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Generalization of some theorems on sets of multiples and primitive sequences

by

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1. Introduction. The main results of this paper are generalizations of a theorem of Besicovitch on primitive sequences and of a theorem of Davenport and Erdös on sets of multiples. For these theorems and a survey of related results we refer to the final chapter of Halberstam and Roth [3].

By a *system* σ we mean a non-empty set of finite, non-empty sets of positive integers. The system σ is called *homogeneous*, if for each $n \in N$ (set of positive integers)

$$S \in \sigma \text{ implies } nS = \{ns\colon s \in S\} \in \sigma.$$

The set $A \subset N$ is said to be σ-free, if it does not contain a subset belonging to σ. For a given homogeneous system σ we discuss the question of the 'greatest possible density' a σ-free set may have. We investigate natural densities and logarithmic densities of σ-free sets.

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2. Natural densities of σ-free sets. First we introduce some notations. For real numbers α, β we define the interval $[\alpha, \beta] = \{n\colon n \in N,\ \alpha \leqslant n \leqslant \beta\}$. If A is a finite set, then $|A|$ denotes the number of elements in A. The counting function of $A \subset N$ is $A(n) = |A \cap [1, n]|$. The limit $d(A) = \lim_{n\to\infty} A(n)/n$, if it exists, is called the natural density of A. The lower and upper natural densities $\underline{d}(A)$ and $\bar{d}(A)$ are defined by the lim inf and lim sup of the same expression. The system σ is characterized by

$$\tau_\sigma(n) = \max\{A(n)\colon A\ \sigma\text{-free}\},$$

$$\underline{\tau}_\sigma = \liminf_{n\to\infty} \tau_\sigma(n)/n, \qquad \bar{\tau}_\sigma = \limsup_{n\to\infty} \tau_\sigma(n)/n.$$

If $\underline{\tau}_\sigma = \bar{\tau}_\sigma$ let $\tau_\sigma = \underline{\tau}_\sigma = \bar{\tau}_\sigma$. Furthermore, we define

$$\underline{d}(\sigma) = \sup\{\underline{d}(A)\colon A\ \sigma\text{-free}\}, \qquad \bar{d}(\sigma) = \sup\{\bar{d}(A)\colon A\ \sigma\text{-free}\}.$$

If $\underline{d}(\sigma)$ and $\bar{d}(\sigma)$ coincide, the common value is denoted by $d(\sigma)$.

Every system σ_0 generates a homogeneous system σ,

$$\sigma = N\sigma_0 = \{T\colon\ T = nS,\ n\in N,\ S\in\sigma_0\}.$$

The investigation of a homogeneous system is facilitated by a small generating system.

THEOREM 1. *Suppose that the homogeneous system σ is generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. Let $a_1, a_2, \ldots, a_k$ be coprime integers greater than 1 and*

$$U = \{u\colon\ u = a_1^{r_1}a_2^{r_2}\ldots a_k^{r_k},\ r_i\in\{0\}\cup N\}.$$

If $S_i \subset U$ for each i then τ_σ exists and

$$\tau_\sigma(n) = \tau_\sigma n + O(\log^k n).$$

Here τ_σ is less than 1. There is a σ-free set A with $\bar{d}(A) = \tau_\sigma$.

Proof. Denote by V the sequence of positive integers which are not a multiple of any of the numbers a_j. It is well-known that

$$(1)\qquad V(n) = nd(V) + O(1), \quad \text{where} \quad d(V) = \prod_{j=1}^{k}\left(1-\frac{1}{a_j}\right).$$

We have

$$(2)\qquad \sum_{u\in U}\frac{1}{u} = \frac{1}{d(V)} \quad \text{and} \quad U(n) \leqslant \left(1+\frac{\log n}{\log 2}\right)^k.$$

Every positive integer has a unique representation of the form uv, $u\in U$, $v\in V$. Therefore, it follows from (1) and (2)

$$n = \sum_{u\leqslant n} V\left(\frac{n}{u}\right) = nd(V)\sum_{u\leqslant n}\frac{1}{u} + O\big(U(n)\big),$$

$$(3)\qquad \sum_{u>n}\frac{1}{u} = O\left(\frac{U(n)}{n}\right) = O\left(\frac{\log^k n}{n}\right),$$

where summation is taken over the numbers $u\in U$.

