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A reciprocity theorem and a three-term relation for generalized Dedekind–Rademacher sums

by

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To the memory of Professor Paul Turán

1. Introduction. For real x, put

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & (x \neq \text{integer}), \\ 0 & (x = \text{integer}). \end{cases}$$

The Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{\mu (\bmod k)} \left(\left(\frac{h\mu}{k}\right)\right)\left(\left(\frac{\mu}{k}\right)\right),$$

where the summation is over a complete residue system $(\bmod k)$. It is well known that $s(h, k)$ satisfies ([11], p. 4) the reciprocity relation

$$(1.1)\qquad 12hk\{s(h, k)+s(k, h)\} = h^2-3hk+k^2+1 \qquad ((h, k) = 1).$$

Rademacher, at the 1963 Number Theory Institute in Boulder, Colorado, proved the following generalization of (1.1). Define

$$s(h, k; x, y) = \sum_{\mu (\bmod k)} \left(\left(h\frac{\mu+y}{k}+x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right),$$

where x, y are arbitrary real numbers. Then

$$(1.2)\qquad s(h, k; x, y)+s(k, h; y, x) = -\frac{1}{4}\delta(x)\,\delta(y)+((x))((y))+ \\ +\frac{1}{2}\left\{\frac{h}{k}\,\bar{B}_2(y)+\frac{1}{hk}\,\bar{B}_2(hy+kx)+\frac{k}{h}\bar{B}_2(x)\right\},$$

where $(h, k) = 1$,

$$\delta(x) = \begin{cases} 1 & (x = \text{integer}), \\ 0 & (x \neq \text{integer}) \end{cases}$$

and

$$\bar{B}_2(x) = B_2(x-[x]), \quad B_2(x) = x^2 - x - \tfrac{1}{2}.$$

For $x = y = 0$, (1.2) reduces to (1.1). Rademacher's proof of (1.2) appeared in [10]. For a simplified proof see [7].

Let $B_n(x)$ be the Bernoulli polynomial of degree n defined by

$$\frac{ue^{xu}}{e^u-1} = \sum_{n=0}^{\infty} B_n(x)\frac{u^n}{n!}, \qquad B_n = B_n(0),$$

and let $\bar{B}_n(x)$ be the Bernoulli function defined by

$$\bar{B}_n(x) = \bar{B}_n(x-[x]).$$

Apostol ([1], [2]) introduced the generalized Dedekind sum

$$s_p(h, k) = \sum_{\mu \pmod k} \bar{B}_p\left(\frac{h\mu}{k}\right)\bar{B}_1\left(\frac{\mu}{k}\right)$$

and proved the reciprocity theorem

$$(1.3)\qquad (p+1)\{hk^p s_p(h, k) + kh^p s_p(h, k)\} = (hB+kB)^{p+1} + pB_{p+1},$$

where $(h, k) = 1$, p odd, $p > 1$. A proof of a different kind was given by the present writer [3].

Rademacher's definition of $s(h, k; x, y)$ suggests that we define

$$s_p(h, k; x, y) = \sum_{\mu \pmod k} \bar{B}_p\left(h\frac{\mu+y}{k}+x\right)\bar{B}_1\left(\frac{\mu+y}{k}\right),$$

which reduces to $s_p(h, k)$ when $x = y = 0$. Since $\bar{B}_n(x+1) = \bar{B}_n(x)$, there is no loss in generality in assuming that

$$(1.4)\qquad 0 \leqslant x < 1, \quad 0 \leqslant y < 1.$$

The writer ([4], [5]) has proved the following

THEOREM 1. *Let* $(h, k) = 1$ *and assume that* x, y *satisfy* (1.4). *Then*

$$(1.5)\qquad (p+1)\{hk^p s_p(h, k; x, y) + kh^p s_p(k, h; y, x)\} = (hB+kB+hy+kx)^{p+1} + p\bar{B}_{p+1}(hy+kx)$$

for all $p \geqslant 0$.

We may replace (1.5) by the following equivalent formulation in which (1.4) is not assumed:

$$(1.6)\qquad (p+1)\{hk^p s_p(h, k; x, y) + kh^p s_p(k, h; y, x)\} = \left(h\bar{B}(y)+k\bar{B}(x)\right)^{p+1} + p\bar{B}_{p+1}(hy+kx).$$

It is to be understood that

$$\left(h\bar{B}(y)+k\bar{B}(x)\right)^{p+1} = \sum_{r=0}^{p+1}\binom{p+1}{r} h^r k^{p-r+1} \bar{B}_r(y)\bar{B}_{p-r+1}(x).$$

Both of the earlier proofs of (1.5) require considerable computation. In the present paper we give a simplified proof that makes use of the following

LEMMA 1. *Let* $(h, k) = 1$, $hk > 1$, $0 \leqslant z < h+k$. *Put* $\zeta = z-[z]$, *the fractional part of* z. *Then we have the identity*

$$(1.7)\qquad \sum \lambda^{hr+ks+z} = \lambda^z \frac{1-\lambda^{hk}}{(1-\lambda^h)(1-\lambda^k)} - \frac{\lambda^{hk+\zeta}}{1-\lambda},$$

where the summation is over all r, s *such that* $0 \leqslant r < k$, $0 \leqslant s < h$, $hr + ks + z < hk$.

