Therefore we may assume that $M'^2-3N' = \pm 3^v$. Using $\varepsilon = (-1)^k$ and (7), we see this can be written as

$$\pm 3^v + 2^{k+1} = (-1)^{k+1}.$$

The sign surely must be negative; and so by a theorem of LeVeque ([4], [2]), we have $(v, k) = (2, 2), (1, 1)$, or $(1, 0)$. We have already seen that $k \geqslant 2$. So $k = 2$ and the corresponding values of N' and M' are

$$N' = 5 \quad \text{and} \quad M' = \pm\sqrt{6}.$$

But then $V_6' = 236$ which is divisible by 59. If $\sqrt{59} \mid U_r'$, then $\sqrt{59} \mid (U_{12}', U_r') = |U_{(12,r)}'| = |M'|$. But then since $59 \mid V_6' = \{M'(M'^2-3N')\}^2 - 2N'^3$, we have $59 \mid N'$ contrary to $(M'^2, N') = 1$. So $59 \nmid Nd$, and Theorem 2 shows that $\{a_{12n}\}$ contains no more than two occurrences of d; therefore $m(d) \leqslant 4$.

The only remaining case is that in which $3 \mid N'$. By (7), we see then that $\varepsilon = (-1)^{k+1}$ and so

$$M'^2 = 2\big(2^{k-1}+(-1)^{k+1}\big) \quad \text{and} \quad N' = 2^k + (-1)^{k+1}.$$

If $k = 2$, then $m(d) \leqslant 4$ by Lemma 13. For $k \geqslant 3$, the result follows from Lemma 20, and so the proof of the theorem is complete.

References

[1] R. Alter and K. K. Kubota, *Multiplicities of second order linear recurrences*, Trans. Amer. Math. Soc. 178 (1973), pp. 271–284.

[2] J. W. S. Cassels, *On the diophantine equation $a^x - b^y = 1$*, Amer. J. Math. 75 (1953), pp. 159–162.

[3] K. K. Kubota, *On a conjecture of Morgan Ward, I*, Acta Arith., this volume, pp. 11–28.

[4] W. J. LeVeque, *On the equation $a^x - b^y = 1$*, Amer. J. Math. 74 (1952), pp. 325–331.

[5] W. Ljunggren, *Noen setninger om ubestemte likninger av formen $(x^n-1)/(x-1) = y^q$*, Norsk Mat. Tidsskr. 25 (1943), pp. 17–20.

[6] L. J. Mordell, *Diophantine Equations*, Academic Press, London 1969.

[7] — *The diophantine equation $y^2 = Dx^4+1$*, J. London Math. Soc. 39 (1964), pp. 161–164.

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Proper solutions of the imbedding problem with restricted ramification

by

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Let k be a field, $\bar{k}$ its separable algebraic closure with the Galois group $\mathfrak{G} = \mathrm{Gal}(\bar{k}/k)$. An imbedding problem is defined by a diagram

$$\text{(1)} \qquad \begin{array}{ccccccccc} & & & & & & \mathfrak{G} & & \\ & & & & & & \downarrow{\scriptstyle\varphi} & & \\ \{1\} & \longrightarrow & A & \longrightarrow & G & \xrightarrow{j} & F & \longrightarrow & \{1\} \end{array}$$

where A, G, F denote finite groups. All arrows are group homomorphisms and the horizontal sequence is exact. We assume φ surjective. Hence, the kernel of φ, $\mathfrak{G}_0 = \mathrm{Ker}\,\varphi$, determines a finite normal extension K/k with $\mathrm{Gal}(K/k) \simeq F$. A solution of the imbedding problem (1) is by definition a homomorphism $\psi\colon \mathfrak{G} \to G$ satisfying the condition $j \circ \psi = \varphi$. ψ is called a *proper solution* if and only if it is surjective.

Let k be a global field, i.e., a finite algebraic number field or an algebraic function field of one variable over a finite constant field. By k_S we denote the maximal normal extension of k unramified outside the given set of primes S. Let $\mathfrak{G}_S$ be the group $\mathrm{Gal}(k_S/k)$. If S contains all ramification points of the extension K/k occurring in the diagram (1), we can factorize φ through the group $\mathfrak{G}_S$:

$$\text{(1}_S\text{)} \qquad \begin{array}{ccccccccc} & & & & \mathfrak{G} & \xrightarrow{\pi_S} & \mathfrak{G}_S & & \\ & & & & & & \downarrow{\scriptstyle\varphi_S} & & \\ \{1\} & \longrightarrow & A & \longrightarrow & G & \xrightarrow{j} & F & \longrightarrow & \{1\}. \end{array}$$

We say (1) admits a solution ψ unramified outside S if and only if (1_S) admits a solution $\psi_S\colon \mathfrak{G}_S \to G$ with $j \circ \psi_S = \varphi_S$ and $\psi = \psi_S \circ \pi_S$ where π_S denotes the canonical epimorphism $\mathfrak{G} \to \mathfrak{G}_S$.

