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A note on a formula of van der Pol

by

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0. Introduction. In [4] van der Pol stated that

$$\sum_{n=-\infty}^{\infty} \frac{2^n}{a^{2^n}+1} = \frac{1}{\log a} \quad (a>1). \tag{1}$$

Replacing a by e^{2^t}, it follows from (1) that

$$\sum_{n=-\infty}^{\infty} \frac{2^{n+t}}{e^{2^{n+t}}+1} = 1 \quad (t\in \mathbf{R}), \tag{2}$$

showing that the evidently periodic function in the left-hand side of (2) is actually a constant function.

After this observation it seems natural to ask for all positive constants α and β such that the periodic function $f_{\alpha,\beta}\colon \mathbf{R}\to\mathbf{R}$ defined by

$$f_{\alpha,\beta}(t) = \sum_{n=-\infty}^{\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}}+1} \quad (t\in \mathbf{R}), \tag{3}$$

is actually constant as a function of t.

In Section 1 we present an elementary proof of (1) and make a remark on the functional equation for the zeta-function of Riemann.

In Section 2, using some theorems of Bohr, Landau and Putnam concerning the non-trivial zeros of Riemann's zeta-function, we prove the main result of this note, namely that $f_{\alpha,\beta}$ is constant if and only if $\alpha=\beta=(\log 2)/k$ for some fixed positive integer k.

1. Elementary proof of van der Pol's identity and a remark on the functional equation for $\zeta(s)$. Because of the identity

$$(1-e^{-x/2^n})\prod_{k=1}^{n}(1+e^{-x/2^k}) = 1-e^{-x} \tag{4}$$

we have for $x>0$

$$\sum_{k=1}^{n}\log(1+e^{-x/2^k}) = \log(1-e^{-x})-\log(1-e^{-x/2^n}). \tag{5}$$

Differentiating both sides of (5) we obtain

$$\sum_{k=1}^{n}\frac{-2^{-k}e^{-x2^{-k}}}{1+e^{-x2^{-k}}}=\frac{e^{-x}}{1-e^{-x}}-\frac{2^{-n}e^{-x2^{-n}}}{1-e^{-x2^{-n}}}\qquad (x>0), \tag{6}$$

or, equivalently,

$$\sum_{k=1}^{n}\frac{2^{-k}}{e^{x2^{-k}}+1}=\frac{1}{x}\cdot\frac{x2^{-n}}{e^{x2^{-n}}-1}-\frac{1}{e^{x}-1}\qquad (x>0). \tag{7}$$

Letting $n\to\infty$ we find

$$\sum_{k=1}^{\infty}\frac{2^{-k}}{e^{2^{-k}x}+1}=\frac{1}{x}-\frac{1}{e^{x}-1}\qquad (x>0). \tag{8}$$

In a similar manner one may derive

$$\sum_{n=0}^{\infty}\frac{2^{n}}{e^{2^{n}x}+1}=\frac{1}{e^{x}-1}\qquad (x>0), \tag{9}$$

from the identity

$$(1-e^{-x})\prod_{k=0}^{n-1}(1+e^{-2^{k}x})=1-e^{-2^{n}x}. \tag{10}$$

Combining (8) and (9) it follows that

$$\sum_{n=-\infty}^{+\infty}\frac{2^{n}}{e^{2^{n}x}+1}=\frac{1}{x}\qquad (x>0), \tag{11}$$

which is equivalent to (1).

Remark. Formula (8), which may serve to make the well-known formula (cf. [3], p. 23)

$$\Gamma(s)\zeta(s)=\int_{0}^{\infty}\left(\frac{1}{e^{x}-1}-\frac{1}{x}\right)x^{s-1}dx\qquad (0<\operatorname{Re}s<1) \tag{12}$$

almost trivial, may also be proved as follows. From

$$\cot z=\frac{1}{2}\left\{\cot\frac{z}{2}-\tan\frac{z}{2}\right\} \tag{13}$$

one obtains

$$\cot z=-\sum_{n=1}^{\infty}\frac{1}{2^{n}}\tan\frac{z}{2^{n}}+\frac{1}{z},$$

and from this, using Euler's formulas for $\sin z$ and $\cos z$, it is easily seen that (8) holds.