In the next place, let a, b, c be three positive integers that satisfy

$$(1.8)\qquad (b, c) = (c, a) = (a, b) = 1.$$

Rademacher [9] has proved the following three-term relation:

$$(1.9)\qquad s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right),$$

where a', b', c' are defined by

$$aa' \equiv 1 \pmod{bc}, \quad bb' \equiv 1 \pmod{ca}, \quad cc' \equiv 1 \pmod{ab}.$$

The present writer, in extending (1.9) to $s(h, k; x, y)$, defined the sum [6]

$$s(a, b, c; x, y, z) = \sum_{t \pmod c} \bar{B}_1\left(a\frac{t+z}{c}-x\right)\bar{B}_1\left(y-b\frac{t+z}{c}\right).$$

Despite the presence of the additional parameters, $s(a, b, c; x, y, z)$ is really no more general than $s(h, k; x, y)$. It was proved that

$$(1.10)\qquad s(a, b, c; x, y, z) + s(b, c, a; y, z, x) + s(c, a, b; z, x, y) = \delta - \frac{a}{2bc}\bar{B}_2(cy-bz) - \frac{b}{2ca}\bar{B}_2(az-cx) - \frac{c}{2ab}\bar{B}_2(bx-ay),$$

where $\delta = 1$ if integers r, s, t exist such that

$$\frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c};$$

$\delta = 0$ otherwise.

Mordell ([8]; [11], p. 39) has proved the following result analogous to (1.9):

$$s(bc,a)+s(ca,b)+s(ab,c) \tag{1.11}$$
$$=\frac{1}{6}abc+\frac{1}{4}(bc+ca+ab)+\frac{1}{4}(a+b+c)+$$
$$+\frac{1}{12}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)+\frac{1}{12abc}-2-N_3(a,b,c),$$

where $N_3(a,b,c)$ denotes the number of lattice points in the tetrahedron

$$0\leqslant r<a,\quad 0\leqslant s<b,\quad 0\leqslant t<c,\quad 0<\frac{r}{a}+\frac{s}{b}+\frac{t}{c}<1. \tag{1.12}$$

We shall prove the following more general theorem.

THEOREM 2. *Let a, b, c be three positive integers that are relatively prime in pairs and let p be an arbitrary positive integer. Let x, y, z be real numbers, $0\leqslant x<1$, $0\leqslant y<1$, $0\leqslant z<1$. Then we have*

$$(abc)\{a^{p-1}s_p(bc,a;\ cy+bz,x)+b^{p-1}s_p(ca,b;\ az+cx,y)+ \tag{1.13}$$
$$+c^{p-1}s_p(ab,c;\ bx+cy,z)\}$$
$$=\frac{3}{2}abcB_p(\omega)+\frac{p}{p+1}B_{p+1}(\omega)+$$
$$+\frac{1}{p+1}\{(abcB+B+\omega)^{p+1}-(abcB+B+abc+\omega)^{p+1}\}+$$
$$+\frac{1}{(p+1)(p+2)abc}\{(bcB+caB+abB+w)^{p+2}-$$
$$-(bcB+caB+abB+w-abc)^{p+2}\}-p(abc)^p\sum_{\sigma<1}(\sigma-1)^{p-1},$$

where

$$w=bcx+cay+abx,\qquad \omega=w-[w],\qquad \sigma=\frac{r+x}{a}+\frac{s+y}{b}+\frac{t+z}{c},$$

and the final summation is over all r, s, t such that $0\leqslant r<a$, $0\leqslant s<b$, $0\leqslant t<c$, $0\leqslant\sigma<1$.

The proof of Theorem 2 makes use of the following

LEMMA 2. *Let a, b, c be three positive integers that are relatively prime in pairs and let w be a real number, $0\leqslant w<bc+ca+ab$. Put*

$$S_1=\sum_{bcr+cas+abt+w<abc}x^{bcr+cas+abt+w},$$
$$S_2=\sum_{bcr+cas+abt+w<2abc}x^{bcr+cas+abt+w},$$

where it is understood that

$$0\leqslant r<a,\quad 0\leqslant s<b,\quad 0\leqslant t<c.$$

Then

$$x^{abc}S_1+S_2=\frac{x^w(1-x^{abc})^2}{(1-x^{bc})(1-x^{ca})(1-x^{ab})}-\frac{x^{2abc+\omega}}{1-x}, \tag{1.14}$$

where $\omega=w-[w]$, the fractional part of w.

Some special cases of Theorem 2 are discussed in the last section of the paper. See in particular Theorems 3 and 4.

While Theorem 2 generalizes (1.11), it does not of course generalize (1.10). Thus a generalization of (1.10) remains an open question.

