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Transference theorems in completions of A-fields of non-zero characteristic

by

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Let k be an A-field of characteristic $p \neq 0$, u a place of k and K the completion of k at u. Let $\mathfrak{o}$ be the ring of u-exceptional integers of k, i.e., those elements x of k such that $\operatorname{ord}_v(x) \geqslant 0$ for all places $v \neq u$ of k. When $k = F_q(T)$ and u the place of k for which $|T|_u > 1$, $\mathfrak{o}$ is no other than $F_q[T]$, and K is the field $F_q((T^{-1}))$ of series

$$\xi = a_\nu T^\nu + a_{\nu-1} T^{\nu-1} + \ldots;$$

in this set-up, Aggarwal [1] obtained analogues of certain transference theorems ([2], [3], [6]) in the usual set-up of $\boldsymbol{Z}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$. The object of this note is to indicate how these results of Aggarwal can be generalized when $F_q(T)$, $F_q((T^{-1}))$ and $F_q[T]$ are replaced respectively by k, K and $\mathfrak{o}$. We shall prove only a few typical results, the deduction of the remaining ones being then a routine matter. The unexplained notations and results will be as in Weil [7]; in particular, F_q is the field of constants of k, g is the genus of k, and d is the degree of u. For any a in K, we put

$$\|a\| = \inf_{x\in\mathfrak{o}} |x-a|;$$

here and elsewhere $|\ |$ means the normalized valuation in the local field K.

Let $\varphi_\lambda(z)$, $1 \leqslant \lambda \leqslant l$, be l linear forms over K in the l variables $z_1, \ldots, z_l$ of determinant $\Delta \neq 0$, and let

$$|\Delta| = q^\delta. \tag{1}$$

Theorem 1. *Suppose that $\varrho_1, \ldots, \varrho_l$ are integers such that*

$$l(g-1)+\delta = d(\varrho_1+\ldots+\varrho_l).$$

The following conditions are equivalent.

(A) *The inequalities*

$$(2)\qquad |\varphi_\lambda(z)| \leqslant q_u^{\varrho_\lambda}, \quad 1 \leqslant \lambda \leqslant l,$$

have no non-zero solution in $\mathfrak{o}^l$.

(B) *For every choice of* $\beta_1, \dots, \beta_l$ *in* K *the inequalities*

$$(3)\qquad |\varphi_\lambda(z) - \beta_\lambda| \leqslant q_u^{\varrho_\lambda}$$

can be solved in $\mathfrak{o}^l$.

Proof. Let $L = (L_v)_v$ be the coherent system of lattices belonging to $E = k^l$, defined by

$$L_v = \begin{cases} \mathfrak{r}_v^l, & \text{if } v \neq u, \\ \text{the } K\text{-lattice given by (2)}, & \text{if } v = u. \end{cases}$$

As

$$q^{-\delta(L)} = \text{measure of } \prod_v L_v = \text{measure of } L_u = q^{d(\varrho_1+\dots+\varrho_l)-\delta} = q^{l(g-1)},$$

we have

$$(4)\qquad \lambda(L) = \lambda(L') - \delta(L) - l(g-1) = \lambda(L').$$

Condition (A) is equivalent to saying that

$$\Lambda(L) = E \cap \prod_v L_v = 0,$$

i.e., $\lambda(L) = 0$. Writing $\beta_1, \dots, \beta_l$ as $\varphi_1(\zeta), \dots, \varphi_l(\zeta)$ with ζ in $E_u = K^l$, we see that condition (B) amounts to: given $\zeta = (\zeta_1, \dots, \zeta_l)$ in E_u there exists z in $\mathfrak{o}^l$ such that $z - \zeta$ is in L_u. Interpreting it in the language of adeles, this amounts to saying that E_u, and hence $E + E_u$, is contained in $E + \prod_v L_v$; as $E + E_u$ is dense in E_A, and as $E + \prod_v L_v$ is an open, and hence closed subgroup of E_A, it follows that condition (B) is equivalent to:

$$E + \prod_v L_v = E_A;$$

this, in turn, is equivalent to the condition that $\lambda(L') = 0$. The theorem is now obvious by (4).

