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# On the distribution of integers with a given number of prime factors

by

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**1. Notation.** The letters $i, j, k, l, m, n, q$ with and without subscripts denote natural numbers. The letter $p$ with and without subscript denotes a prime. The letter $c$ stands for a fixed positive constant, though not necessarily the same constant when used in different contexts.

Let $n = p_1^{j_1} \ldots p_l^{j_l}$ be the standard factorization of $n$. Then

$$\sigma_k(x) = \left|\{n \mid j_1 + \ldots + j_l = k,\ n \leqslant x\}\right|,$$

$$\pi_k(x) = \left|\{n \mid j_1 = \ldots = j_l,\ l = k,\ n \leqslant x\}\right|,$$

$$\varrho_k(x) = \left|\{n \mid l = k,\ n \leqslant x\}\right|,$$

$$s = \sigma + it,\quad \sigma, t \text{ real},$$

$$P(s) = \sum_p p^{-s},$$

$$g(s) = P(s) - \log\frac{1}{s-1},$$

$$g(s; m, j_1, \ldots, j_q) = g(s)^m P(j_1 s) \ldots P(j_q s),$$

$$a_n(m; j_1, \ldots, j_q) = \frac{d^n}{ds^n}\left(\frac{1}{s}\, g(s; m, j_1, \ldots, j_q)\right)\Big|_{s=1}$$

$$l_2(x) = \log\log(x)$$

$$r_t = \left(\log(|t|+9)\right)^{-2/3}\left(l_2(|t|+9)\right)^{-1/3},$$

$$C_{n,m} = \int_0^\infty t^n e^{-t}(\log t)^m\,dt = \Gamma^{(m)}(n+1),$$

$$A_{n,m,j} = (-1)^{m-j}\binom{m}{j} C_{n,m-j},$$

$$x \text{ is a real number} \geqslant 10,$$

$$a = 1 + \frac{1}{\log x},$$

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$$k_0 = c_0(\log x)^{3/5} l_2(x)^{-6/5},$$
$$N = [c\log x]-1,$$
$$T = \exp\left(c\left(\log^3 x/l_2(x)\right)^{1/5}\right),$$

$R$ stands for an error term with a bound of the form

$$x\exp\left(-c\left(\log^3 x/l_2(x)\right)^{1/5}\right).$$

**2. Introduction.** L. G. Sathe [2] obtained asymptotic formulas for $\sigma_k(x)$, $\pi_k(x)$ and $\varrho_k(x)$ which are valid for $k \ll l_2(x)$. His expressions are of the form

$$\frac{x}{(k-1)!\log x}\sum_{j=0}^{k} a_j l_2(x)^j + O\left(\frac{x l_2(x)^k}{k!(\log x)^2}\right), \tag{2.1}$$

where $a_j$ are constants independent of $x$. A. Selberg [3] gave a much simpler proof of the same results.

In this paper we prove that the above functions $\sigma_k$, $\pi_k$, $\varrho_k$ can be expressed in the form

$$\frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1} b_{nm}\frac{l_2(x)^m}{(\log x)^n} + R \tag{2.2}$$

for all $k \leqslant k_0$, where the constants $b_{nm}$ depend on the function but are independent of $x$.

In case $k=1$ the formulas (2.2) are the known ones for $\pi(x)$. As in the case $k=1$ the remainder term can be improved to $O(x^{1/2+\varepsilon})$ under the assumption of the Riemann Hypothesis. If $k$ is small compared to $l_2(x)$ then the leading term in (2.2) is $x l_2(x)^{k-1}/((k-1)!\log x)$ as in (2.1). If $k$ is large compared to $l_2(x)$ this is no longer the case, but the remainder term remains small compared to the principal term.

**3. Lemmas.** We need the following Lemma 3.12 of [4], p. 53.

**3.1.** Lemma. *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converge absolutely for $\sigma > 1$ and let $|a_n| \leqslant \Phi(n)$ where $\Phi(n)$ is a positive nondecreasing function. Assume that there exists an $a > 0$ such that $f(\sigma) \ll (\sigma-1)^{-\alpha}$ for $\sigma > 1$. Then for every $a > 1$ we have*

$$\sum_{n\leqslant x} a_n = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} f(s)\frac{x^s}{s}\,ds + O\left(\frac{x^a}{T(a-1)^\alpha} + \Phi(x) + \frac{\Phi(2x)\log x}{T}\right). \tag{3.2}$$

For our applications we may restrict attention to the case where $0 \leqslant a_n \leqslant 1$ and where $a$, $T$ are the values of Section 1. In these cases we have



$$\sum_{n\leqslant x} a_n = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} f(s)\frac{x^s}{s}\,ds + R. \tag{3.3}$$

**3.4.** Lemma. *Let $\Omega$ be a region in the strip $1/2+\varepsilon < \sigma < 1$ which contains no zeros of $\zeta(s)$. Then $g(s) = P(s) - \log(1/s-1)$ is analytic in $\Omega$. If $\Omega$ lies in a region $\sigma > 1-cr_t$ then $g(s) \ll \log(|t|+9)$ in $\Omega$.*

Proof. We have

$$\log\zeta(s) = \log\prod_p \frac{1}{1-p^{-s}} = \sum_{m=1}^{\infty}\frac{1}{m}\sum_p\frac{1}{p^{ms}} = \sum_{m=1}^{\infty}\frac{1}{m}P(ms).$$

Thus

$$P(s) = \log\zeta(s) - \sum_{m=2}^{\infty}\frac{1}{m}P(ms)$$

and

$$g(s) = \log\left((s-1)\zeta(s)\right) - \sum_{m=2}^{\infty}\frac{1}{m}P(ms) \tag{3.5}$$

where the first function on the right is analytic in any region in which $\zeta(s) \neq 0$ and the second function is analytic for $\sigma > 1/2$. This proves the first part of the lemma. To prove the second part we note first that

$$\sum_{m=2}^{\infty}\frac{1}{m}P(ms) \leqslant \sum_{m=2}^{\infty}\frac{1}{m}\sum_p\frac{1}{p^{m(1/2+\varepsilon)}} \ll 1, \quad s \in \Omega. \tag{3.6}$$

In order to estimate $\log\left((s-1)\zeta(s)\right)$ in $\Omega$ we need the facts (see (11.7), p. 87 of [1]) that $\zeta(s) \neq 0$ if $\sigma \geqslant 1-cr_t$ where $c$ is some absolute constant and that

$$\zeta(s) \ll \left(\log(|t|+9)\right)^{2/3} \quad \text{for} \quad \sigma \geqslant 1-2cr_t. \tag{3.7}$$