2. Proof of (1.7). Put

$$S=\sum\lambda^{hr+ks+z}, \tag{2.1}$$

where the summation is over all r, s satisfying

$$0\leqslant r<k,\quad 0\leqslant s<h,\quad hr+ks+z<hk. \tag{2.2}$$

If we divide both sides of (1.7) by λ^{ζ}, it is clear that we may, without loss in generality assume that z is an integer, $0\leqslant z<h+k$. Also since (1.7) is symmetric in h, k we may assume that $k<h$. It follows that, if $0\leqslant z<h$, the inequality $hr+ks+z<hk$ can be satisfied for all r, $0\leqslant r\leqslant k-1$; however, if $h\leqslant z<h+k$, the value $r=k-1$ must be deleted.

Hence, for z an integer, $0\leqslant z<h$, we have

$$S=\sum_{r=0}^{k-1}\lambda^{hr+z}\sum_{ks<h(k-r)-z}\lambda^{ks}=\sum_{r=0}^{k-1}\lambda^{hr+z}\sum_{0\leqslant s\leqslant h-\left[\frac{hr+z}{k}\right]}\lambda^{ks}$$
$$=\sum_{r=0}^{k-1}\lambda^{hr+z}\frac{1-\lambda^{hk-k[(hr+z)/k]}}{1-\lambda^k}$$
$$=\frac{\lambda^z}{1-\lambda^k}\frac{1-\lambda^{hk}}{1-\lambda^h}-\frac{\lambda^{hk}}{1-\lambda^k}\sum_{r=0}^{k-1}\lambda^{hr+z-k[(hr+z)/k]}.$$

The exponent on the extreme right is evidently the remainder obtained in dividing $hr+z$ by k. Hence the set of numbers

$$\{hr+z-k[(hr+z)/k],\ 0\leqslant r\leqslant k-1\}$$

is identical, except for order, with the set $\{0,1,2,\ldots,k\}$. It follows that

$$\sum_{r=0}^{k-1} \lambda^{hr+z-k[(hr+z)/k]} = \frac{1-\lambda^k}{1-\lambda}$$

and therefore

(2.3) $$S = \lambda^z \frac{1-\lambda^{hk}}{(1-\lambda^h)(1-\lambda^k)} - \frac{\lambda^{hk}}{1-\lambda} \quad (0 \leqslant z < h).$$

Now let $h \leqslant z < h+k$. Then, excluding the value $r = k-1$, we have

$$S = \sum_{r=0}^{k-2} \lambda^{hr+z} \sum_{ks < h(k-r)-z} \lambda^{ks} = \sum_{r=0}^{k-2} \lambda^{hr+z} \frac{1-\lambda^{hk-k[(hr+z)/k]}}{1-\lambda^k}$$

$$= \frac{\lambda^z}{1-\lambda^k} \frac{1-\lambda^{h(k-1)}}{1-\lambda^h} - \frac{\lambda^{hk}}{1-\lambda^k} \sum_{r=0}^{k-2} \lambda^{hr+z-k[(hr+z)/k]}.$$

The set of numbers

$$\{hr+z-k[(hr+z)/k],\ 0 \leqslant r \leqslant k-2\}$$

excludes the number $z-h$ from the set $\{0, 1, 2, \ldots, k-1\}$.

It follows that

$$S = \lambda^z \frac{1-\lambda^{hk-h}}{(1-\lambda^h)(1-\lambda^k)} - \frac{\lambda^{hk}}{1-\lambda^k}\left\{\frac{1-\lambda^k}{1-\lambda} - \lambda^{z-h}\right\}$$

$$= \lambda^z \frac{1-\lambda^{hk}}{(1-\lambda^h)(1-\lambda^k)} - \frac{\lambda^{hk}}{1-\lambda},$$

so that (2.3) holds in this case also.

This completes the proof of (1.7).

3. Proof of Theorem 1. Let

(3.1) $$S_p = hk^p s_p(h, k; x, y) + kh^p s_p(k, h; y, x).$$

Then exactly as in [5], § 2, we have

(3.2) $$S_p = (hk)^p \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \left\{\bar{B}_1\left(\frac{\mu+y}{k}\right) + \bar{B}_1\left(\frac{\nu+x}{h}\right)\right\} \bar{B}_p\left(\frac{\mu}{k} + \frac{\nu}{h} + \frac{y}{k} + \frac{x}{h}\right).$$

We may assume, with no loss in generality, that

(3.3) $$0 \leqslant x < 1, \quad 0 \leqslant y < 1.$$

Put

(3.4) $$\sigma = \frac{\mu}{k} + \frac{\nu}{h} + \frac{y}{k} + \frac{x}{h},$$

so that

(3.5) $$0 \leqslant \sigma < 2 \quad (0 \leqslant \mu < k,\ 0 \leqslant \nu < h).$$

Since

$$\bar{B}_1(x) = x - \tfrac{1}{2} \quad (0 \leqslant x < 1),$$

it follows from (3.3) that

(3.6) $$\bar{B}_1(x) + \bar{B}_1(y) = \bar{B}_1(x+y) + \tfrac{1}{2} f(x+y),$$

where

(3.7) $$f(x) = \begin{cases} -1 & (0 \leqslant x < 1), \\ +1 & (1 \leqslant x < 2). \end{cases}$$

Thus (3.2) becomes

$$S_p = (hk)^p \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \{\bar{B}_1(\sigma) + \tfrac{1}{2} f(\sigma)\} \bar{B}_p(\sigma)$$

and therefore

(3.8) $$S_p = (hk)^p \Big\{ \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_1(\sigma) \bar{B}_p(\sigma) + \tfrac{1}{2} \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) - \tfrac{1}{2} \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) \Big\}.$$