It is well known that (12) may be used to prove the functional equation for $\zeta(s)$, by observing that the function

$$f(x)=\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}}$$

is self-reciprocal for sine transforms, i.e.

$$f(t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin tx\,dx.$$

This well known fact, often proved by means of complex integration methods, may be proved in an elementary way as follows:

$$\begin{aligned}
\int_{0}^{\infty}f(x)\sin tx\,dx &= \int_{0}^{\infty}\left(\frac{1}{e^{x\sqrt{2\pi}}-1}-\frac{1}{x\sqrt{2\pi}}\right)\sin tx\,dx \\
&= -\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sin tx}{x}dx+\int_{0}^{\infty}\frac{\sin tx}{e^{x\sqrt{2\pi}}-1}dx \\
&= -\frac{1}{\sqrt{2\pi}}\cdot\frac{\pi}{2}+\int_{0}^{\infty}\frac{e^{itx}-e^{-itx}}{2i}\left(\sum_{k=1}^{\infty}e^{-kx\sqrt{2\pi}}\right)dx \\
&= -\frac{\sqrt{2\pi}}{4}+\sum_{k=1}^{\infty}\frac{t}{k^{2}\cdot 2\pi+t^{2}}=-\frac{\sqrt{2\pi}}{4}+\sum_{k=1}^{\infty}\frac{t2\pi}{4k^{2}\pi^{2}+(t\sqrt{2\pi})^{2}} \\
&= -\frac{\sqrt{2\pi}}{4}+\frac{\sqrt{2\pi}}{2}\left\{\frac{1}{e^{t\sqrt{2\pi}}-1}-\frac{1}{t\sqrt{2\pi}}+\frac{1}{2}\right\}=\sqrt{\frac{\pi}{2}}f(t).
\end{aligned} \tag{14}$$

The relation

$$\frac{1}{e^{u}-1}-\frac{1}{u}+\frac{1}{2}=\sum_{k=1}^{\infty}\frac{2u}{4k^{2}\pi^{2}+u^{2}} \tag{15}$$

which is crucial in (14) may be obtained directly from (13) as is shown by Euler and Schröter ([1], pp. 204–207). Putting things together we see, following Titchmarsh ([3], p. 23), that the functional equation for $\zeta(s)$ may be based almost entirely upon (13).

2. Fourier analytical approach to van der Pol's identity. In the previous section it turned out that the periodic function

$$f(t)=\sum_{n=-\infty}^{\infty}\frac{2^{n+t}}{e^{2^{n+t}}+1}\qquad (t\in\boldsymbol{R}),$$

is actually constant.

The question arises whether there are any other series of a similar type with a constant sum.

In this note we will only consider this question for the functions $f = f_{\alpha,\beta}$ defined by (3). It is easily seen that the functions $f_{\alpha,\beta}$ are well defined on $\boldsymbol{R}$, are periodic with period 1 and may be represented on $\boldsymbol{R}$ by their Fourier series.

Before carrying out any Fourier analysis on the functions $f_{\alpha,\beta}$ we will show directly that there are indeed some constant functions $f_{\alpha,\beta}$ different from the case $\alpha = \beta = \log 2$.

PROPOSITION. *If $f_{\alpha,\beta}$ is constant, then the functions $\varphi_{k,\alpha,\beta}\colon \boldsymbol{R} \to \boldsymbol{C}$ $(k = 1, 2, 3, \ldots)$, defined by*

$$\varphi_{k,\alpha,\beta}(t) = \sum_{n=-\infty}^{+\infty} c_n \frac{e^{\frac{\alpha}{k}(n+t)}}{e^{e^{\frac{\beta}{k}(n+t)}}+1} \qquad (t \in \boldsymbol{R}),$$

where

$$c_m \in \boldsymbol{C} \quad \text{and} \quad c_{m+k} = c_m \quad \text{for all } m \in \boldsymbol{Z},$$

are also constant.