We now give some applications of the theory of successive minima, developed by us in [4], to transference problems. Thus, let $\varphi_\lambda(z)$ be as above, and let L be the K-lattice defined by

$$(5)\qquad N(z) = \max_\lambda |\varphi_\lambda(z)| \leqslant 1,$$

so that the measure of L is $|\Delta|^{-1} = q^{-\delta}$. If $\sigma_1, \dots, \sigma_l$ denote the successive minima of L, then, by [4], we have

$$(6)\qquad q^\delta \leqslant \sigma_1 \dots \sigma_l \leqslant q^{\delta + l(g-1+d)}.$$

Let $z^{(1)}, \dots, z^{(l)}$ be vectors in $\mathfrak{o}^l$ which are independent over K and which are such that

$$N(z^{(\lambda)}) = \sigma_\lambda, \quad 1 \leqslant \lambda \leqslant l.$$

Finally, let

$$(7)\qquad \sigma = \sup_{\zeta \in K^l} \inf_{z \in \mathfrak{o}^l} N(z - \zeta).$$

Take any ζ in $E_u = K^l$ and write it as

$$\zeta = \sum_\lambda \beta_\lambda z^{(\lambda)}, \quad \beta_\lambda \in K, \ 1 \leqslant \lambda \leqslant l;$$

by Lemma 2 of [5], find b_λ in $\mathfrak{o}$ such that

$$|\beta_\lambda - b_\lambda| \leqslant q^{2g-2+d}$$

and put

$$z = \sum_\lambda b_\lambda z^{(\lambda)};$$

then

$$N(z - \zeta) = N\Big(\sum_\lambda (\beta_\lambda - b_\lambda) z^{(\lambda)}\Big) \leqslant q^{2g-2+d} \sigma_l$$

and it follows that

$$(8)\qquad \sigma \leqslant q^{2g-2+d} \sigma_l.$$

Next, let t be the least integer such that $td > 2g-2$, $td \geqslant g$; thus

$$t = \begin{cases} 0 & \text{if } g = 0, \\ \left[\dfrac{2g-2}{d}\right] + 1 & \text{otherwise;} \end{cases}$$

since $\lambda(tu) = td - g + 1 \geqslant 1$, and $\lambda((t+1)u) = (t+1)d - g + 1$, it follows that there exists a non-unit β in $\mathfrak{o}$ such that

$$\operatorname{ord}_u(\beta) = -(t+1)$$

and hence

$$(9)\qquad |\beta| \leqslant q^e$$

where

$$(9')\qquad e = \begin{cases} d & \text{if } g = 0, \\ 2(g+d-1) & \text{if } g \geqslant 1. \end{cases}$$

Now call $\beta^{(\lambda)}$ the vector in $E_u = K^l$ having β^{-1} as its λth coordinate and having zero for its remaining coordinates. By the definition of σ, we can find a vector $x^{(\lambda)}$ in $\mathfrak{o}^l$ such that

$$N(\beta^{(\lambda)} - x^{(\lambda)}) \leqslant \sigma.$$

The vector

$$y^{(\lambda)} = \beta(\beta^{(\lambda)} - x^{(\lambda)})$$

is in $\mathfrak{o}^l$, and

$$N(y^{(\lambda)}) \leqslant q^e \sigma, \quad 1 \leqslant \lambda \leqslant l. \tag{10}$$

The matrix $(y_\mu^{(\lambda)})$ has entries $\equiv 0 \bmod \beta$ in the non-diagonal places and entries $\equiv 1 \bmod \beta$ in the diagonal ones; in particular,

$$\det(y_\mu^{(\lambda)}) \neq 0,$$

and hence $y^{(1)}, \ldots, y^{(l)}$ are independent over K. Consequently, by (10), we have

$$\sigma_l \leqslant q^e \sigma. \tag{11}$$

Suppose now that the inequalities

$$|\varphi_\lambda(z)| < 1, \quad 1 \leqslant \lambda \leqslant l, \tag{12}$$

have no non-zero solution in $\mathfrak{o}^l$. Then $\sigma_1 \geqslant 1$ and hence, by (6) and (8),

$$\sigma \leqslant q^{2g-2+d} \sigma_l \leqslant q^s \tag{13}$$

with

$$s = (l+2)(g-1) + (l+1)d + \delta. \tag{13'}$$

If now

$$\beta_\lambda = \varphi_\lambda(\zeta), \quad 1 \leqslant \lambda \leqslant l,$$

are arbitrary elements of K, then by (13) and the definition (7) of σ, there exists z in $\mathfrak{o}^l$ such that $N(z-\zeta) \leqslant q^s$, i.e.,

$$|\varphi_\lambda(z) - \beta_\lambda| \leqslant q^s, \quad 1 \leqslant \lambda \leqslant l. \tag{14}$$

Thus, we have proved

THEOREM 2. *If the inequalities* (12) *have no non-zero solution in* $\mathfrak{o}^l$, *then for every choice of* $\beta_1, \ldots, \beta_l$ *in* K, *the inequalities* (14) *have a solution* z *in* $\mathfrak{o}^l$.