If we define

$$\tau_\sigma^U(n) = \max\{A(n)\colon\ A\subset U,\ A\ \sigma\text{-free}\},$$

and if R denotes the unique subset of U having the counting function $R(n) = \tau_\sigma^U(n)$ then

$$\tau_\sigma(n) = \sum_{v\in V}\tau_\sigma^U\left(\frac{n}{v}\right) = \sum_{v\in V}R\left(\frac{n}{v}\right).$$

Thus, by (1),

$$\tau_\sigma(n) = \sum_{r\in R}V\left(\frac{n}{r}\right) = nd(V)\sum\frac{1}{r} - nd(V)\sum_{r>n}\frac{1}{r} + O\big(R(n)\big),$$

where summation is over $r\in R$. By $R\subset U$, it now follows from (2) and (3) that

$$(4)\qquad \tau_\sigma(n) = \tau_\sigma n + O(\log^k n), \quad \text{where} \quad \tau_\sigma = d(V)\sum_{r\in R}\frac{1}{r}.$$

Since σ and the sets S_i are non-empty by definition, R is a proper subset of U. We have

$$\sum_{r\in R}\frac{1}{r} < \sum_{u\in U}\frac{1}{u} = \frac{1}{d(V)},$$

whence $\tau_\sigma < 1$. The existence of a σ-free set A with $\bar{d}(A) = \tau_\sigma$ is ensured by the following lemma.

LEMMA 1. *Suppose that the homogeneous system σ is generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. If $M = \{z\colon\ z = \max S_i,\ S_i\in\sigma_0\}$ has natural density 0, then there is a σ-free set A with $\bar{d}(A) = \bar{\tau}_\sigma$.*

Proof. Let ε_j $(j = 1, 2, \ldots)$ be positive numbers satisfying $0 < \varepsilon_j < 1$ and $\lim\limits_{j\to\infty}\varepsilon_j = 0$. There is a sequence of integers x_j starting with $x_0 = 0$ and having the following properties for $j > 0$.

(a) $x_j > \dfrac{1}{\varepsilon_j}x_{j-1}$,

(b) $\tau_\sigma(x_j) > (\bar{\tau}_\sigma - \varepsilon_j)x_j$,

(c) if $T_{j-1} = \{mn\colon\ m\in M,\ n\in[1, x_{j-1}]\}$ then $T_{j-1}(x_j) < \varepsilon_j x_j$.

Let A_j' be a σ-free set in $[1, x_j]$ with $|A_j'| > (\bar{\tau}_\sigma - \varepsilon_j)x_j$. Using the notation $B\urcorner C = \{z\colon\ z\in B,\ z\notin C\}$ we define

$$A_j = A_j'\urcorner([1, x_{j-1}]\cup T_{j-1}), \qquad A = \bigcup_{j=1}^{\infty}A_j.$$

The sets A_j are disjoint and σ-free. From (a), (b), (c) we obtain

$$A(x_j) \geqslant A_j(x_j) > (\bar{\tau}_\sigma - 3\varepsilon_j)x_j,$$

hence $\bar{d}(A) \geqslant \bar{\tau}_\sigma$. Assume now that A contains a set nS_i. Let $d = \min S_i$ and $D = \max S_i$. Since the sets A_j are σ-free, we must have

$$nd\in A_k, \quad nD\in A_q, \quad k < q.$$

From $n \leqslant x_k \leqslant x_{q-1}$ and $D\in M$ follows $nD\in T_{q-1}$, which contradicts the definition of A_q. Therefore, A is σ-free.

LEMMA 2. *Suppose that the homogeneous system σ is generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. Let $d_j = \min S_j$ and $D_j = \max S_j$. If $\lim\limits_{j\to\infty} d_j/D_j = 0$ then τ_σ exists.*

Proof. Denote by σ_j the homogeneous system generated by $\{S_1, S_2, \ldots$ $\ldots, S_j\}$. By Theorem 1, the density τ_{σ_j} exists. Moreover, $\lim\limits_{j\to\infty}\tau_{\sigma_j} = \tau$

exists, because $\tau_{\sigma_j} \geqslant \tau_{\sigma_{j+1}}$. If for positive ε the integer j is chosen so large that $d_k/D_k < \varepsilon$ for each $k > j$ then

$$\tau_{\sigma_j}(n) - \varepsilon n \leqslant \tau_\sigma(n) \leqslant \tau_{\sigma_j}(n), \qquad \tau_{\sigma_j} - \varepsilon \leqslant \underline{\tau}_\sigma \leqslant \overline{\tau}_\sigma \leqslant \tau_{\sigma_j}.$$

For $j \to \infty$ and $\varepsilon \to 0$ we obtain $\underline{\tau}_\sigma = \overline{\tau}_\sigma = \tau$.