Proof.

$$\varphi_{k,\alpha,\beta}(t) = \sum_{r=1}^{k} \sum_{n=-\infty}^{+\infty} c_{nk+r} \frac{e^{\frac{\alpha}{k}(nk+r+t)}}{e^{e^{\frac{\beta}{k}(nk+r+t)}}+1}$$

$$= \sum_{r=1}^{k} c_r \sum_{n=-\infty}^{\infty} \frac{e^{\alpha\left(n+\frac{r}{k}+\frac{t}{k}\right)}}{e^{e^{\beta\left(n+\frac{r}{k}+\frac{t}{k}\right)}}+1}$$

$$= \sum_{r=1}^{k} c_r f_{\alpha,\beta}\left(\frac{r}{k}+\frac{t}{k}\right) = f_{\alpha,\beta}(t) \sum_{r=1}^{k} c_r. \blacksquare$$

COROLLARY. *If $\alpha = \beta = \log 2/k$ for some positive integer k then $f_{\alpha,\beta}$ is constant.*

We now observe that $f_{\alpha,\beta}$ is constant if and only if all Fourier coefficients

$$\int_0^1 f(t) e^{2\pi\nu t i} dt \qquad (\nu \in \boldsymbol{Z} \setminus \{0\})$$

are simultaneously equal to zero.

These Fourier coefficients may be computed as follows:

$$\int_0^1 f(t) e^{2\pi\nu t i} dt = \int_0^1 e^{2\pi\nu t i} \left(\sum_{n=-\infty}^{+\infty} \frac{e^{\alpha(n+t)}}{e^{e^{\beta(n+t)}}+1} \right) dt$$

$$= \sum_{n=-\infty}^{+\infty} \int_0^1 \frac{e^{\alpha(n+t)+2\pi\nu t i}}{e^{e^{\beta(n+t)}}+1} dt = \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{\alpha u+\nu(u-n)2\pi i}}{e^{e^{\beta u}}+1} du$$

$$= \sum_{n=-\infty}^{+\infty} \int_n^{n+1} \frac{e^{\alpha u+\nu u 2\pi i}}{e^{e^{\beta u}}+1} du = \int_{-\infty}^{+\infty} \frac{e^{\alpha u+\nu u 2\pi i}}{e^{e^{\beta u}}+1} du$$

$$= \frac{1}{\beta} \int_0^{\infty} \frac{w^{\left(\frac{\alpha}{\beta}+\frac{2\pi\nu}{\beta}i\right)-1}}{e^w+1} dw = \frac{1}{\beta} \Gamma(s)\,(1-2^{1-s})\,\zeta(s),$$

where $s = \dfrac{\alpha}{\beta} + \dfrac{2\pi\nu}{\beta} i$.

Hence, in order to have that $f_{\alpha,\beta}$ is constant we must have

$$(1-2^{1-s})\zeta(s) = 0 \qquad \left(s = \frac{\alpha}{\beta} + \frac{2\pi\nu}{\beta} i;\ \nu \in \boldsymbol{Z} \setminus \{0\}\right),$$

since the Γ-function has no zeros at all. This may be rephrased by saying that $f_{\alpha,\beta}$ is constant if and only if the set of zeros of $(1-2^{1-s})\zeta(s)$ contains a subset of the form $\left\{\dfrac{\alpha}{\beta} + n\dfrac{2\pi i}{\beta}\right\}_{n \in \boldsymbol{Z}\setminus\{0\}}$.

Since the zeros of $(1-2^{1-s})\zeta(s)$ come in conjugate pairs we may just as well say that $f_{\alpha,\beta}$ is constant if and only if the set of zeros of $(1-2^{1-s}) \times \zeta(s)$ contains an arithmetical progression $\left\{\dfrac{\alpha}{\beta} + n\dfrac{2\pi i}{\beta}\right\}_{n=1}^{\infty}$. Denoting $\dfrac{\alpha}{\beta}$ by σ we consider the following five cases.

Case 1. $\sigma > 1$. Since $\zeta(s) \neq 0$ and $1-2^{1-s} \neq 0$ for $\operatorname{Re} s > 1$ it is clear that $f_{\alpha,\beta}$ *cannot* be a constant function.