It follows from the multiplication theorem

$$\bar{B}_n(kx) = k^{1-n} \sum_{\mu \pmod k} \bar{B}_n\left(x + \frac{\mu}{k}\right)$$

that

$$\sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) = (hk)^{1-p} \bar{B}_p(hy+kx)$$

Thus (3.8) becomes

(3.9) $$S_p = (hk)^p T_p - (hk)^p U_p + \tfrac{1}{2} hk \bar{B}_p(hy+kx),$$

where

(3.10) $$T_p = \sum_{\mu=0}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_1(\sigma) \bar{B}_p(\sigma)$$

and

(3.11) $$U_p = \sum_{\substack{\mu=0 \\ \sigma-1}}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma) = \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \bar{B}_p(\sigma).$$

It is proved in [5] that

(3.12) $$(hk)^p T_p = \frac{p}{p+1} B_{p+1}(\zeta) + \frac{1}{p+1} (Bhk + B + \zeta)^{p+1} + \frac{1}{2} hk B_p(\zeta),$$

where

$$(3.13)\qquad z = hy + kx, \qquad \zeta = z - [z],$$

so that ζ is the fractional part of z.

To evaluate U_p we consider

$$\sum_{p=0}^{\infty} U_p \frac{(hku)^p}{p!} = \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \sum_{p=0}^{\infty} B_p(\sigma) \frac{(hku)^p}{p!} = \sum_{\substack{\mu=0 \\ \sigma<1}}^{k-1} \sum_{\nu=0}^{h-1} \frac{hku}{e^{hku}-1} e^{hk\sigma u}.$$

Since by (3.4) and (3.13)

$$hk\sigma = h\mu + k\nu + z,$$

we get

$$(3.14)\qquad \sum_{p=0}^{\infty} U_p \frac{(hku)^p}{p!} = \frac{hku}{e^{hku}-1} \sum_{\substack{\mu=0 \\ h\mu+k\nu+z<hk}}^{k-1} \sum_{\nu=0}^{h-1} e^{(h\mu+k\nu+z)u}.$$

By (1.7) the double sum on the right is equal to

$$\frac{e^{(hk+\zeta)u}}{e^u-1} - e^{zu} \frac{e^{hku}-1}{(e^{hu}-1)(e^{ku}-1)}.$$

Thus (3.14) becomes

$$\sum_{p=0}^{\infty} U_p \frac{(hku)^p}{p!} = \frac{hku}{e^{hku}-1} \frac{e^{(hk+\zeta)u}}{e^u-1} - \frac{hkue^{zu}}{(e^{hu}-1)(e^{ku}-1)}.$$

Now multiply both sides by u and we have

$$\sum_{p=1}^{\infty} p(hk)^{p-1} U_{p-1} \frac{u^p}{p!} = \frac{hkue^{hku}}{e^{hku}-1} \frac{ue^{\zeta u}}{e^u-1} - \frac{hu}{e^{hu}-1} \frac{ku}{e^{ku}-1} e^{zu}.$$

Hence, equating coefficients, we get

$$(3.15)\qquad (p+1)(hk)^p U_p = (hkB + hk + B + \zeta)^{p+1} - (hB + kB + z)^{p+1}.$$

We now substitute from (3.12) and (3.15) in (3.9) and get (1.5). This completes the proof of Theorem 1.

4. Proof of (1.14). Let a, b, c be three positive integers that are relatively prime in pairs:

$$(4.1)\qquad (b, c) = (c, a) = (a, b) = 1.$$

Without loss of generality we may assume that w is also an integer, $0 \leqslant w < bc + ca + ab$.

Put

$$(4.2)\qquad S_1 = \sum_{bcr+cas+abt+w<abc} x^{bcr+cas+abt+w}$$

and

$$(4.3)\qquad S_2 = \sum_{bcr+cas+abt+w<2abc} x^{bcr+cas+abt+w},$$

where it is understood in such sums that

$$(4.4)\qquad 0 \leqslant r < a, \qquad 0 \leqslant s < b, \qquad 0 \leqslant t < c.$$

By (4.2) we have

$$S_1 = \sum_{br+as+w/c<ab} x^{c(br+as+w/k)} \sum_{t<c-\frac{c}{ab}(br+as+w/c)} x^{abt}$$

$$= \sum_{br+as+w/c<ab} x^{c(br+as+w/c)} \frac{1-x^{ab(c-[c(br+as+w/c)/ab])}}{1-x^{ab}}$$

$$= \frac{1}{1-x^{ab}} \sum_{br+as+w/c<ab} x^{c(br+as+w/k)} - \frac{x^{abc}}{1-x^{ab}} \sum_{br+as+w/c<ab} x^{R(c(br+as+w/c)/ab)},$$

where $R(m/ab)$ denotes the remainder obtained in dividing m by ab.