Also, if $\sigma_0 = 1+(c/3)r_t$, $s_0 = \sigma_0 + it$ then

$$|\zeta(s_0)| = \left|\prod_p \frac{1}{1-p^{-s_0}}\right| \geqslant \prod_p \frac{1}{1+p^{-\sigma_0}} = \frac{\zeta(2\sigma_0)}{\zeta(\sigma_0)} \gg \sigma_0 - 1 \gg r_t. \tag{3.8}$$

Finally we need the estimate

$$\left|\frac{\zeta'}{\zeta}(s_0)\right| \leqslant \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}} = \int_1^{\infty}\frac{d\psi(x)}{x^{\sigma_0}} = \sigma_0\int_1^{\infty}\frac{\psi(x)}{x^{\sigma_0+1}}\,dx \ll \frac{1}{\sigma_0-1} \ll r_t^{-1}. \tag{3.9}$$

We now wish to apply Lemma $\gamma$ from [4], p. 50.

LEMMA $\gamma$. *Let $f(s)$ be analytic in $|s-s_0|<r$ and satisfy*

$$|f(s)/f(s_0)|<e^M, \quad M>1 \quad \text{in } |s-s_0|<r.$$

*If $|f'(s_0)/f(s_0)|<M/r$ and $f(s)\neq 0$ in the domain $\sigma>\sigma_0-2r'$, $|s-s_0|<r$ where $0<r'<r/4$ then*

$$\left|\frac{f'(s)}{f(s)}\right| \ll \frac{M}{r} \quad \text{for} \quad |s-s_0|\le r'.$$

We set $f(s)=\zeta(s)$, $r=2cr_t$, $r'=r/5$. Then by (3.7), (3.8) and (3.9) we can choose $M\ll l_2(|t|+9)$ and obtain

$$\left|\frac{\zeta'}{\zeta}(s)\right| \ll r_t^{-1} l_2(|t|+9) \quad \text{for} \quad \sigma>1-\frac{c}{15}r_t. \tag{3.10}$$

Integrating (3.10) we get

$$|\log\zeta(s)-\log\zeta(s_0)| \ll |s-s_0|r_t^{-1}l_2(|t|+9) \ll l_2(|t|+9)$$

and hence, using (3.8), we have

$$\begin{aligned}\big|\log\big((s-1)\zeta(s)\big)\big| &\ll |\log(s-1)|+\big|\log|\zeta(s_0)|\big|+l_2(|t|+9)\\ &\ll \log(|t|+9).\end{aligned} \tag{3.11}$$

The last part of the lemma now follows if we substitute (3.6) and (3.11) in (3.5). ∎

**3.12.** LEMMA. *Let $0<\alpha<1$. Then*

$$\int_\alpha^1 \left(\log\frac{1}{1-s}\right)^m (1-s)^n x^s ds = \frac{x}{(\log x)^{n+1}} \sum_{j=0}^m A_{nmj} l_2(x)^j + S_{nm} \tag{3.13}$$

*where*

$$A_{nmj} = (-1)^{m-j}\binom{m}{j} C_{n,m-j} \tag{3.14}$$

*and*

$$|S_{nm}| < 2^{m+1}\frac{x^\alpha}{\log x} l_2(x)^m. \tag{3.15}$$

Proof. Make the substitution $1-s=\tau/\log x$ to get

$$\begin{aligned}\int_\alpha^1 \left(\log\frac{1}{1-s}\right)^m (1-s)^n x^s ds &= \frac{x}{(\log x)^{n+1}} \int_0^{(1-\alpha)\log x} \left(\log\frac{1}{\tau}+l_2(x)\right)^m \tau^n e^{-\tau} d\tau\\ &= \frac{x}{(\log x)^{n+1}} \sum_{j=0}^m l_2(x)^j (-1)^{m-j}\binom{m}{j} \int_0^{(1-\alpha)\log x} (\log\tau)^{m-j}\tau^n e^{-\tau} d\tau.\end{aligned} \tag{3.16}$$



Now write

$$\begin{aligned}&\int_0^{(1-\alpha)\log x} (\log\tau)^{m-j}\tau^n e^{-\tau} d\tau\\ &= \int_0^\infty (\log\tau)^{m-j}\tau^n e^{-\tau}d\tau - \int_{(1-\alpha)\log x}^\infty (\log\tau)^{m-j}\tau^n e^{-\tau}d\tau = C_{n,m-j}-I_{n,m-j}.\end{aligned} \tag{3.17}$$

If we integrate by parts we get

$$\begin{aligned}I_{n,m-j} &= \int_{(1-\alpha)\log x}^\infty (\log\tau)^{m-j}\tau^n e^{-\tau}d\tau\\ &= -e^{-\tau}\tau^n(\log\tau)^{m-j}\Big|_{(1-\alpha)\log x}^\infty + nI_{n-1,m-j}+(m-j)I_{n-1,m-j-1}\\ &= x^{\alpha-1}(1-\alpha)^n(\log x)^n\big(\log(1-\alpha)+l_2(x)\big)^{m-j}+\\ &\qquad + nI_{n-1,m-j}+(m-j)I_{n-1,m-j-1}.\end{aligned} \tag{3.18}$$

Now

$$I_{n-1,m-j} = \int_{(1-\alpha)\log x}^\infty \tau^{n-1}(\log\tau)^{m-j}e^{-\tau}d\tau < \frac{1}{(1-\alpha)\log x} I_{n,m-j} \tag{3.19}$$

and

$$I_{n-1,m-j-1} < \frac{1}{(1-\alpha)\log x\big(\log(1-\alpha)+l_2(x)\big)} I_{n,m-j}. \tag{3.20}$$

Thus, if $n\le 1/4(1-\alpha)\log x$ and $m\le 1/4(1-\alpha)\log x\big(\log(1-\alpha)+l_2(x)\big)$, it follows that $nI_{n-1,m-j}+(m-j)I_{n-1,m-j-1}<\frac{1}{2}I_{n,m-j}$ and (3.18) yields

$$\begin{aligned}I_{n,m-j} &< 2x^{\alpha-1}(1-\alpha)^n(\log x)^n\big(\log(1-\alpha)+l_2(x)\big)^{m-j}\\ &< 2x^{\alpha-1}(\log x)^n l_2(x)^{m-j}.\end{aligned} \tag{3.21}$$

