = -\psi(n)$) for all integers n. We note that if both k and ψ are even (or odd), we have

(7)
$$\sum_{n=-\infty}^{\infty}{}'\frac{\psi(n)}{n^k} = 2\sum_{n=1}^{\infty}\frac{\psi(n)}{n^k}.$$

We shall need the following lemma in the next section, which follows easily from the well-known Frobenius determinant relation (cf. [7], p. 284, Th. 5).

LEMMA. *Let G be a finite abelian group and let H be a subgroup of G. Let λ be a character of H and let Λ be the set of all characters of G whose restriction to H is equal to λ. Then for each (complex valued) function f on G with*
$$f(ah) = \lambda(h)f(a) \quad (a \in G,\ h \in H),$$
we have
$$\det_{a,b\in T} f(a^{-1}b) = \prod_{\chi\in\Lambda}\Big(\sum_{a\in T}\overline{\chi}(a)f(a)\Big),$$
where T is a complete representative system of G by H and $\overline{\chi}$ is the complex conjugate of χ.

3. Proof of Theorem. Let ζ denote a primitive qth root of unity. Then the Galois group of $\boldsymbol{Q}(\zeta)$ over $\boldsymbol{Q}$ is given by the mappings $\sigma_a\colon \zeta \mapsto \zeta^a$, where a runs through a complete set of residues prime to q. Since $B_k(x)$ is a polynomial in x with coefficients in $\boldsymbol{Q}$, it follows from (4) that $P_k(x) \in \boldsymbol{Q}$ for all $x \in \boldsymbol{Q}$. Hence the equation (5) shows that $F_k\left(\frac{b}{q}\right) \in \boldsymbol{Q}(\zeta)$ for all integers b and that $\left(F_k\left(\frac{b}{q}\right)\right)^{\sigma_a} = F_k\left(\frac{ab}{q}\right)$. To prove our theorem it suffices to show that $F_k\left(\frac{b}{q}\right)$, $b \in T$ are linearly independent over $\boldsymbol{Q}$.

Suppose that there exist $C_b \in \boldsymbol{Q}$ such that
$$\sum_{b\in T} C_b F_k\left(\frac{b}{q}\right) = 0.$$
Then applying the mappings $\sigma_{\bar a}$ $(a \in T)$, we have

(8)
$$\sum_{b\in T} C_b F_k\left(\frac{\bar a b}{q}\right) = 0,$$

where $\bar{a}$ is defined by $\bar{a}a \equiv 1 \pmod{q}$. Now (8) together with (6) calls for the application of our lemma with

G: the group of reduced residue classes mod q,

$H = \{1, -1\}$,

$\lambda(-1) = (-1)^k$,

Δ: the set of all even or odd Dirichlet characters mod q according as k is even or odd,

$f(b) = F_k\left(\frac{b}{q}\right)$,

T: the set occurring in our theorem.

We obtain

$$\det_{a,b\in T} F_k\left(\frac{\bar{a}b}{q}\right) = \prod_{\chi\in\Delta}\left(\sum_{a\in T}\bar{\chi}(a)F_k\left(\frac{a}{q}\right)\right). \tag{9}$$

Here from (2) and (7) we have for any $\chi\in\Delta$

$$\sum_{a\in T}\bar{\chi}(a)F_k\left(\frac{a}{q}\right) = \frac{1}{2}\sum_{m=0}^{q-1}\bar{\chi}(m)F_k\left(\frac{m}{q}\right) = -\frac{k!\,q^k}{(2\pi i)^k}L(k,\bar{\chi}).$$

From this and (9) we have

$$\det_{a,b\in T} F_k\left(\frac{\bar{a}b}{q}\right) = \left(-\frac{k!\,q^k}{(2\pi i)^k}\right)^{\varphi(q)/2}\prod_{\chi\in\Delta}L(k,\bar{\chi}) \neq 0.$$

This together with (8) shows that $C_b = 0$ for all $b\in T$, which completes the proof of our theorem.

4. Corollaries. Let Φ_q denote the qth cyclotomic polynomial.

Corollary 1 (cf. Baker–Birch–Wirsing [3], Th. 1). *If ψ is a non-vanishing arithmetical function with period q such that* (i) *ψ is even or odd according as k is even or odd,* (ii) *$\psi(n) = 0$ if $(n,q) > 1$,* (iii) *Φ_q is irreducible over $Q(\psi(1),\ldots,\psi(q))$, then*

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{n^k} \neq 0.$$

Proof. We have from (7), (2), and the conditions (i), (ii)

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{n^k} = \frac{1}{2}\sum_{n=-\infty}^{\infty}{}'\frac{\psi(n)}{n^k} = -\frac{(2\pi i)^k}{k!\,q^k}\frac{1}{2}\sum_{m=0}^{q-1}\psi(m)F_k\left(\frac{m}{q}\right)$$

$$= -\frac{(2\pi i)^k}{k!\,q^k}\frac{1}{2}\sum_{b\in T}\left\{\psi(b)F_k\left(\frac{b}{q}\right) + \psi(-b)F_k\left(\frac{-b}{q}\right)\right\}$$

$$= -\frac{(2\pi i)^k}{k!\,q^k}\sum_{b\in T}\psi(b)F_k\left(\frac{b}{q}\right).$$

We see from the condition (iii) that the qth cyclotomic field $Q(\zeta)$ and the field $Q(\psi(1),\ldots,\psi(q))$ are linearly disjoint over Q, so our theorem implies that $F_k\left(\frac{b}{q}\right)$, $b\in T$, are linearly independent over $Q(\psi(1),\ldots,\psi(q))$. Hence

$$\sum_{b\in T}\psi(b)F_k\left(\frac{b}{q}\right) \neq 0,$$

as required.

Corollary 2 (cf. [3]; Cor. 1 to Th. 1). *Let $(q,\varphi(q)) = 1$ and let Δ be the set of all even or odd Dirichlet characters* mod q *according as k is even or odd. Then the numbers $L(k,\chi)$, $\chi\in\Delta$ are linearly independent over Q.*

Proof. This follows immediately from Corollary 1 on noting that any

$$\psi = \sum_{\chi\in\Delta}a_\chi\chi$$

with rational a_χ fulfills the conditions of Corollary 1, since Φ_q is irreducible over the $\varphi(q)$-th cyclotomic number field and the matrix $[\chi(a)]$ $(a\in T,\ \chi\in\Delta)$ is nonsingular.

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