On the other hand, suppose that the inequalities

$$|\varphi_\lambda(z) - \beta_\lambda| \leqslant 1, \quad 1 \leqslant \lambda \leqslant l, \tag{15}$$

can be solved for z in $\mathfrak{o}^l$, for every choice of $\beta_1, \ldots, \beta_l$ in K. Then $\sigma \leqslant 1$; therefore, by (11), $\sigma_l \leqslant q^e$, and hence by (6),

$$\sigma_1 \geqslant q^{\delta-(l-1)e}.$$

In other words:

THEOREM 3. *Suppose that for every choice of* $\beta_1, \ldots, \beta_l$ *in* K, *the inequalities* (15) *can be solved for* z *in* $\mathfrak{o}^l$. *Then the inequalities*

$$|\varphi_\lambda(z)| < q^{\delta-(l-1)e}, \quad 1 \leqslant \lambda \leqslant l,$$

have no non-zero solution in $\mathfrak{o}^l$.

Let now L' denote the lattice dual to L:

$$L' = \{w \in E_u = K^l \colon |z \cdot w| \leqslant 1 \text{ for all } z \text{ in } L\};$$

choose linear forms $\psi_\lambda(w)$, $1 \leqslant \lambda \leqslant l$, such that

$$\sum_\lambda \varphi_\lambda(z)\psi_\lambda(w) = \sum_\lambda z_\lambda w_\lambda;$$

then L' is given by

$$N'(w) = \max_\lambda |\psi_\lambda(w)| \leqslant 1.$$

If $\sigma_1', \ldots, \sigma_l'$ denote the successive minima of L', then [4],

$$1 \leqslant \sigma_\lambda \sigma'_{l-\lambda+1} \leqslant q^{l(g-1+d)}.$$

Combining this with (8) and (11) we get

$$q^{-e} \leqslant \sigma_1' \sigma \leqslant q^{(l+2)(g-1)+(l+1)d}.$$

This connects the homogeneous problem for L' with the inhomogeneous problem for L. For instance, the following result is an easy consequence of these considerations.

THEOREM 4. *Let* $\theta_1, \ldots, \theta_l$ *be elements of* K *which, together with* 1, *are linearly independent over* k. *Then, for any* $\beta_1, \ldots, \beta_l$ *in* K *and any* ϱ, *the inequalities*

$$\|\theta_\lambda z - \beta_\lambda\| \leqslant q_u^{-\varrho}, \quad 1 \leqslant \lambda \leqslant l,$$

can be solved for z *in* $\mathfrak{o}$.

Remark. The referee has remarked that the relation (6) can be improved to

$$\sigma_1 \ldots \sigma_l = q^{\delta + l\delta(-u)}$$

where $\delta(-u)$ denotes the dimension of the space of differentials with divisors $\geqslant -u$.

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Uniform distribution of third order linear recurrence sequences

by

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1. Introduction. Let $\{u_n\}$ be defined by

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \ldots + a_w u_{n-w} \quad \text{for} \quad n \geqslant w \tag{1}$$

and $u_0, u_1, \ldots, u_{w-1}$ given, where $u_0, u_1, \ldots, u_{w-1}, a_1, a_2, \ldots, a_w$ are all integers and $a_w \neq 0$. This is called a *linear recurrence* of order w.

A sequence is said to be *uniformly distributed modulo* m, written u.d. mod m, provided each residue modulo m appears with an asymptotic density of $1/m$.

Uniform distribution of recurrence sequences was first considered in the special case of the Fibonacci numbers. Kuipers and Shiue [2] showed that 5 is the only prime for which the Fibonacci numbers are uniformly distributed, and Niederreiter [6] showed that they are uniformly distributed mod 5^h for $h \geqslant 1$. Kuipers and Shiue [3] obtained sufficient conditions for a general second order recurrence to be uniformly distributed mod p^k. This question was completely settled when both necessary and sufficient conditions were obtained independently by Bumby [1], Nathanson [5], Long and Webb [7].

In this paper we consider uniform distribution of higher order sequences. The principal result, Theorem 3, gives necessary and sufficient conditions for a third order recurrence sequence $\{u_n\}$ to be uniformly distributed modulo M, where M is divisible only by primes $p > 5$.

2. General results on uniform distribution. The sequence $\{u_n\}$ is periodic modulo m for every m and is purely periodic mod m provided $(m, a_w) = 1$. It follows that $\{u_n\}$ is u.d. mod m if and only if each residue modulo m appears equally often in every period modulo m. Notice that in this paper, a period will not necessarily mean a least period.

The recurrence given in (1) has corresponding characteristic polynomial

$$c(x) = x^w - a_1 x^{w-1} - a_2 x^{w-2} - \ldots - a_w$$