The main result of the present paper is the following

THEOREM. *Suppose k is a global field, K/k is a given finite normal extension with $F = \mathrm{Gal}(K/k)$, A a finite F-module with* $\mathrm{char}\,k \nmid \mathrm{card}\,A$. *Then every imbedding problem* (1) *with at least one solution admits a proper solution unramified outside a suitable finite set of primes $S \cup \{\mathfrak{q}_1, \ldots, \mathfrak{q}_m\}$. Here S depends only on n = exponent of A and on the extension $K(\zeta_n)/k$ (ζ_n denotes a primitive n-th root of unity). The set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_m\}$ is disjoint to S and consists of m different primes totally decomposed in the extension $k(A, \zeta_n)/k$, m being the length of a F-composition series of A. (By $k(A, \zeta_n)/k$ we denote the field $k(A)(\zeta_n)$ where $k(A)$ is the field corresponding to the fix group of A.)*

The theorem constitutes a strengthened version of results of Ikeda [3] and Iškhanov [4].

For the proof of the theorem, we introduce the following notations: $k_\mathfrak{p}$ – the $\mathfrak{p}$-adic closure of k; $\bar{k}_\mathfrak{p}$ — a separable algebraic closure of $k_\mathfrak{p}$; $k_\mathfrak{p}^{nr}$ — the maximal unramified extension of $k_\mathfrak{p}$ contained in $\bar{k}_\mathfrak{p}$; $T_\mathfrak{p} = \mathrm{Gal}(\bar{k}_\mathfrak{p}/k_\mathfrak{p}^{nr})$; $H^1_{nr}(k_\mathfrak{p}, A) = \mathrm{Im}[\mathrm{inf}\colon H^1(k_\mathfrak{p}^{nr}/k_\mathfrak{p}, A^{T_\mathfrak{p}}) \to H^1(\bar{k}_\mathfrak{p}/k_\mathfrak{p}, A)]$.

The following key proposition is derived from Tate's global duality theorem ([9], [2]).

PROPOSITION (see [6], Behauptung). *Under the assumptions of our theorem there exists a finite set of primes S containing all ramification points of K/k and depending only on n = exponent of A and on the extension $K(\zeta_n)/k$ such that for any finite set T of primes disjoint to S and any finite F-module A with* $\mathrm{char}\,k \nmid \mathrm{card}\,A$ *the canonical map*

$$H^1(k_{S\cup T}/k, A) \to \sum_{\mathfrak{p}\in T} H^1(k_\mathfrak{p}, A)/H^1_{nr}(k_\mathfrak{p}, A)$$

is surjective.

In the paper [6] the author described a class of sets S satisfying the conditions of this proposition.

Proof of the theorem. We fix a diagram (1) with at least one solution. Suppose that the corresponding group extension is given by the cohomology class $\varepsilon \in H^2(F, A)$. Let be $\mathfrak{G}_0 = \mathrm{Ker}\,\varphi = \mathrm{Gal}(\bar{k}/K)$. It is well-known that the existence of a solution of (1) amounts to the existence of a certain element $\chi \in H^1(\mathfrak{G}_0, A)^F$ going over *via* the transgression map $H^1(\mathfrak{G}_0, A)^F \to H^2(F, A)$ into the given class $\varepsilon \in H^2(F, A)$. The whole set of all solutions of (1) is described by the sums $\chi + \omega \in H^1(\mathfrak{G}_0, A)^F$ where $\omega = \mathrm{Res}\,\alpha$, $\alpha \in H^1(\mathfrak{G}, A)$, Res: $H^1(\mathfrak{G}, A) \to H^1(\mathfrak{G}_0, A)^F$ (cf. Neukirch [5], § 1). The analogous facts hold for the solutions of (1_S).

