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Some consequences of the Riemann hypothesis

by

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In memory of Paul Turán

The main object of this note is to show that the remainder term in the prime number theorem, assuming the Riemann hypothesis, can be reduced from

$$\psi(x) = x + O(x^{1/2}\log^2 x) \tag{1}$$

to

$$\psi(x) = x + O\big(x^{1/2}(\log\log x)^2\big), \tag{2}$$

except on a set of finite logarithmic measure.

We also give short proofs of Cramér's conditional estimates ([1], [2]) of the mean value of the remainder term

$$\int_1^X \big(\psi(x)-x\big)^2 \frac{dx}{x} \ll X, \tag{3}$$

and

$$\int_1^X \big(\psi(x)-x\big)^2 \frac{dx}{x^2} \sim C\log X. \tag{4}$$

It follows from (3) that for each function $\varphi = \varphi_x$ for which $\varphi_x \to \infty$ as $x \to \infty$, we have

$$\psi(x) = x + O(x^{1/2}\varphi_x)$$

for almost all x, i.e. except on a set whose intersection with the interval $[1, X]$ has measure $o(X)$. The proof that for $\varphi_x = (\log\log x)^2$ the exceptional set has finite logarithmic measure is a combination of the arguments which prove (1) and (3). A similar method yields a short proof of Selberg's conditional result [6] on the normal density of primes in short intervals:

(5) $$\int_1^X \left(\psi(x+h_x)-\psi(x)-h_x\right)^2 dx = o(h_X^2 X),$$

for increasing functions $h = h_x$ which satisfy

$$h_x \leqslant x, \quad h_x/\log^2 x \to \infty.$$

J. H. Mueller and the author have recently shown [4] that the Riemann hypothesis together with Montgomery's conjecture [5] on the pair density of zeros implies that O may be replaced by o in (1).

(6) $$\psi(x) = x - \sum_{|\gamma|\leqslant X} \frac{x^\varrho}{\varrho} + O(\log^2 X),$$

valid for $X \leqslant x \ll X$. Here the sum is over the complex zeros $\varrho = \frac{1}{2}+i\gamma$ of the Riemann zeta function. We will use the fact that the number of terms with $t < \gamma \leqslant t+1$ is $\ll \log t$ for $t \geqslant 2$. It follows from this that the contribution of the terms with $|\gamma| \leqslant \log X$ is $\ll X^{1/2}(\log\log X)^2$.

For $T \leqslant X$, we have

(7) $$\int_X^{eX} \Big| \sum_{T<|\gamma|\leqslant X} \frac{x^\varrho}{\varrho}\Big|^2 \frac{dx}{x^2} \ll \frac{\log^2 T}{T}.$$

In fact, Lemma 1 of [3] shows that the integral is

$$\int_{\log X}^{1+\log X} \Big| \sum_{T<|\gamma|\leqslant X} \frac{e^{i\gamma u}}{\varrho}\Big|^2 du \ll \int_{T-1}^{X} \Big(\sum_{t<\gamma\leqslant t+1} \frac{1}{|\varrho|}\Big)^2 dt,$$

and the bound in (7) now follows from the fact mentioned above.

It follows from (7) that the logarithmic measure of the set of x in $[X, eX]$ for which

$$\Big| \sum_{T<|\gamma|\leqslant X} \frac{x^\varrho}{\varrho}\Big| \geqslant x^{1/2}(\log\log x)^2$$

is

$$\ll \frac{\log^2 T}{T(\log\log X)^4} = \frac{1}{T\log^2 T} \quad \text{for} \quad T = \log X.$$

Choosing $X = e^T$ with $T = 2, 3, \ldots$, we see that (2) holds except on a set whose total logarithmic measure is finite.

2. From (6) and (7) with $T = 2$, we get

(8) $$\int_X^{eX} (\psi(x)-x)^2 \frac{dx}{x^2} \ll 1,$$

from which (3) follows easily by a splitting-up argument.