Put

$$(4.5)\qquad \begin{aligned} U &= \{u|\ u = c(br+as)+w,\ c(br+as)+w < abc\}, \\ V &= \{v|\ v = c(br+as)+w,\ c(br+as)+w > abc\}. \end{aligned}$$

Thus

$$(4.6)\qquad S_1 = \frac{x^{abc}}{1-x^{ab}} \sum_{u\in U} x^u - \frac{x^{abc}}{1-x^{ab}} \sum_{u\in U} x^{R(u/ab)}.$$

In the next place we take $S_2 = S_2' + S_2''$, where

$$S_2' = \sum_{\substack{bcr+cas+abt+w<2abc \\ br+as+w/c<ab}} x^{c(br+as+w/c)+abt},$$

$$S_2'' = \sum_{\substack{bcr+cas+abt+w<2abc \\ br+as+w/c>ab}} x^{c(br+as+w/c)+abt}.$$

Clearly

$$(4.7)\qquad S_2' = \sum_{br+as+w/c<ab} x^{c(br+as+w/c)} \sum_{t=0}^{c-1} x^{abt} = \frac{1-x^{abc}}{1-x^{ab}} \sum_{u\in U} x^u.$$

As for S_2'' we have

$$(4.8)\qquad S_2'' = \sum_{br+as+w/c>ab} x^{c(br+as+w/c)} \sum_{t<2c-\frac{c}{ab}(br+as+w/c)} x^{abt}$$

$$= \sum_{v\in V} \sum_{t<2c-v/(ab)} x^{abt} = \sum_{v\in V} \frac{1-x^{ab(2c-[v/ab])}}{1-x^{ab}}$$

$$= \frac{1}{1-x^{ab}} \sum_{v\in V} x^v - \frac{x^{2abc}}{1-x^{ab}} \sum_{v\in V} x^{R(v/ab)}.$$

It follows from (4.6) and (4.8) that

$$x^{abc}S_1 + S_2'' = \frac{x^{abc}}{1-x^{ab}} \sum_{u\in U} x^u + \frac{1}{1-x^{ab}} \sum_{v\in V} x^v - $$

$$- \frac{x^{2abc}}{1-x^{ab}} \Big\{ \sum_{u\in U} x^{R(u/ab)} + \sum_{v\in V} x^{R(v/ab)} \Big\}.$$

Since

$$\sum_{u\in U} x^{R(u/ab)} + \sum_{v\in V} x^{R(v/ab)} = \sum_{m=0}^{ab-1} x^m = \frac{1-x^{ab}}{1-x},$$

we have

$$(4.9)\qquad x^{abc}S_1 + S_2'' = \frac{x^{abc}}{1-x^{ab}} \sum_{u\in U} x^u + \frac{1}{1-x^{ab}} \sum_{v\in V} x^v - \frac{x^{2abc}}{1-x}.$$

Hence, by (4.7) and (4.9),

$$x^{abc}S_1 + S_2' + S_2'' = \Big\{ \frac{1-x^{abc}}{1-x^{ab}} + \frac{x^{abc}}{1-x^{ab}} \Big\} \sum_{u\in U} x^u + \frac{1}{1-x^{ab}} \sum_{v\in V} x^v - \frac{x^{2abc}}{1-x}$$

$$= \frac{1}{1-x^{ab}} \Big\{ \sum_{u\in U} x^u + \sum_{v\in V} x^v \Big\} - \frac{x^{2abc}}{1-x}$$

$$= \frac{1}{1-x^{ab}} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} x^{c(br+as)+w} - \frac{x^{2abc}}{1-x}$$

$$= \frac{x^w}{1-x^{ab}} \, \frac{1-x^{abc}}{1-x^{bc}} \, \frac{1-x^{abc}}{1-x^{ac}} - \frac{x^{2abc}}{1-x}.$$

Thus

$$(4.10)\qquad x^{abc}S_1 + S_2 = \frac{x^w(1-x^{abc})^2}{(1-x^{bc})(1-x^{ca})(1-x^{ab})} - \frac{x^{2abc}}{1-x}.$$

5. Proof of Theorem 2. Put

$$(5.1)\qquad S_p = a^{p-1}s_p(bc, a;\, cy+bz, x) + b^{p-1}s_p(ca, b;\, az+cx, y) + $$
$$+ c^{p-1}s_p(ab, c;\, bx+ay, z).$$

Then by (1.11) and the multiplication theorem for $\bar{B}_p(x)$ we have

$$(5.2)\qquad S_p = (abc)^{p-1} \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{t=0}^{c-1} \bar{B}_p\Big(\frac{r+x}{a} + \frac{s+y}{b} + \frac{t+z}{c} \Big) \times$$
$$\times \Big\{ \bar{B}_1\Big(\frac{r+x}{a}\Big) + \bar{B}_1\Big(\frac{s+y}{b}\Big) + \bar{B}_1\Big(\frac{t+z}{c}\Big) \Big\}.$$

If, for brevity, we put

$$(5.3)\qquad \xi = \frac{r+x}{a}, \quad \eta = \frac{s+y}{b}, \quad \zeta = \frac{t+z}{c},$$

(5.2) may be written compactly in the form

$$(5.4)\qquad S_p = (abc)^{p-1} \sum_{\xi,\eta,\zeta} \bar{B}_p(\xi+\eta+\zeta)\{\bar{B}_1(\xi) + \bar{B}_1(\eta) + \bar{B}_1(\zeta)\}.$$