Substituting (3.17) in (3.16) we get

$$\begin{aligned}&\int_\alpha^1 \left(\log\frac{1}{1-s}\right)^m (1-s)^n x^s ds\\ &= \frac{x}{(\log x)^{n+1}} \sum_{j=0}^m (-1)^{m-j}\binom{m}{j}(C_{n,m-j}-I_{n,m-j})l_2(x)^j\\ &= \frac{x}{(\log x)^{n+1}} \sum_{j=0}^m A_{nmj}l_2(x)^j + S_{nm}\end{aligned} \tag{3.22}$$

where, using (3.21), we have

$$|S_{nm}| < \frac{x}{(\log x)^{n+1}} 2x^{\alpha-1}(\log x)^n \sum \binom{m}{j} l_2(x)^{m-j}l_2(x)^j = 2^{m+1}\frac{x^\alpha}{\log x} l_2(x)^m. \quad ∎$$

**3.23.** LEMMA. *Let $f(s)$ be analytic in the closure of the domain $\mathcal{D}$ which is bounded by the arcs $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ where $0 < a \leqslant 1 - cr_0$ and*

$$\Gamma_1 = \{s = a+it \mid -T \leqslant t \leqslant T\},$$

$$\Gamma_2 = \{s = \sigma+iT \mid 1-cr_T \leqslant \sigma \leqslant a\},$$

$$\Gamma_3 = \{s = \sigma(t)+it \mid \sigma(t) = \min\{1-cr_t,\ a+\tfrac{1}{2}t^2\},\ -T \leqslant t \leqslant T\},$$

$$\Gamma_4 = \{s = \sigma-iT \mid 1-cr_t \leqslant \sigma \leqslant a\}.$$

*Assume that $|f(s)| \leqslant M$ in the closure of $\mathcal{D}$ and let*

$$\left(\frac{f(s)}{s}\right)^{(n)}\Bigg|_{s=1} = a_n; \qquad n = 0, 1, \ldots, N;$$

$$\left|\left(\frac{f(s)}{s}\right)^{(N+1)}\right| \leqslant a_{N+1} \quad \textit{for} \quad a \leqslant s \leqslant 1.$$

*Then*

$$(3.24)\quad I = \frac{1}{2\pi i}\int_{\Gamma_1}\left(\log\frac{1}{s-1}\right)^k \frac{x^s f(s)}{s}\,ds = \frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1} B_{nm}\frac{l_2(x)^m}{(\log x)^n} +$$

$$+\ O\Bigg(MR + \frac{x^a}{\log x}\sum_{n=0}^{N}\frac{|a_n|}{n!}(2l_2(x)+\pi)^k +$$

$$+\frac{x}{(\log x)^{N+2}}\,\frac{a_{N+1}}{(N+1)!}\,(\pi+1+l_2(x))^k \max_{0\leqslant m<k}|C_{N+1,m}|\Bigg)$$

*where*

$$B_{nm} = (-1)^n\frac{a_n}{n!}\sum_{l=0}^{k/2}\binom{k}{2l+1}(-1)^l\pi^{2l}A_{n,k-2l-1,m}.$$

Proof. The integrand in $I$ is analytic and single valued in the domain $\mathcal{D}'$ obtained from $\mathcal{D}$ by deleting a circular disk $|s-1| \leqslant \varepsilon$, $\varepsilon < \min\{1-a, a-1\}$ and a branch-cut $a \leqslant s \leqslant 1-\varepsilon$. We normalize $\log(1/s-1)$ so that $\log(1/a-1) > 0$. Then $\arg(1/s-1) = -\pi$ on the upper edge and $\arg(1/s-1) = \pi$ on the lower edge of the cut. Now

$$\int_{\partial\mathcal{D}'}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds = 0$$



and therefore

$$I = -\frac{1}{2\pi i}\int_a^{1-\varepsilon}\left[\left(\log\frac{1}{1-\sigma}-\pi i\right)^k - \left(\log\frac{1}{1-\sigma}+\pi i\right)^k\right]\frac{x^\sigma f(\sigma)}{\sigma}\,d\sigma +$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\left(\log\frac{1}{\varepsilon}-i\theta\right)^k\frac{x^{1+\varepsilon e^{i\theta}}f(1+\varepsilon e^{i\theta})}{1+\varepsilon e^{i\theta}}\,\varepsilon e^{i\theta}d\theta +$$

$$+O\left(\left|\sum_{j=2}^{4}\int_{\Gamma_j}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds\right|\right).$$

If we let $\varepsilon \to 0$ then the second integral tends to 0 like $\varepsilon(\log 1/\varepsilon)^k$ and we get

$$(3.25)\quad I = \sum_{l=0}^{k/2}\binom{k}{2l+1}(\pi i)^{2l}\int_a^1\left(\log\frac{1}{1-\sigma}\right)^{k-2l-1}\frac{x^\sigma f(\sigma)}{\sigma}\,d\sigma +$$

$$+O\left(\left|\sum_{j=2}^{4}\int_{\Gamma_j}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds\right|\right).$$

We first estimate the error term in (3.25)

$$(3.26)\quad \left|\int_{\Gamma_2}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds\right|$$

$$\leqslant \int_{\Gamma_2}\left|\frac{x^{\sigma+iT}(\log(1/\sigma-1+iT))^k f(\sigma+iT)}{\sigma+iT}\right|d\sigma$$

$$\leqslant \frac{Mx^a(\log T)^k}{T}(a-1+cr_t) \ll MR.$$

Similarly

$$(3.27)\quad \left|\int_{\Gamma_4}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds\right| \ll MR.$$

Also

$$(3.28)\quad \left|\int_{\Gamma_3}\left(\log\frac{1}{s-1}\right)^k\frac{x^s f(s)}{s}\,ds\right|$$

$$\ll M\int_{-T}^{T}\left|\frac{x^{1-\sigma(t)+it}}{1-\sigma(t)+it}\right|\,|\log(\sigma(t)-1+it)|^k dt$$

$$\ll \frac{M}{k+1}x^{1-cr_T}(\log T)^{k+1} \ll MR.$$

Thus the error term in (3.25) is of the form $MR$.

