

A note on large gaps between prime numbers

by

M. N. HUXLEY (Cardiff)

In memory of Professor Paul Turán

1. Introduction. Let p_n denote the n th prime number, $d_n = p_{n+1} - p_n$. As $n \rightarrow \infty$, $d_n/\log p_n$ has mean value 1, and numerical evidence is consistent with Cramér's conjecture [2]:

$$(1.1) \quad d_n \leq (1 + o(1)) \log^2 p_n,$$

the best bound known is

$$(1.2) \quad d_n \ll p_n^{7/12 + \varepsilon}.$$

(We use ε to denote a positive number which may be chosen arbitrarily small, but ε need not denote the same number in each formula. The symbol \ll denotes an inequality in which an unspecified positive constant, depending only on ε , times the right-hand side exceeds the modulus of the left-hand side.) The Riemann hypothesis implies

$$(1.3) \quad d_n \ll p_n^{1/2} \log p_n.$$

Erdős suggested that the proportion of the interval $[1, N]$ covered by long gaps between consecutive prime numbers tended to zero as $N \rightarrow \infty$. Wolke [17] proved that

$$(1.4) \quad \sum_{\substack{p_n < N \\ d_n > N^{1/2}}} d_n \ll N^\theta,$$

with $\theta = 29/30$. The exponent has been improved by Ivić [10] to $\theta = 23/25$, and by R. J. Cook [1] to $85/98 + \varepsilon$. Related results have been given by Moreno [14], Warlimont [16], Heppner [6], Ivić [10] and Heath-Brown [5]. In this note we use more powerful zero-density theorems for $\zeta(s)$. Our results depend on the 'Lindelöf exponent' c for which

$$(1.5) \quad \zeta\left(\frac{1}{2} + it\right) \ll |t|^c.$$

Lindelöf's hypothesis requires that (1.5) be valid for every $\epsilon > 0$. The exponent in (1.4) can be taken as

$$(1.6) \quad \theta = \frac{1}{2} + 2 \max(c, 41/266) + \epsilon.$$

The number $41/266 = 0.1541 \dots$ could be improved slightly, but not below $3/20$ by our methods. Note that $41/266$ is less than Kolesnik's estimate [12] $c = 173/1067 + \epsilon = 0.1621 \dots$ for the Lindelöf exponent. Assuming the Lindelöf hypothesis we may, however, take

$$(1.7) \quad \theta = 3/4 + \epsilon.$$

If we replace the condition $d_n > N^{1/2}$ in (1.4) by

$$(1.8) \quad d_n > N^{1/2 + \epsilon_1}$$

for any $\epsilon_1 > 0$, we have (1.4) with

$$(1.9) \quad \theta = \frac{1}{2} + 2 \cdot \frac{41}{266} + \epsilon_2 = \frac{215}{266} + \epsilon_2,$$

and on the Lindelöf hypothesis

$$(1.10) \quad \theta = \frac{1}{2} + \epsilon_2,$$

where ϵ_2 can be made arbitrarily small. Results can be obtained for longer or shorter gaps. Thus if we sum over gaps for which

$$(1.11) \quad d_n > N^{7/12},$$

we can take

$$\theta = 3/4 + \epsilon.$$

As in Ivić [10] our method fails for $d_n \leq N^{1/6}$.

I would like to thank Cook, Heath-Brown, Ivić and Wolke for making unpublished manuscripts available to me. Heath-Brown also has new bounds for $\zeta(s)$ which may lead to new estimates for the exponent θ of (1.4).

2. Wolke's method. For simplicity we consider the range

$$(2.1) \quad N \leq p_n < p_{n+1} \leq 2N,$$

where N is sufficiently large. A result of the type (1.4) may be deduced by letting N run through the powers of 2 and summing over the various ranges. Let as usual

$$(2.2) \quad \psi(x) = \sum_{a \geq 1} \sum_{\substack{p \text{ prime} \\ p^a \leq x}} \log p.$$