We assume in what follows that x, y, z satisfy the inequalities

$$(5.5)\qquad 0 \leqslant x < 1, \quad 0 \leqslant y < 1, \quad 0 \leqslant z < 1,$$

so that

$$(5.6)\qquad 0 \leqslant \xi < 1, \quad 0 \leqslant \eta < 1, \quad 0 \leqslant \zeta < 1.$$

It follows from (5.6) and the definition of $\bar{B}_1(x)$ that

$$\bar{B}_1(\xi) + \bar{B}_1(\eta) + \bar{B}_1(\zeta) = \xi + \eta + \zeta - \tfrac{3}{2}.$$

Hence we have

$$(5.7)\qquad \bar{B}_1(\xi) + \bar{B}_1(\eta) + \bar{B}_1(\zeta) = \bar{B}_1(\sigma) + \varepsilon,$$

where $\sigma = \xi + \eta + \zeta$ and

$$(5.8)\qquad \varepsilon = \varepsilon(\sigma) = \begin{cases} -1 & (0 \leqslant \sigma < 1), \\ 0 & (1 \leqslant \sigma < 2), \\ +1 & (2 \leqslant \sigma < 3). \end{cases}$$

Thus (5.4) becomes

$$(5.9)\qquad (abc)^{1-p}S_p$$

$$= \sum_{\xi,\eta,\zeta} \bar{B}_p(\sigma)\{\bar{B}_1(\sigma) + \varepsilon(\sigma)\}$$

$$= \sum_{\sigma<1} \bar{B}_p(\sigma)\big(\bar{B}_1(\sigma) - 1\big) + \sum_{1\leqslant\sigma<2} \bar{B}_p(\sigma)\,\bar{B}_1(\sigma) + \sum_{\sigma>2} \bar{B}_p(\sigma)\big(\bar{B}_1(\sigma) + 1\big)$$

$$= \sum_{\xi,\eta,\zeta} \bar{B}_p(\sigma)\,\bar{B}_1(\sigma) + \sum_{\xi,\eta,\zeta} \bar{B}_p(\sigma) - \Big\{ 2 \sum_{\sigma<1} \bar{B}_p(\sigma) + \sum_{1\leqslant\sigma<2} \bar{B}_p(\sigma) \Big\}$$

$$= R_p + T_p - U_p, \text{ say.}$$

Clearly

$$R_p = \sum_{\xi,\eta,\zeta} \bar{B}_p(\xi+\eta+\zeta)\,\bar{B}_1(\xi+\eta+\zeta)$$

$$= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1} \bar{B}_p\left(\frac{r+x}{a}+\frac{s+y}{b}+\frac{t+z}{c}\right)\bar{B}_1\left(\frac{r+x}{a}+\frac{s+y}{b}+\frac{t+z}{c}\right)$$

$$= \sum_{m \,(\mathrm{mod}\, abc)} \bar{B}_p\left(\frac{m}{abc}+\frac{w}{abc}\right)\bar{B}_1\left(\frac{m}{abc}+\frac{w}{abc}\right)$$

$$= \sum_{m=0}^{abc-1} \bar{B}_p\left(\frac{m}{abc}+\frac{\omega}{abc}\right)\bar{B}_1\left(\frac{m}{abc}+\frac{\omega}{abc}\right),$$

where

$$(5.10)\qquad w = bcx+cay+abz, \qquad \omega = w-[w].$$

Then, exactly as in the proof of (3.12), we get, first,

$$(5.11)\qquad (abc)^p R_p = \frac{p}{p+1}B_{p+1}(\omega)+\frac{1}{p+1}(Babc+B+\omega)^{p+1}+\tfrac{1}{2}abcB_p(\omega).$$

As for T_p, by the multiplication theorem for $\bar{B}_p(x)$, we have

$$T_p = \sum_{\xi,\eta,\zeta}\bar{B}_p(\xi+\eta+\zeta) = (abc)^{1-p}\bar{B}_p(bcx+cay+abz).$$

Thus, by (5.10),

$$(5.12)\qquad (abc)^p T_p = abc\,\bar{B}_p(w) = abcB_p(\omega).$$

To evaluate U_p we take

$$(5.13)\qquad \sum_{p=0}^{\infty} U_p\frac{(abcu)^p}{p!} = \sum_{p=0}^{\infty}\frac{(abcu)^p}{p!}\Big\{2\sum_{\sigma<1}\bar{B}_p(\sigma)+\sum_{1\leqslant\sigma<2}\bar{B}_p(\sigma)\Big\}$$

$$= \sum_{p=0}^{\infty}\frac{(abcu)^p}{p!}\Big\{2\sum_{\sigma<1}B_p(\sigma)+\sum_{1\leqslant\sigma<2}B_p(\sigma-1)\Big\}$$

$$= 2\sum_{\sigma<1}\frac{abcu}{e^{abcu}-1}e^{abc\sigma u}+\sum_{1\leqslant\sigma<2}\frac{abcu}{e^{abcu}-1}e^{abc(\sigma-1)u}$$