Now the principal term in (3.25) is

$$(3.29)\quad \sum_{l=0}^{k/2}\binom{k}{2l+1}(-1)^l\pi^{2l}\Bigg(\sum_{n=0}^{N}\frac{(-1)^n a_n}{n!}\int_a^1\Big(\log\frac{1}{1-\sigma}\Big)^{k-2l-1}(1-\sigma)^n x^\sigma d\sigma+ +\theta\frac{a_{N+1}}{(N+1)!}\int_a^1\Big(\log\frac{1}{1-\sigma}\Big)^{k-2l-1}(1-\sigma)^{N+1}x^\sigma d\sigma\Bigg)$$

where $|\theta|\leqslant 1$. Now, by Lemma 3.12

$$(3.30)\quad \sum_{l=0}^{k/2}\binom{k}{2l+1}(-1)^l\pi^{2l}\Bigg(\sum_{n=0}^{N}\frac{(-1)^n a_n}{n!}\int_a^1\Big(\log\frac{1}{1-\sigma}\Big)^{k-2l-1}(1-\sigma)^n x^\sigma d\sigma+ +\theta\frac{a_{N+1}}{(N+1)!}\int_a^1\Big(\log\frac{1}{1-\sigma}\Big)^{k-2l-1}(1-\sigma)^{N+1}x^\sigma d\sigma\Bigg)$$
$$=\sum_{l=0}^{k/2}\binom{k}{2l+1}(-1)^l\pi^{2l}\sum_{n=0}^{N}\frac{(-1)^n a_n}{n!}\frac{x}{(\log x)^{n+1}}\sum_{j=0}^{k-2l-1}A_{n,k-2l-1,j}\,l_2(x)^j+E,$$

where the error term satisfies, according to (3.15) and (3.16),

$$(3.31)\quad E\ll\sum_{l=0}^{k/2}\binom{k}{2l+1}\pi^{2l}\Bigg(\sum_{n=0}^{N}\frac{|a_n|}{n!}2^{k-2l-1}\frac{x^a}{\log x}l_2(x)^{k-2l-1}+ +\frac{a_N}{(N+1)!}\frac{x}{(\log x)^{N+2}}\sum_{j=0}^{k-2l-1}\binom{k-2l-1}{j}|C_{N+1,k-2l-j-1}|\,l_2(x)^j\Bigg)$$
$$\ll\frac{x^a}{\log x}(2l_2(x)+\pi)^k\sum_{n=0}^{N}\frac{|a_n|}{n!}+\frac{x}{(\log x)^{N+2}}(l_2(x)+1+\pi)^k\frac{a_N}{(N+1)!}\times \times\max_{0\leqslant m<k}|C_{N+1,m}|.$$

Substituting (3.30) and (3.31) in (3.25) we get (3.24).

**3.32.** Definition. Given a finite or infinite set of variables $x_1, x_2, \ldots$ then we define the symmetric formal power series

$$S_{a_1,a_2,\ldots,a_l}=\sum x_{i_1}^{a_1}x_{i_2}^{a_2}\ldots x_{i_l}^{a_l}$$

where the sum is extended over all $l$-tuples $i_1,\ldots,i_l$ of distinct indices. Note that if some of the exponents $a_1,\ldots,a_l$ are equal then each monomial occurs several times in $S_{a_1,\ldots,a_l}$.

We therefore define

$$C(a_1,a_2,\ldots,a_l)=\frac{1}{m_1!m_2!\ldots m_j!}$$



when the set $\{a_1,\ldots,a_l\}$ consists of $j$ distinct elements occurring with frequencies $m_1, m_2, \ldots, m_j$ respectively, and

$$S^*_{a_1,\ldots,a_l}=C(a_1,\ldots,a_l)S_{a_1,\ldots,a_l}$$

as the symmetrization of the monomial $x_1^{a_1}x_2^{a_2}\ldots x_l^{a_l}$.

**3.33.** Lemma. *We have*

$$(3.34)\quad S_{a_1,\ldots,a_l}=\sum_{m=1}^{l}\frac{(-1)^{m+l}}{m!}\sum_{j_1+\ldots+j_m=l}(j_1-1)!\ldots(j_m-1)!\sum S_{A_1}S_{A_2}\ldots S_{A_m},$$

*where the last sum is over all partitions of the set $\{a_1,\ldots,a_l\}$ into $m$ sets with $j_1,\ldots,j_m$ elements respectively and $A_1,\ldots,A_m$ are the sums of the elements in those sets.*

Proof. By induction on $l$. The lemma is obvious for $l=1$. Now assume the lemma true for $l-1$. We clearly have

$$S_{a_l}S_{a_1,\ldots,a_{l-1}}=S_{a_1,\ldots,a_{l-1},a_l}+S_{a_1+a_l,a_2,\ldots,a_{l-1}}+\ldots+S_{a_1,\ldots,a_{l-2},a_{l-1}+a_l}$$

and hence

$$(3.35)\quad S_{a_1,\ldots,a_l}=S_{a_l}S_{a_1,\ldots,a_{l-1}}-S_{a_1+a_l,a_2,\ldots,a_{l-1}}-\ldots-S_{a_1,a_{l-2},a_{l-1}+a_l}.$$

Using the induction hypothesis on the right side of (3.35) we get all the terms in (3.34) in which one of the factors $S_{A_i}$ is $S_{A_l}$ from the first term on the right of (3.35) while each term with a factor $S_{A_i}$ where $A_i=a_{i_1}+\ldots \ldots+a_{i_j}+a_l$ occurs in exactly $j$ of the latter terms in (3.35). Hence

$$(3.36)\quad S_{a_1,\ldots,a_l}=S_{a_l}\sum_{m=1}^{l-1}\frac{(-1)^{m+l-1}}{m!}\sum_{j_1+\ldots+j_m=l-1}(j_1-1)!\ldots(j_m-1)!\sum S_{A_1}\ldots S_{A_m}-$$
$$-\sum_{m=1}^{l-1}\frac{(-1)^{m+l-1}}{m!}\sum_{j_1+\ldots+j_m=l-1}(j_1-1)!\ldots(j_m-1)!\times$$
$$\times\sum(j_1S_{A_1+a_l}S_{A_2}\ldots S_{A_m}+\ldots+j_mS_{A_1}\ldots S_{A_{m-1}}S_{A+a_l})$$

where $A_1,\ldots,A_m$ refer to sums of partitions of $\{a_1,\ldots,a_{l-1}\}$.

Reordering the terms on the right of (3.36) we get (3.34) where the terms in the first sum are $S_{A_1}\ldots S_{A_m}S_{A_{m+1}}$ with $A_{m+1}=a_l$ while the terms in the second sum are $S_{A_1}\ldots S_{A_{i-1}}S_{A_i+a_l}S_{A_{i+1}}\ldots S_{A_m}$, yielding all partitions of $\{a_1,\ldots,a_l\}$.