To get (4), we observe first that for each fixed T, and $X \to \infty$,

$$\int_T^X \Big| \sum_{|\gamma|\leqslant T} \frac{x^\varrho}{\varrho}\Big|^2 \frac{dx}{x^2} = \int_{\log T}^{\log X} \Big| \sum_{|\gamma|\leqslant T} \frac{x^{i\gamma u}}{\varrho}\Big|^2 du \sim \Big({\sum_{|\gamma|\leqslant T}}' \frac{m^2(\varrho)}{|\varrho|^2}\Big) \log X,$$

where the dash indicates that the sum is over distinct zeros, and $m(\varrho)$ is the multiplicity of the zero at the point ϱ. It follows that

$$\int_T^X \Big| \sum_{|\gamma|\leqslant T} \frac{x^\varrho}{\varrho}\Big|^2 \frac{dx}{x^2} \sim C\log X, \quad C = \sum_{\text{all }\varrho} \frac{m^2(\varrho)}{|\varrho|^2},$$

where $T = T_X$ is a suitable function such that $T \to \infty$ and $\log T = o(\log X)$ as $X \to \infty$. From (7) we get that, for such T,

$$\int_T^X \Big| \sum_{T<|\gamma|\leqslant X} \frac{x^\varrho}{\varrho}\Big|^2 \frac{dx}{x^2} = o(\log X).$$

It follows that

$$\int_T^X (\psi(x)-x)^2 \frac{dx}{x^2} \sim C\log X$$

as $X \to \infty$, for such T. Combining this with

$$\int_1^T (\psi(x)-x)^2 \frac{dx}{x^2} \ll \log T = o(\log X),$$

which follows from (8), we get (4).

3. To prove (5), we use (6) to express $\psi(x+h)-\psi(x)-h$ as

$$-\int_x^{x+h} \Big(\sum_{|\gamma|\leqslant T} y^{\varrho-1}\Big) dy + \sum_{T<|\gamma|\leqslant X} \frac{x^\varrho}{\varrho} - \sum_{T<|\gamma|\leqslant X} \frac{(x+h)^\varrho}{\varrho} + O(\log^2 X),$$

for $X < x \leqslant eX$, $h \leqslant X$, and $T \leqslant X$. Putting

$$S_1(y) = \sum_{|\gamma|\leqslant T} y^{i\gamma}, \quad S_2(y) = \sum_{T<|\gamma|\leqslant X} \frac{y^{i\gamma}}{\varrho},$$

this may be written as

$$-\int_x^{x+h} S_1(y) \frac{dy}{y^{1/2}} + x^{1/2}S_2(x) - (x+h)^{1/2}S_2(x+h) + O(\log^2 X).$$

A simple argument shows that

$$\int_X^{eX} \left| \int_x^{x+h} S_1(y) \frac{dy}{y^{1/2}} \right|^2 dx \ll h_{eX}^2 \int_X^{2eX} |S_1(y)|^2 \frac{dy}{y}.$$

The same method as in § 1 shows that the integral on the right is

$$\ll \int_{-(T+1)}^{T} \Big(\sum_{t<\gamma\leqslant t+1} 1 \Big)^2 dt \ll T\log^2 T.$$

Also,

$$\int_X^{eX} |x^{1/2} S_2(x)|^2 dx \ll X^2 \int_X^{eX} |S_2(x)|^2 \frac{dx}{x} \ll X^2 \frac{\log^2 T}{T},$$

by (7); similarly,

$$\int_X^{eX} |(x+h)^{1/2} S_2(x+h)|^2 dx \ll X \int_X^{eX} |S_2(x+h)|^2 dx,$$

and since h is increasing and $\leqslant x$, this is also

$$\ll X \int_X^{2eX} |S_2(y)|^2 dy \ll X^2 \frac{\log^2 T}{T}.$$

It follows that for $T \leqslant X$,

$$\int_X^{eX} \big(\psi(x+h) - \psi(x) - h\big)^2 dx \ll h_{eX}^2 T\log^2 T + X^2 \frac{\log^2 T}{T} + X\log^4 X.$$

For $T = X/h_{eX}$, this is $o(Xh_{eX}^2)$, provided $h_{eX}^2/\log^2 X \to \infty$. A simple splitting-up argument completes the proof of (5).

References

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