$$= \frac{abcue^{-abcu}}{e^{abcu}-1}\Big\{2e^{abcu}\sum_{\sigma<1}e^{abc\sigma u}+\sum_{1\leqslant\sigma<2}e^{abc\sigma u}\Big\}$$

$$= \frac{abcue^{-abcu}}{e^{abcu}-1}\Big\{(e^{abcu}-1)\sum_{\sigma<1}e^{abc\sigma u}+$$

$$+e^{abcu}\sum_{\sigma<1}e^{abc\sigma u}+\sum_{\sigma<2}e^{abc\sigma u}\Big\}$$

$$= abcu\sum_{\sigma<1}e^{abc(\sigma-1)u}+$$

$$+\frac{abcue^{-abcu}}{e^{abcu}-1}\Big\{e^{abcu}\sum_{\sigma<1}e^{abc\sigma u}+\sum_{\sigma<2}e^{abc\sigma u}\Big\}.$$

By (4.10), with

$$abc\sigma = abc(\xi+\eta+\zeta) = bcr+cas+abt+w, \qquad w = bcx+cay+abt,$$

we have

$$e^{abcu}\sum_{\sigma<1}e^{abc\sigma u}+\sum_{\sigma<2}e^{abc\sigma u} = \frac{e^{2abcu+\omega u}}{e^u-1}-\frac{e^{wu}(e^{abcu}-1)^2}{(e^{bcu}-1)(e^{cau}-1)(e^{abcu}-1)}.$$

Then it follows from (5.13) that

$$\sum_{p=0}^{\infty}U_p\frac{(abcu)^p}{p!}$$

$$= abcu\sum_{\sigma<1}e^{abc(\sigma-1)u}+\frac{abcue^{abcu+\omega u}}{(e^u-1)(e^{abcu}-1)}-\frac{abcue^{wu}(1-e^{-abcu})}{(e^{bcu}-1)(e^{cau}-1)(e^{abu}-1)}.$$

Equating coefficients of $u^p/p!$, we get

$$(5.14)\qquad (abc)^pU_p = p(abc)^p\sum_{\sigma<1}(\sigma-1)^{p-1}+\frac{1}{p+1}(abcB+B+abc+\omega)^{p+1}-$$

$$-\frac{1}{(p+1)(p+2)abc}(bcB+caB+abB+w)^{p+2}+$$

$$+\frac{1}{(p+1)(p+2)abc}(bcB+caB+abB+w-abc)^{p+2}.$$

Thus, by (5.9), (5.11), (5.12) and (5.14), we have

$$(5.15)\qquad abcS_p = \frac{p}{p+1}B_{p+1}(\omega)+\frac{3}{2}abcB_p(\omega)-p(abc)^p\sum_{\sigma<1}(\sigma-1)+$$

$$+\frac{1}{p+1}(abcB+B+\omega)^{p+1}-\frac{1}{p+1}(abcB+B+abc+\omega)^{p+1}+$$

$$+\frac{1}{(p+1)(p+2)abc}(bcB+caB+abB+w)^{p+2}-$$

$$-\frac{1}{(p+1)(p+2)abc}(bcB+caB+abB+w-abc)^{p+2}.$$

This completes the proof of Theorem 2.

6. Some special cases. To begin with we take $p = 0$ in Theorem 2. We assume that

$$0 \leqslant x < 1, \quad 0 \leqslant y < 1, \quad 0 \leqslant z < 1. \tag{6.1}$$

Then by (5.1) we find that

$$abcS_0 = bcx + cay + abz - \tfrac{1}{2}(bc + ca + ab). \tag{6.2}$$

As for (5.15), we have

$$abcS_0 = (-\tfrac{1}{2}abc - \tfrac{1}{2} + \omega) + \tfrac{3}{2}abc + (\tfrac{1}{2}abc + \tfrac{1}{2} - abc - \omega) -$$
$$- \frac{1}{2abc}(-bc - ca - ab + 2w)(-abc) - \tfrac{1}{2}abc,$$

which reduces to the right hand side of (6.2).

The special case $p = 1$ takes more computation. By (5.15) we have

$$abcS_1 = \tfrac{1}{2}B_2(\omega) + \tfrac{1}{2}(abcB + B + \omega)^2 + \tfrac{3}{2}abcB_1(\omega) - abc \sum_{\sigma<1} 1 - \tag{6.3}$$
$$- \tfrac{1}{2}(abcB + B + abc + \omega)^2 + \frac{1}{6abc}(bcB + caB + abB + w)^3 -$$
$$- \frac{1}{6abc}(bcB + caB + abB + w - abc)^3.$$

Clearly, the sum

$$\sum_{\sigma<1} 1 = N_3(a, b, c) + 1 \tag{6.4}$$

is equal to the number of lattice points in the tetrahedron

$$0 \leqslant r < a, \quad 0 \leqslant s < b, \quad 0 \leqslant t < c, \quad 0 \leqslant \frac{r}{a} + \frac{s}{b} + \frac{t}{c} < 1. \tag{6.5}$$