We write, using $1 \leq a_1 \leq \ldots \leq a_l$

(3.37) $$S^*_{a_1,\ldots,a_l} = \sum_{m=1}^{l} \sum_{\substack{A_1+\ldots+A_m=a_1+\ldots+a_l \\ a_1\leq A_1\leq\ldots\leq A_m}} D(a_1, \ldots, a_l; A_1, \ldots, A_m) S_{A_1} \ldots S_{A_m}$$

where

(3.38) $$D(a_1, \ldots, a_l; A_1, \ldots, A_m) = C(a_1, \ldots, a_l)(-1)^{m+l} \sum (j_1-1)! \ldots (j_m-1)!.$$

The sum being extended over all partitions of $\{a_1, \ldots, a_l\}$ into sets $\mathcal{A}_1, \ldots, \mathcal{A}_m$ where the sum of the elements of $\mathcal{A}_i$ is $A_i$ $(i = 1, \ldots, m)$ and $j_i$ is the cardinality of $\mathcal{A}_i$.

From now on we are interested in the case $x_i = p_i^{-s}$; $i = 1, 2, \ldots$ and define

(3.39) $$P_{a_1,\ldots,a_l}(s) = S^*_{a_1,\ldots,a_l} = \sum_{m=1}^{l} \sum_{\substack{A_1+\ldots+A_m=a_1+\ldots+a_l \\ a_1\leq A_1\leq\ldots\leq A_m}} D(a_1, \ldots, a_l; A_1, \ldots, A_m) P(A_1 s) \ldots P(A_m s)$$

since $P_a(s) = P(as)$.

**3.40.** Corollary. *We have*

$$f_\pi(s) = \sum_{n=1}^{\infty} \frac{\pi_k(n)-\pi_k(n-1)}{n^s} = P_{\underbrace{1,\ldots,1}_{k}}(s)$$

$$= \frac{1}{k!} \sum_{m=1}^{k} \frac{(-1)^{m+k}}{m!} \sum_{j_1+\ldots+j_m=k} (j_1-1)!\ldots(j_m-1)! \frac{k!}{j_1!\ldots j_m!} P(j_1 s) \ldots P(j_m s)$$

$$= \sum_{m=1}^{k} \frac{(-1)^{m+k}}{m!} \sum_{j_1+\ldots+j_m=k} \frac{P(j_1 s) \ldots P(j_m s)}{j_1 \ldots j_m}$$

*or, if we write* $D(\underbrace{1, \ldots, 1}_{k}; j_1, \ldots, j_m) = D_k(j_1, \ldots, j_m)$, *then*

(3.41) $$f_\pi(s) = \sum_{m=1} \sum_{\substack{1\leq j_1\leq\ldots\leq j_m \\ j_1+\ldots+j_m=k}} D_k(j_1, \ldots, j_m) P(j_1 s) \ldots P(j_m s),$$

$$D_k(j_1, \ldots, j_m) = (-1)^{m+k} \frac{1}{j_1 \ldots j_m}.$$



**3.42.** Corollary. *We have*

$$f_\sigma(s) = \sum_{n=1}^{\infty} \frac{\sigma_k(n)-\sigma_k(n-1)}{n^s} = \sum P_{a_1,\ldots,a_l}(s)$$

*where the last sum is extended over all partitions of $k$ into positive integers $a_1 + \ldots + a_l$ $(a_1 \leq \ldots \leq a_l)$.*

*Thus, by* (3.37), (3.38) *we have*

(3.43) $$f_\sigma(s) = \sum_{a} \sum_{A} D(a_1, \ldots, a_l; A_1, \ldots, A_m) P(A_1 s) \ldots P(A_m s) = \sum_{A} E(A_1, \ldots, A_m) P(A_1 s) \ldots P(A_m s)$$

*where the last sum is over all partitions $A_1 + \ldots + A_m$ $(1 \leq A_1 \leq \ldots \leq A_m)$ of $k$ and*

(3.44) $$E(A_1, \ldots, A_m) = \sum_{a} D(a_1, \ldots, a_l; A_1, \ldots, A_m)$$

*where the sum is over all $a_1 + \ldots + a_l = k$ $(1 \leq a_1 \leq \ldots \leq a_l)$ which are sums of partitions of $A_1, A_2, \ldots, A_m$.*

**3.45.** Corollary. *We have*

$$f_\varrho(s) = \sum_{n=1}^{\infty} \frac{\varrho_k(n)-\varrho_k(n-1)}{n^s} = \sum_{1<a_1\leq\ldots\leq a_k} P_{a_1,\ldots,a_k}(s)$$

$$= \sum_{a} \sum_{A} D(a_1, \ldots, a_k; A_1, \ldots, A_m) P(A_1 s) \ldots P(A_m s)$$

$$= \sum_{A} F(A_1, \ldots, A_m) P(A_1 s) \ldots P(A_m s)$$

*where the last sum is extended over all $m$-tuples $1 \leq A_1 \leq \ldots \leq A_m$, $1 \leq m \leq k$ and*

(3.46) $$F(A_1, \ldots, A_m) = \sum_{a} D(a_1, \ldots, a_k; A_1, \ldots, A_m)$$

*where the sum is over all $a_1 + \ldots + a_k = A_1 + \ldots + A_m$ $(1 \leq a_1 \leq \ldots \leq a_k)$ which are sums of partitions of $A_1, A_2, \ldots, A_m$.*

**3.47.** Lemma. *For each $k < k_0$ we have*

(3.48) $$\sum_{\substack{1\leq j_1\leq\ldots\leq j_m \\ j_1+\ldots+j_m=k \\ 1\leq m\leq k}} |D_k(j_1, \ldots, j_m)| \leq \sqrt{T},$$

$$\sum_{\substack{A_1+\ldots+A_m=k\\ 1\le A_1\le\ldots\le A_m}} |E(A_1,\ldots,A_m)| \ll \sqrt{T}, \tag{3.49}$$

$$\sum_{\substack{A_1+\ldots+A_m\le A\\ 1\le m\le k\\ 1\le A_1\le\ldots\le A_m}} |F(A_1,\ldots,A_m)| \ll \sqrt{T}. \tag{3.50}$$