Case 2. $\sigma = 1$. Since $\zeta(s) \neq 0$ for $\operatorname{Re} s = 1$ and $1-2^{1-s} = 0$ only at the points $s = 1 + \dfrac{2\pi r i}{\log 2}$, $r \in \boldsymbol{Z}$, it is easily seen that $f_{\alpha,\beta}$ is a constant function only if $(\alpha =)\,\beta = \dfrac{\log 2}{k}$ for some positive integer k.

Case 3. $\frac{1}{2} < \sigma < 1$. Since $1-2^{1-s} \neq 0$ for $\operatorname{Re} s < 1$ and since the number of zeros of $\zeta(s)$ on the line $\operatorname{Re} s = \sigma$ having ordinates (in absolute value) not exceeding $T > 0$, is $o(T)$ ([3], p. 202, Theorem 9.19(A)), we may conclude that the set of non-trivial zeros of $\zeta(s)$ at the right of the critical line $\operatorname{Re} s = \frac{1}{2}$ does *not* contain an arithmetical progression. It follows that $f_{\alpha,\beta}$ is *not* constant.

Case 4. $\sigma = \frac{1}{2}$. In this case we invoke a theorem of Putnam [2] saying that the set of zeros of $\zeta(s)$ on the critical line $\operatorname{Re} s = \frac{1}{2}$ does not contain an arithmetical progression. It follows that also in this case $f_{\alpha,\beta}$ *cannot* be constant.

Case 5. $0 < \sigma < \frac{1}{2}$. Because of the functional equation for $\zeta(s)$ this case may be reduced to Case 3.

Summarizing, we have the following

THEOREM. *If α and β are positive constants and $f_{\alpha,\beta}$: $\boldsymbol{R}\to\boldsymbol{R}$ is defined by* (3) *then $f_{\alpha,\beta}$ is a constant function only in case $\alpha = \beta = \dfrac{\log 2}{k}$, where k is any positive integer.*

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References

[1] K. Knopp, *Theory and application of infinite series*, Blackie and Son Ltd., 1928.

[2] C. R. Putnam, *On the non-periodicity of the zeros of the Riemann zeta-function*, Amer. J. Math. 76 (1954), pp. 97–99.

[3] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

[4] *Wiskundige Opgaven*, Noordhoff N. V., Groningen, 19, I (1950), pp. 308–311.

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Some results in number theory, I

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Dedicated to the memory of Professor Paul Turán

Let $\varphi(n)$ denote Euler's totient function and $V(n)$ the number of distinct prime factors of n. In this paper, we shall study the quantity $V\big((n, \varphi(n))\big)$ which arises naturally in group theory. For example, letting $G(n)$ denote the number of non-isomorphic groups of order n, we have by a classical result of Burnside that $G(n) = 1$ if and only if $V\big(n, \varphi(n)\big) = 0$ (i.e. $\big(n, \varphi(n)\big) = 1$). Erdös [1] showed that the number $F_1(x)$ of $n \leqslant x$ satisfying the latter condition is

$$F_1(x) = \big(1+o(1)\big)xe^{-\gamma}/\log_3 x \tag{1}$$

where γ is Euler's constant and we write $\log_1 x = \log x$, $\log_a x = \log(\log_{a-1} x)$. More generally, we can define $F_k(x)$ to be the number of $n \leqslant x$ for which $G(n) = k$. The authors [2] have shown that for each k,

$$F_k(x) \ll x/\log_4 x.$$

The proof depended essentially on a weak form of the following result stated by Erdös in [1]: for each $\varepsilon > 0$, the number of $n \leqslant x$ that fail to satisfy

$$(1-\varepsilon)\log_4 n < V\big(n, \varphi(n)\big) < (1+\varepsilon)\log_4 n$$

is $o(x)$. (A proof of this was supplied by the authors in [2].)

It is an interesting number-theoretic problem to estimate the number $A_k(x)$ of $n \leqslant x$ for which $V\big(n, \varphi(n)\big) = k$. Our main result here is the following theorem.

THEOREM. *For each $k \geqslant 0$, we have*

$$A_k(x) = \frac{\big(1+o(1)\big)xe^{-\gamma}(\log_4 x)^k}{k!\log_3 x}. \tag{2}$$

The proof will require several lemmas and intermediate results. The first two lemmas are due to Erdös [1].