We now specialize further by taking $x = y = z = 0$, $w = \omega = 0$. Thus (6.3) reduces to

$$abcS_1 = \tfrac{1}{12} + \tfrac{1}{2}(abcB + B)^2 - \tfrac{1}{4}abc - abc\,(N_3(a, b, c) + 1) -$$
$$- \tfrac{1}{2}(abcB + B + abc)^2 + \frac{1}{6abc}(bcB + caB + abB)^3 -$$
$$- \frac{1}{6abc}(bcB + caB + abB - abc)^3.$$

Simplifying, we get

$$s_1(bc, a) + s_1(ca, b) + s_1(ab, c) \tag{6.6}$$
$$= -\frac{5}{4} + \frac{1}{12abc} + \frac{1}{6}abc + \frac{1}{4}(a + b + c) + \frac{1}{4}(bc + ca + ab) +$$
$$+ \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) + N_3(a, b, c).$$

Since $s_1(h, k) = s(h, k) + \frac{1}{4}$, it is evident that (6.6) is identical with Mordell's theorem (1.11), thus furnishing a partial check on Theorem 2. Moreover (6.3) yields the following direct generalization of Mordell's theorem.

THEOREM 3. *Let a, b, c be positive integers that are relatively prime in pairs. Let x, y, z be real numbers, $0 \leqslant x < 1$, $0 \leqslant y < 1$, $0 \leqslant z < 1$. Let*

$$w = bcx + cay + abz, \qquad \omega = w - [w].$$

Then we have

$$s_1(bc, a;\ cy + bz, x) + s_1(ca, b;\ az + cx, y) + s_1(ab, c;\ bx + ay, z) \tag{6.7}$$
$$= \frac{3}{2}B_1(\omega) + \frac{1}{2abc}B_2(\omega) - N_3(a, b, c) - 1 +$$
$$+ \frac{1}{2abc}\{(abcB + B + \omega)^2 - (abcB + B + ab + c + \omega)^2\} +$$
$$+ \frac{1}{6(abc)^2}\{(bcB + caB + abB + w)^3 - (bcB + caB + abB + w - abc)^3\}.$$

Finally we state the special case of Theorem 2 with $x = y = z = 0$.

THEOREM 4. *Let a, b, c satisfy the usual requirements and $p \geqslant 0$. Then we have*

$$a^{p-1}s_p(bc, a) + b^{p-1}s_p(ca, b) + c^{p-1}s_p(ab, c) \tag{6.8}$$
$$= \frac{3}{2}B_p \frac{p}{(p+1)(abc)} B_{p+1} +$$
$$+ \frac{1}{(p+1)(abc)}\{(abcB + B)^{p+1} - (acbB + B + abc)^{p+1}\} +$$
$$+ \frac{1}{(p+1)(p+2)(abc)^2}\{(bcB + caB + abB)^{p+2} - (bcB + caB + abB - abc)^{p+2}\}$$
$$- p(abc)^{p-1} \sum_{\sigma<1} \left(\frac{r}{a} + \frac{s}{b} + \frac{t}{c} - 1\right)^{p-1},$$

where the summation is over all r, s, t satisfying

$$0 \leqslant r < a, \quad 0 \leqslant s < b, \quad 0 \leqslant t < c, \quad 0 \leqslant \frac{r}{a} + \frac{s}{b} + \frac{t}{c} < 1.$$

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Quadratic diophantine equations with parameters

by

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To the memory of Paul Turán

1. In an earlier paper [3] written in collaboration with the late Harold Davenport we proved:

Theorem A. *Let $a(t)$, $b(t)$ be polynomials with integral coefficients. Suppose that every arithmetical progression contains an integer τ such that the equation $a(\tau)x^2+b(\tau)y^2 = z^2$ has a solution in integers x, y, z, not all 0. Then there exist polynomials $x(t)$, $y(t)$, $z(t)$ in $Z[t]$, not all identically 0, such that $a(t)x(t)^2+b(t)y(t)^2 \equiv z(t)^2$ identically in t.*

From this result we derived:

Theorem B. *Let $F(x, y, t)$ be a polynomial with integral coefficients which is of degree at most 2 in x and y. Suppose that every arithmetical progression contains an integer τ such that the equation $F(x, y, \tau) = 0$ is soluble in rational numbers for x and y. Then there exist rational functions $x(t)$, $y(t)$ in $Q(t)$ such that $F(x(t), y(t), t) \equiv 0$ identically in t.*

Earlier, one of us asked [6] whether a result similar to Theorem B holds if $F(x, y, t)$ is replaced by *any* polynomial $F(x, y, t_1, \ldots, t_r)$ and the stronger assumption is made that for all integral r-tuples $\tau_1, \ldots, \tau_r$, the equation $F(x, y, \tau_1, \ldots, \tau_r) = 0$ is soluble in the rational numbers for x and y. The stronger assumption is needed since the hypothesis analogous to the one of Theorem B involving arithmetical progressions is not sufficient already for $F(x, y, t) = x^2-y^3-t$. We shall show here that if F is of degree at most 2 in x and y a hypothesis analogous to the one of Theorem B suffices for any number of parameters t_i. We shall also indicate an equation of an elliptic curve over $Q(t)$ for which the stronger assumption involving all integers t does not seem to suffice.

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