Proof. From (3.41) we get

$$\sum |D_k(j_1,\ldots,j_m)| = \sum \frac{1}{j_1\ldots j_m} < \sum_{m=1}^{k}\Big(\sum_{j=1}^{k}\frac{1}{j}\Big)^m$$

$$\ll \sum_{m=1}^{k_0-1}(\log k_0)^m \ll (\log k_0)^{k_0} \ll \exp\big(k_0 l_2(k_0)\big) \ll \sqrt{T}$$

which proves (3.48). From (3.44) we get

$$\sum |E(A_1,\ldots,A_m)| \ll k^k \max |E(A_1,\ldots,A_m)|$$

$$= k^k \max_{A} \sum_{\substack{a_1+\ldots+a_l=k\\ 1\le a_1\le\ldots\le a_l\\ 1\le l\le k}} |D(a_1,\ldots,a_l;A_1,\ldots,A_m)|$$

$$\ll k^k\cdot k^k k! \ll k_0^{3k_0} \ll \exp(3k_0\log k_0) \ll \sqrt{T}$$

which proves (3.49). From (3.50) we get

$$\sum |F(A_1,\ldots,A_m)| \ll A^k \max |F(A_1,\ldots,A_m)|$$

$$\ll A^k \max\Big|\sum_{\substack{a_1+\ldots+a_k\le A\\ 1\le a_1\le\ldots\le a_k}} D(a_1,\ldots,a_k;A_1,\ldots,A_m)\Big|$$

$$\ll A^k A^k k! \ll A^{2k} k_0^{k_0} \ll \sqrt{T}. \ \blacksquare$$

## 4. Theorems

**4.1.** THEOREM. *For* $k < k_0$ *we have*

$$\pi_k(x) = \frac{x}{\log x}\sum_{k=0}^{N}\sum_{m=0}^{k-1} b_{nm}(\pi_k)\frac{l_2(x)^m}{(\log x)^n} + R$$

*where*

$$b_{nm}(\pi_k) = \sum_{l=1}^{k}\sum_{q=0}^{l}\binom{l}{q}\sum_{\substack{j_1+\ldots+j_u=k-l\\ 1<j_1\le\ldots\le j_u\\ 0\le u\le(k-l)/2}} D_k(\underbrace{1,\ldots,1}_{l}, j_1,\ldots,j_u)\, B_{nm}(q;l;j_1,\ldots,j_u)$$



*and*

$$B_{nm}(q;l;j_1,\ldots,j_u) = \frac{(-1)^n}{n!}a_n(l-q;j_1,\ldots,j_u)\sum_{i=0}^{q/2}\binom{q}{2i+1}(-1)^i\pi^{2i}A_{n,q-2i-1,m}.$$

Proof. We first note that all three functions $f_\pi, f_\sigma, f_\varrho$ satisfy the conditions for $f(s)$ in Lemma 3.1 since $|\pi_k(n)-\pi_k(n-1)|\le 1$, $|\sigma_k(n)-\sigma_k(n-1)|\le 1$ and $|\varrho_k(n)-\varrho_k(n-1)|\le 1$ and $|f_\pi(s)|\le|\zeta(\sigma)|$, $|f_\sigma(s)|\le|\zeta(\sigma)|$, $|f_\varrho(s)|\le|\zeta(\sigma)|$ and $|\zeta(\sigma)|\ll 1/(\sigma-1)$ for $\sigma>1$.

Thus, by (3.3) we have

$$\pi_k(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} f_\pi(s)\frac{x^s}{s}ds + R. \tag{4.2}$$

By (3.41) this yields

$$\pi_k(x) = \frac{1}{2\pi i}\sum_{\substack{j_1\le\ldots\le j_u\\ j_1+\ldots+j_u=k}} D_k(j_1,\ldots,j_u)\int_{a-iT}^{a+iT} P(j_1 s)\ldots P(j_u s)\frac{x^s}{s}ds + R. \tag{4.3}$$

For those terms on the right of (4.3) with $j_1\ge 2$ we have

$$\Big|\int_{a-iT}^{a+iT} P(j_1 s)\ldots P(j_u s)\frac{x^s}{s}ds\Big| \tag{4.4}$$

$$= \Big|\Big(-\int_{3/4-iT}^{a-iT} + \int_{3/4-iT}^{3/4+iT} + \int_{3/4+iT}^{a+iT}\Big)P(j_1 s)\ldots P(j_u s)\frac{x^s}{s}ds\Big|$$

$$\ll \frac{x^a}{T} + x^{3/4}\log T,$$

since $|P(js)|\le\sum 1/p^{2\sigma}\le P(3/2)\le\log\zeta(3/2)<1$ when $\sigma\ge 3/4$, $j\ge 2$. Thus the contribution of these terms is bounded by

$$\sum |D_k(j_1,\ldots,j_u)|\cdot\Big(\frac{x}{T}+x^{3/4}\log T\Big) \ll \sqrt{T}\Big(\frac{x}{T}+x^{3/4}\log T\Big) \ll R$$

by (3.48).

Thus (4.3) becomes

$$\pi_k(x) = \frac{1}{2\pi i}\sum_{l=1}^{k}\sum_{\substack{j_1+\ldots+j_u=k-l\\ 1<j_1\le\ldots\le j_u}} D_k(\underbrace{1,\ldots,1}_{l}, j_1,\ldots,j_u)\times \tag{4.5}$$

$$\times\int_{a-iT}^{a+iT} P(s)^l P(j_1 s)\ldots P(j_u s)\frac{x^s}{s}ds + R.$$

We now apply Lemma 3.4 to obtain

$$(4.6)\quad I_l(j_1,\dots,j_u) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} P(s)^l P(j_1 s)\dots P(j_u s)\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\int_{a-iT}^{a+iT}\Big(\log\frac{1}{s-1}+g(s)\Big)^l P(j_1 s)\dots P(j_u s)\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\sum_{q=0}^{l}\binom{l}{q}\int_{a-iT}^{a+iT}\Big(\log\frac{1}{s-1}\Big)^q g(s;l-q;j_1,\dots,j_u)\frac{x^s}{s}\,ds.$$