By results of Iškhanov [4] and the author [6], there exists a finite set of primes S with all properties required in our proposition such that our soluble imbedding problem (1) has a solution, ψ (say), unramified outside S. Starting from this solution we proceed by induction on m (= length of the F-module A). Choose the F-submodule A_1 of A such

that A/A_1 is an irreducible F-module $\neq \{0\}$. Obviously, A_1 is a normal subgroup in G, and the canonical epimorphism $G \to G/A_1$ yields a new imbedding problem:

(2)
$$\begin{array}{ccccccccc} & & & & G & \xleftarrow{\ \psi\ } & \mathfrak{G} & & \\ & & & & \downarrow & \swarrow_{\bar{\psi}} & \downarrow{\scriptstyle\varphi} & & \\ \{1\} & \longrightarrow & A/A_1 & \longrightarrow & G/A_1 & \longrightarrow & F & \longrightarrow & \{1\}. \end{array}$$

Every solution ψ of (1) gives us a solution $\bar{\psi}$ of (2).

Let $\mathfrak{q}_1 = \mathfrak{q}$ be a prime of the ground field k totally decomposed in the extension $k(A, \zeta_n)/k$ and different from all primes lying in S. By the well-known facts (see an elementary proof by Dress [1]), such a prime ideal always exists. Set $L_\mathfrak{q} = k_\mathfrak{q}(\sqrt[n]{\pi_\mathfrak{q}})$ where $\pi_\mathfrak{q}$ denotes a prime element for $\mathfrak{q}$. $L_\mathfrak{q}/k_\mathfrak{q}$ is a totally ramified cyclic extension of degree n. Let $a \in A$ be an arbitrary element of A with $a \notin A_1$. If we map a generator of $\mathrm{Gal}(L_\mathfrak{q}/k_\mathfrak{q})$ onto the element a, we get a non-trivial homomorphism

$$\eta_\mathfrak{q}\colon \mathrm{Gal}(L_\mathfrak{q}/k_\mathfrak{q}) \to A$$

whereas the composition with the canonical map $A \to A/A_1$ is still a non-trivial homomorphism

$$\bar{\eta}_\mathfrak{q}\colon \mathrm{Gal}(L_\mathfrak{q}/k_\mathfrak{q}) \to A/A_1.$$

By the choice of $\mathfrak{q}$ the group $\mathrm{Gal}(\bar{k}_\mathfrak{q}/k_\mathfrak{q})$ acts trivially on A, and we get elements

$$\eta_\mathfrak{q} \in H^1(k_\mathfrak{q}, A), \qquad \eta_\mathfrak{q} \notin H^1_{nr}(k_\mathfrak{q}, A),$$
$$\bar{\eta}_\mathfrak{q} \in H^1(k_\mathfrak{q}, A/A_1) \qquad \bar{\eta}_\mathfrak{q} \notin H^1_{nr}(k_\mathfrak{q}, A/A_1).$$

By the proposition stated above, there is an element α' from $H^1(\mathfrak{G}_{S\cup\{\mathfrak{q}\}}, A)$ which localizes at $\mathfrak{q}$ just to the given class

$$\eta'_\mathfrak{q} = \eta_\mathfrak{q} \bmod H^1_{nr}(k_\mathfrak{q}, A) = \alpha'.$$

Set $\mathfrak{G}' = \mathfrak{G}_{S\cup\{\mathfrak{q}\}}$, $\mathfrak{G}'_0 = \mathrm{Gal}(k_{S\cup\{\mathfrak{q}\}}/K)$. To the solution ψ of (1) unramified outside S corresponds a solution ψ_S of (1_S) and we can associate to ψ_S an element $\chi_S \in H^1(k_S/K, A)^F$. Then

$$\omega' = \mathrm{Res}(\mathfrak{G}' \to \mathfrak{G}'_0)\,\alpha' \in H^1(k_{S\cup\{\mathfrak{q}\}}/K, A)^F$$

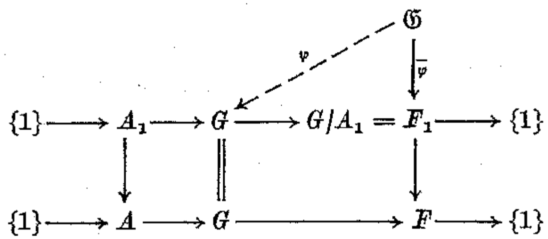
and

$$\chi'_S = \mathrm{inf}\big(H^1(k_S/K, A)^F \to H^1(k_{S\cup\{\mathfrak{q}\}}/K, A)^F\big)\chi_S$$

give a new solution ψ' of (1) corresponding to $\chi'_S + \omega' \in H^1(k_{S\cup\{\mathfrak{q}\}}/K, A)^F$ and a solution $\bar{\psi}'$ of (2) corresponding to $(\chi'_S + \omega') \in H^1(k_{S\cup\{\mathfrak{q}\}}/K, A/A_1)^F$. Thus we have solutions ψ' resp. $\bar{\psi}'$ unramified outside $S \cup \{\mathfrak{q}_1\}$. This last solution induces a proper solution of (2) because localization at $\mathfrak{q}$ and the map $A \to A/A_1$ show that the fixed field, L (say), of the kernel of $\overline{(\chi'_S + \omega')}$

does not coincide with K. On the other hand, $\mathrm{Gal}(L/K)$ is a F-submodule of A/A_1, hence, in virtue of the F-irreducibility of A/A_1, it must be isomorphic to A/A_1. In this way, the solution $\bar{\psi}$ defines a third imbedding problem:

(3)



F_1 acts *via* the canonical epimorphism $F_1 \to F$ on the F-module A_1. The F_1-length of the F_1-module A_1 is not greater than $(m-1)$. Now by induction the proof is complete because for the new module A_1 the field $k(A_1, \zeta_n)$ is contained in the field $k(A, \zeta_n)$.

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References

[1] A. Dress, *Zu einem Satz aus der Theorie der algebraischen Zahlen*, J. Reine Angew. Math. 216 (1964), pp. 218–219.

[2] K. Haberland, *Der Tatesche Dualitätssatz aus der Galois-Kohomologie über Zahlkörpern*, Dissertation, Berlin 1975.

[3] M. Ikeda, *Zur Existenz eigentlicher galoisscher Körper beim Einbettungsproblem*, Abh. Math. Sem. Univ. Hamburg 24 (1960), pp. 126–131.

[4] В. В. Ишханов, *Задача погружения с ограниченным ветвлением*, Известия АН СССР, серия матем. 36: 4 (1972), pp. 742–748.

[5] J. Neukirch, *Über das Einbettungsproblem der algebraischen Zahlentheorie*, Invent. Math. 21 (1973), pp. 59–116.

[6] O. Neumann, *Über das Einbettungsproblem für globale Körper bei beschränkter Verzweigung*, Math. Nachr. 71 (1976), pp. 147–152.

[7] G. Poitou, *Cohomologie Galoisienne des modules finis*, Paris 1967.

[8] J.-P. Serre, *Cohomologie Galoisienne*, Lecture notes in mathematics 5, Berlin–Göttingen–Heidelberg 1964.

[9] J. Tate, *Duality theorems in Galois cohomology over number fields*, Proceed. Intern. Congr. Math. Stockholm, 1962, pp. 288–292.

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A new equidistribution property of norms of ideals in given classes

by

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0. Introduction. In [4] the author obtained the following theorem:

Let K be a finite extension of Q, the rational field. If $\{\mathscr{C}_j\}_{j\in J}$ is any non-empty collection of narrow ideal classes of K, then the number of natural numbers $\leqslant x$ which are norms of integral ideals in $\bigcup_{j\in J} \mathscr{C}_j$ is asymptotically

$$D(K, J)x(\log x)^{E(K)-1}\{1 + O_{K,J}(\log x)^{-C(K,J)}\}, \tag{0.1}$$

where $D(K, J)$ and $C(K, J)$ are positive and $E(K)$ is the Dirichlet density of the set of rational primes admitting in K at least one prime ideal factor of residual degree unity.

Owing to the great complexity of the proof of (0.1) it was not feasible in [4] to attempt a discussion of the relations between the $D(K, J)$, as J varies. It is natural to expect that $D(K, J_1)$ equals $D(K, J_2)$ if J_1 and J_2 are singletons, since the weighted sums

$$\sum_{\mathfrak{a}\in\mathscr{C}, N\mathfrak{a}\leqslant x} 1 \tag{0.2}$$

are well-known to be asymptotically the same for all classes $\mathscr{C}$. However, the unweighted sums in (0.1) are much more difficult to handle. In this paper, we shall prove the following results:

THEOREM 1. *For singletons J_i, $D(K, J_1) = D(K, J_2)$.*

THEOREM 2. *If K/Q is normal, then all but a proportion*

$$O_K\big((\log\log x)^{A(K)}/(\log x)^{B(K)}\big)$$

of the integers $\leqslant x$ which are norms of integral ideals in a given class $\mathscr{C}$ are norms of integral ideals of each class in the coset $\mathscr{C}H$, where H is the group of narrow classes containing fractional ideals of norm unity. (The constant $B(K)$ is positive.)

We remark that if $n = N\mathfrak{a} = N\mathfrak{b}$, where $\mathfrak{a}\in\mathscr{C}$ and $\mathfrak{b}\in\mathscr{D}$, then $\mathscr{C}\mathscr{D}^{-1}\in H$, so $\mathscr{C}H = \mathscr{D}H$, and this indicates the strength of Theorem 2. We also prove