Our starting point is the approximate formula ([15], Satz 4.5), valid for $N \leq x \leq 2N$ and for $T \leq N$,

$$(2.3) \quad \psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + O\left(\frac{Nl^2}{T}\right).$$

where $l = \log 2N$, and the sum is over zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 \leq \beta \leq 1$, $|\gamma| \leq T$. To discuss gaps with

$$(2.4) \quad d_n = p_{n+1} - p_n \geq h = 2\delta N$$

we let

$$(2.5) \quad E(x) = \psi(x + \delta x) - \psi(x) - \delta x = - \sum_{\rho} x^{\rho} \delta(\rho) + O(Nl^2/T),$$

where

$$(2.6) \quad \delta(\rho) = \int_1^{1+\delta} u^{\rho-1} du,$$

$$(2.7) \quad |\delta(\rho)| \leq \delta.$$

By (1.2) $d_n < N^{2/3}$, so that for $a \geq 3$ there is at most one a th power between p_n and p_{n+1} . The number of squares in an interval of length d contained in the interval (2.1) is

$$(2.8) \quad \leq 1 + d/2N^{1/2},$$

so that

$$(2.9) \quad \psi(x + \delta x) - \psi(x) \leq \delta x l / 2N^{1/2} + 2l^2$$

when

$$(2.10) \quad N \leq p_n \leq x < x + \delta x \leq p_{n+1} \leq 2N.$$

Hence if N is sufficiently large and if $h \geq 8l^2$,

$$(2.11) \quad E(x) \leq -\delta x / 2$$

on a subinterval of $[p_n, p_{n+1}]$ of length at least $d_n/2$. We now write

$$(2.12) \quad E(x) = E_1(x) + E_2(x) + E_3(x),$$

where for some α to be chosen, $E_1(x)$ is the sum over zeros $\rho = \beta + i\gamma$ with $\beta < \alpha$, $E_2(x)$ over zeros with $\beta \geq \alpha$ and $E_3(x)$ is the error term in (2.5). If

$$(2.13) \quad T = BNl^2/h$$

with some sufficiently large constant B , then

$$(2.14) \quad |E_3(x)| \leq \delta N / 6 \leq \delta x / 6$$

for $N \leq x \leq 2N$. To estimate $E_1(x)$ and $E_2(x)$ we write as usual $N(\sigma, T)$

for the number of zeros of $\zeta(s)$ with $\beta \geq \sigma$, $|y| \leq T$. By dividing the interval $\alpha \leq \sigma \leq 1$ into subintervals of width $1/l$, we find

$$(2.15) \quad |E_2(x)| \ll 2\delta l \max_{\alpha \leq \sigma \leq 1} N^\alpha N(\sigma, T).$$

The known bounds for $N(\sigma, T)$ can be thrown into the form

$$(2.16) \quad N(\sigma, T) \ll T^{A(\sigma)(1-\sigma)},$$

where $A(\sigma)$ increases from $\sigma = 1/2$ to $\sigma = 3/4$, then decreases for $\sigma > 3/4$. Also $N(\sigma, T)$ is zero for

$$(2.17) \quad \sigma \geq \sigma_0 = 1 - \frac{D}{l^{2/3} \log^{1/3} l},$$

where D is a constant. Hence if for some $\varepsilon > 0$

$$(2.18) \quad T^{A(\alpha)} \ll N^{1-\varepsilon},$$

then

$$(2.19) \quad (T^{A(\sigma)/N})^{1-\sigma} < 1/12l$$

for $\alpha \leq \sigma \leq \sigma_0$, and

$$(2.20) \quad |E_2(x)| \leq \delta N/6.$$

We now have

$$(2.21) \quad |E_1(x)| \geq \delta N/6$$

on a subinterval of $[p_n, p_{n+1}]$ of length at least $d_n/2$.

We now estimate

$$(2.22) \quad \int_N^{2N} |E_1(x)|^2 dx \\ = \int_{u=1}^{1+\delta} \int_{v=1}^{1+\delta} \sum_q \sum_{q'} \delta(q) \delta(q') \frac{(2N)^{q+q'+1} - N^{q+q'+1}}{q+q'+1} u^{q-1} v^{q'-1} dv du \\ \leq 9\delta^2 \sum_q \sum_{q'} \frac{N^{\theta+\theta'+1}}{|q+q'+1|} \ll \delta^2 l^2 \sum_q N^{2\theta+1} \ll \delta^2 l^3 \max_{1/2 \leq \sigma \leq \alpha} N(\sigma, T) N^{2\sigma+1},$$

where we have used the fact that

$$(2.23) \quad N(0, t+1) - N(0, t) \ll l$$

for $0 \leq t \leq T$. The maximum in (2.22) need not occur at the endpoints of the range $1/2 \leq \sigma \leq \alpha$.