Now for $\sigma > 1-cr_t$, where $c$ is the constant of Lemma 3.4 chosen small enough so that $cr_0 \leqslant 1/8$ we have

$$|g(s;l-q;j_1,\dots,j_u)| \leqslant \big(c_1\log(|t|+9)\big)^{l-q} \leqslant \big(c_1\log(T+9)\big)^l = M$$

where $c_1$ is a constant such that $|\log((s-1)\zeta(s))| \leqslant c_1\log(|t|+9)$. Applying Lemma 3.23 to the right of (4.6) with $f(s)$ replaced by $g(s;l-q;j_1,\dots,j_u)$ we have

$$(4.7)\quad I_l(j_1,\dots,j_u) = \sum_{q=0}^{l}\binom{l}{q}\frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1}B_{nm}(q;l;j_1,\dots,j_u)\frac{l_2(x)^m}{(\log x)^n}+E$$

where

$$(4.8)\quad B_{nm}(q;l;j_1,\dots,j_u) = (-1)^n\frac{a_n(l-q;j_1,\dots,j_u)}{n!}\sum_{i=0}^{q/2}\binom{q}{2i+1}(-1)^i\pi^{2i}A_{n,q-2i-1,m}$$

and

$$(4.9)\quad E \ll R+\frac{x^a}{\log x}\sum_{n=0}^{N}\frac{|a_n(l-q;j_1,\dots,j_u)|}{n!}\big(2l_2(x)+\pi\big)^q + \frac{x}{(\log x)^{N+2}}\frac{|a_{N+1}(l-q;j_1,\dots,j_u)|}{(N+1)!}\cdot\big(l_2(x)+\pi+1\big)^q\max_{0\leqslant i<q}|C_{N+1,i}|.$$

By Cauchy's estimate with $a > 7/8$ and $M_1 = \max_{|s-1|\leqslant 1/4}|\log((s-1)\zeta(s))|$ we have

$$(4.10)\quad |a_n(l-q;j_1,\dots,j_u)| \leqslant M_1^{l-q}n!\,8^n,\qquad n=0,1,\dots,N+1,$$



and

$$(4.11)\quad |C_{N+1,i}| = \left|\frac{d^i}{ds^i}\Gamma(s)\right|\Big|_{N+1} \leqslant \frac{i!(N+1+u)!}{u^i},\qquad 0<u\leqslant N.$$

Since we are interested in the case $i<q\leqslant k<N$ we may pick $u=i$ in (4.11) to get

$$(4.12)\quad |C_{N+1,i}| \leqslant \frac{(N+1)!(+N)^i i!}{i^i} \leqslant (N+1)!\Big(\frac{2N}{e}\Big)^{i+1/2} \leqslant (N+1)!N^q.$$

Substituting (4.10) and (4.12) in (4.9) we get

$$E \ll R+\frac{x^a}{\log x}\sum_{q=0}^{l}\binom{l}{q}\sum_{n=0}^{N+1}8^nM_1^{l-q}(1-a)^n\big(2l_2(x)+\pi\big)^q + \frac{x}{(\log x)^{N+2}}\sum_{q=0}^{l}\binom{l}{q}8^nM_1^{l-q}\big(l_2(x)+\pi+1\big)^2(N+1)!N^q$$
$$\ll R+\frac{x^a}{\log x}\sum_{n=0}^{\infty}\big(8(1-a)\big)^n\sum_{q=0}^{l}\binom{l}{q}M_1^{l-q}\big(2l_2(x)+\pi\big)^q + \frac{x}{(\log x)^{N+2}}8^N(N+1)!\sum_{q=0}^{l}\binom{l}{q}M_1^{l-q}\big(N(l_2(x)+\pi+1)\big)^q$$
$$\ll R+\frac{1}{8a-7}\frac{x^a}{\log x}\big(M_1+2l_2(x)+\pi\big)^l + \frac{x}{(\log x)^{N+2}}8^N(N+1)!\big(M_1+N(l_2(x)+\pi+1)\big)^l.$$

Since $N=[c\log x]-1$, $l\ll k_0 = c(\log x)^{3/5}l_2(x)^{-6/5}$ we get, after choosing $a = 7/8+1/\log x$

$$(4.13)\quad E \ll R+x^{a+\varepsilon}+x^{1+c\log 8+c\log c+cl_2(x)-cl_2(x)+o(1)} \ll R+x^{1-1/100} \ll R,$$

if $c$ is chosen as a sufficiently small constant.

We now substitute (4.7) and (4.13) in (4.5) to get

$$(4.14)\quad \pi_k(x) = \sum_{l=1}^{k}\sum_{\substack{j_1+\dots+j_u=k-l\\ 2\leqslant j_1\leqslant\dots\leqslant j_u}} D_k(\underbrace{1,\dots,1}_{l},j_1,\dots,j_u)I_l(j_1,\dots,j_u)+R = \frac{x}{\log x}\sum_{l=1}^{k}\sum_{q=0}^{l}\binom{l}{q}\sum_{\substack{j_1+\dots+j_u=k-l\\ 2\leqslant j_1\leqslant\dots\leqslant j_u}} D_k(1,\dots,1,j_1,\dots,j_u)\times$$

$$\times \sum_{n=0}^{N}\sum_{m=0}^{k-1} B_{nm}(q; l; j_1, \ldots, j_u)\frac{l_2(x)^m}{(\log x)^n} + E$$

$$= \frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1} b_{nm}(\pi_k)\frac{l_2(x)^m}{(\log x)^n} + E,$$

$$(4.15)\qquad E \ll R\sum_{q=1}^{k}\frac{1}{q!}\sum_{l=1}^{q}\binom{q}{l}\sum_{\substack{j_1+\ldots+j_{q-l}=k-l\\ j_i\geq 2}}\frac{1}{j_1\ldots j_{q-l}}$$

$$\ll R\sum_{q=1}^{k}\frac{1}{q!}\sum_{l=1}^{q}\binom{q}{l}\Big(\sum_{j=2}^{k}\frac{1}{j}\Big)^{q-l}$$

$$\ll R\sum_{q=1}^{\infty}\frac{1}{q!}(\log k)^q \ll kR \ll R. \qquad ∎$$

Note that

$$b_{0,k-1}(\pi_k) = \frac{1}{k!}\,a_0(0)\cdot k\cdot C_{00} = \frac{1}{(k-1)!},$$

$$b_{0,k-2}(\pi_k) = \frac{1}{(k-2)!}\big(-C_{01}+2-g(1)\big)$$

$$= \frac{1}{(k-2)!}\big(\gamma+2+\tfrac{1}{2}P(2)+\tfrac{1}{3}P(3)+\ldots\big) = c(k-1)b_{0,k-1}.$$

Thus, if $k \geq l_2(x)$ then the first term is not the leading term. However, if $k < k_0$ then we have

$$\pi(\sqrt[k]{x}) \geq \frac{k\sqrt[k]{x}}{\log x}$$

and

$$\pi_k(x) \geq \binom{\pi(\sqrt[k]{x})}{k} \geq \Big(\frac{\pi(\sqrt[k]{x})-k}{k}\Big)^k > c^k\Big(\frac{\sqrt[k]{x}}{\log x}-1\Big)^k > \frac{x}{(\log x)^k}e^{-ck}$$

$$> x\exp\big(-c(\log^3 x/l_2(x))^{1/5}\big) \gg R.$$

So the principal term exceeds the error term in this case.

**4.16.** Theorem. *For $k \leq k_0$ we have*

$$\sigma_k(x) = \frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1} b_{nm}(\sigma_k)\frac{l_2(x)^m}{(\log x)^n} + R$$



*where*

$$b_{nm}(\sigma_k) = \sum_{l=1}^{k}\sum_{q=0}^{l}\binom{l}{q}\sum_{\substack{A_1+\ldots+A_u=k-q\\ 2\leq A_1\leq\ldots\leq A_u}} E(\underbrace{1,\ldots,1}_{l}, A_1, \ldots, A_u)B_{nm}(q; l; A_1, \ldots, A_u).$$

Proof. As in the proof of Theorem 4.1 we have, using (3.43),

$$(4.17)\qquad \sigma_k(x) = \frac{1}{2\pi i}\int_{a-iT}^{a+iT} f_\sigma(s)\frac{x^s}{s}\,ds + R$$

$$= \frac{1}{2\pi i}\sum_{\substack{A_1+\ldots+A_u=k\\ 1\leq A_1\leq\ldots\leq A_u}} E(A_1, \ldots, A_u)\int_{a-iT}^{a+iT} P(A_1 s)\ldots P(A_u s)\frac{x^s}{s}\,ds + R.$$

As in the proof of Theorem 4.1, using (3.49), we can include the contribution of all terms on the right of (4.17) with $1 < A_1$ in the error term $R$. Thus

$$(4.18)\qquad \sigma_k(x) = \frac{1}{2\pi i}\times$$

$$\times\sum_{l=1}^{k}\sum_{\substack{A_1+\ldots+A_u=k-q\\ 2\leq A_1\leq\ldots\leq A_u}} E(\underbrace{1,\ldots,1}_{l}, A_1, \ldots, A_u)\int_{a-iT}^{a+iT} P(s)^l P(A_1 s)\ldots P(A_u s)\frac{x^s}{s}$$

$$= \sum_{l=1}^{k} E(1, \ldots, 1, A_1, \ldots, A_u)I_l(A_1, \ldots, A_u) + R.$$

By (4.7) we have

$$(4.19)\qquad \sigma_k(x) = \frac{x}{\log x}\sum_{l=1}^{k}\sum_{A} E(1, \ldots, 1, A_1, \ldots, A_u)\sum_{q=0}^{l}\binom{l}{q}\times$$

$$\times\sum_{n=0}^{N}\sum_{m=0}^{k-1} B_{nm}(q; l; A_1, \ldots, A_u)\frac{l_2(x)^m}{(\log x)^n} + E.$$

The proof that $E \ll R$ proceeds exactly as (4.9) using the estimate (3.49). ∎

The comments made after the proof of Theorem 4.1, that the leading term $b_{0,k-1}(\sigma_n)x l_2(x)^{k-1}/\log x$ dominates if $k$ is small compared to $l_2(x)$ and that the principal term exceeds the error term for $k < k_0$ remain valid for Theorem 4.16.

**4.20.** Theorem. *For $k < k_0$ we have*

$$\varrho_k(x) = \frac{x}{\log x}\sum_{n=0}^{N}\sum_{m=0}^{k-1} b_{nm}(\varrho_k)\frac{l_2(x)^m}{(\log x)^n} + R$$

where

$$b_{nm}(\varrho_k) = \sum_{l=1}^{k} \sum_{q=0}^{l} \binom{l}{q} \times$$

$$\times \sum_{\substack{2 \leqslant A_1 \leqslant \ldots \leqslant A_u \\ l+A_1+\ldots+A_u \leqslant \log x/\log 2}} F(\underbrace{1, \ldots, 1}_{l}, A_1, \ldots, A_u) B_{nm}(q; l; A_1, \ldots, A_u).$$

Proof. As in the proof of Theorem 4.1 we have

$$(4.21)\quad \varrho_k(x) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f_\varrho(s) \frac{x^s}{s} + R$$

$$= \frac{1}{2\pi i} \sum_{1 \leqslant A_1 \leqslant \ldots \leqslant A_u} F(A_1, \ldots, A_u) \int_{a-iT}^{a+iT} P(A_1 s) \ldots P(A_u s) \frac{x^s}{s} ds + R.$$

Note that the sum in (4.21) is infinite. However, all integers $n = p_1^{a_1} \ldots \ldots p_k^{a_k} \leqslant x$ satisfy $a_1 + \ldots + a_k \leqslant \log x/\log 2$. Thus (4.21) remains valid if we restrict the summation on the right of (4.21) to $A_1 + \ldots + A_u \leqslant \log x/\log 2$. If we substitute this bound in (3.50) we get

$$(4.22)\quad \sum_{\substack{A_1+\ldots+A_u \leqslant \log x/\log 2 \\ 1 \leqslant u \leqslant k \\ 1 \leqslant A_1 \leqslant \ldots \leqslant A_u}} |F(A_1, \ldots, A_u)| \ll \left(\frac{\log x}{\log 2}\right)^{k_0} \sqrt{T} \ll T^{2/3}.$$

Using (4.22) we can include the contributions of all terms on the right of (4.21) with $1 < A_1$ in the error term $R$. Thus

$$(4.23)\quad \varrho_k(x)$$

$$= \sum_{l=1}^{k} \sum_{\substack{2 \leqslant A_1 \leqslant \ldots \leqslant A_u \\ l+A_1+\ldots+A_u \leqslant \log x/\log 2}} F(\underbrace{1, \ldots, 1}_{l}, A_1, \ldots, A_u) I_l(A_1, \ldots, A_u) + R$$

$$= \frac{x}{\log x} \sum_{l=1} \sum_{A} F(1, \ldots, 1, A_1, \ldots, A_u) \sum_{q=0}^{l} \binom{l}{q} \times$$

$$\times \sum_{n=0}^{N} \sum_{m=0}^{k-1} B_{nm}(q; l; A_1, \ldots, A_u) \frac{l_2(x)^m}{(\log x)^n} + E$$

where $E \ll R$, using (4.22). ∎

The remarks concerning leading and principal terms are again valid for Theorem 4.20.



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