3. Calculations. With $h = N^{1/2}$ we may choose any α for which $A(\alpha) < 2$. By Jutila [11] we have $A(\alpha) < 2+\varepsilon$ for $\alpha \geq 11/14$. If c is an exponent for

which (1.5) holds, and $c \geq 1/7$, so that $1/2+2c \geq 11/14$, then $A(\alpha) < 2$ for $\alpha > 1/2+2c$. This was shown in formula (10.19) of [8] using Haneke's estimate [4], which gives $c = 6/37+\varepsilon$ for any $\varepsilon > 0$. The later estimate of Kolesnik [12] permits $c = 173/1067+\varepsilon$, and we may sharpen (10.19) of [8] to

$$(3.1) \quad N(\sigma, T) \ll T^{8c(1-\sigma)/(2\sigma-1)+\varepsilon}$$

for an interval of values of σ about the point $1/2+2c$. We must now compute the maximum in (2.22). Since by (2.13)

$$(3.2) \quad T = BN^{1/2} l^2,$$

we want the maximum of

$$(3.3) \quad f(\sigma) = 2\sigma + A(\sigma)(1-\sigma)/2.$$

For $1/2 \leq \sigma \leq 3/4$ we use the classical bound of Ingham [9] with

$$(3.4) \quad A(\sigma) = 3/(2-\sigma) + \varepsilon.$$

Since $A(\sigma) < 4$ and $A(\sigma)$ is increasing, the maximum on this range is $9/5+\varepsilon$ at $\sigma = 3/4$. For $3/4 \leq \sigma \leq 11/14$ we use the estimate

$$(3.5) \quad A(\sigma) = 3/(3\sigma-1) + \varepsilon$$

from [7], which can, however, be improved slightly by the methods of the author and Jutila (see for example [8] and [11]). The maximum of $f(\sigma)$ on this range is again at the right-hand end point, $481/266+\varepsilon$ at $\sigma = 11/14$. As remarked above, we may take $A(\sigma) = 2+\varepsilon$ for $\sigma \geq 11/14$. The best estimates obtainable for $A(\sigma)$ form a continuous function of σ , whilst those we are quoting are discontinuous at $11/14$. The maximum of $f(\sigma)$ for $11/14 \leq \sigma \leq \alpha$ is now $1+\alpha+\varepsilon$ at $\sigma = \alpha$. Hence we have

$$(3.6) \quad \frac{\delta^2 N^2}{36} \sum_{d_n > N^{1/2}} \frac{d_n}{2} \leq \int_N^{2N} |E_1(x)|^2 dx \ll \delta^2 N^{2+\varepsilon} l^3 \max(N^\alpha, N^{215/266}),$$

the sum being over n for which (2.1) holds. Taking $\alpha = 1/2+2c+\varepsilon$, we deduce (1.4) with the exponent θ given by (1.6).

If we assume the Lindelöf hypothesis, after Turán and Halász [3] we may take $\alpha = 3/4+\varepsilon$, and $A(\sigma) = 2+\varepsilon$ for $1/2 \leq \sigma \leq \alpha$. The maximum of $f(\sigma)$ for $1/2 \leq \sigma \leq \alpha$ is $1+\alpha+\varepsilon$ at $\sigma = \alpha$, which gives (1.7).

If we sharpen the condition $d_n \geq h$ by taking $h = N^{1/2+\varepsilon_1}$ in place of $h = N^{1/2}$, we may take any α for which $A(\alpha) < 2+\varepsilon_3$, in particular, $\alpha = 11/14$, and, if the Lindelöf hypothesis be assumed, even $\alpha = 1/2+\varepsilon_4$. This gives the bounds (1.9) and (1.10). Finally with $h = N^{7/12}$ we want



$A(\alpha) < 12/5$, and we can choose $\alpha = 3/4 + \varepsilon$. In place of (3.3) we want the maximum of

$$(3.7) \quad f(\sigma) = 2\sigma + \frac{5}{12} A(\sigma)(1-\sigma)$$

on the range $1/2 \leq \sigma \leq \alpha$ with $A(\sigma)$ given by (3.4), and this is $1 + \alpha + \varepsilon$ at $\sigma = \alpha$. As before, this leads to $\theta = \alpha + \varepsilon$, and the exponent (1.12).

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(1036)

Note added in proof by the editor. In the revised version of [10] that has already appeared in Math. Ann. 241 (1979), pp. 1-9, Ivić proves (1.4) with $\theta = 215/266 + \varepsilon$ and announces the value $\theta = 3/4 + \varepsilon$ due to Heath-Brown.

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