We are now going to state the announced generalization of a theorem of Besicovitch ([3], p. 257) on primitive sequences. We denote by NG the set of multiples $\{ng: n \in N, g \in G\}$.

THEOREM 2. *Let the homogeneous system σ be generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. Suppose that there is a sequence $G = \{g_1, g_2, \ldots\}$ of positive integers satisfying*

(i) *$S_j \cap G \neq \emptyset$ for each j,*

(ii) *$\lim\limits_{j\to\infty} \overline{d}(NG_j) = 0$ if $G_j = \{g_j, g_{j+1}, \ldots\}$.*

Then τ_σ exists and $\overline{d}(\sigma) = \tau_\sigma$. Furthermore, $\tau_\sigma = 0$ is equivalent to $\{1\} \in \sigma$, and $1 \notin G$ implies $\underline{d}(\sigma) > 0$.

Proof. We make use of the following lemma which is easily deduced from an inequality of Behrend ([3], p. 263).

LEMMA 3 (Erdös [2]). *If $1 \notin G$ and $\lim\limits_{j\to\infty} \overline{d}(NG_j) = 0$ then $d(NG)$ exists and is less than 1.*

We note that $N \urcorner NG$ is σ-free by (i). So Lemma 3 implies $\underline{d}(\sigma) > 0$ if $1 \notin G$. Now suppose $1 \in G$ and $G' = G \urcorner \{1\}$ then

$$(N \urcorner NG') \cap (n/2, n]$$

is σ-free for each $n \in N$ if $\{1\} \notin \sigma$. In this case Lemma 3 implies $\underline{\tau}_\sigma > 0$.

It remains to prove the existence of τ_σ and $\overline{d}(\sigma) = \tau_\sigma$. By (ii), we may assume that G is finite, $G = \{g_1, \ldots, g_t\}$. Then the existence of τ_σ follows either from Theorem 1 or Lemma 2.

LEMMA 4. *If m is any positive real number then*

$$\lim_{x\to\infty} d(N[x/m, x]) = 0.$$

This is an immediate consequence of a theorem of Erdös ([3], p. 268). To construct a σ-free set A with $\overline{d}(A) \geqslant \tau_\sigma - \varepsilon$ $(0 < \varepsilon < 1)$, we choose

$$m = \frac{3}{\varepsilon} g_t, \qquad \varepsilon_j = \left(\frac{1}{2}\right)^j \frac{\varepsilon}{3} \qquad (j = 0, 1, 2, \ldots). \tag{5}$$

There is a sequence of integers x_j starting with $x_0 = 0$ and having the following properties for $j > 0$:

(a) $x_j > m x_{j-1}$,

(b) $\tau_\sigma(x_j) > (\tau_\sigma - \frac{1}{3}\varepsilon) x_j$,

(c) if $B_j = N[x_j/m, x_j]$ then $d(B_j) < \varepsilon_j$ and $B_{j-1}(x) < \varepsilon_{j-1} x$ for each $x \geqslant x_j$.

Let A_j' be a σ-free set in $[1, x_j]$ with $|A_j'| > (\tau_\sigma - \frac{1}{3}\varepsilon) x_j$. Define

$$A_j = A_j' \urcorner \Big(\bigcup_{i<j} B_i \cup [1, \tfrac{1}{3}\varepsilon x_j]\Big), \qquad A = \bigcup_{j=1}^{\infty} A_j. \tag{6}$$

The sets A_j are disjoint and σ-free. From (a), (b), (c) we obtain

$$A(x_j) \geqslant A_j(x_j) > \Big(\tau_\sigma - \tfrac{1}{3}\varepsilon - \tfrac{1}{3}\varepsilon - \sum_{i=1}^{j-1} \varepsilon_i\Big) x_j > (\tau_\sigma - \varepsilon) x_j,$$

hence $\overline{d}(A) \geqslant \tau_\sigma - \varepsilon$. To prove that A is σ-free, assume that A contains a set of the form nS_i, $S_i \in \sigma_0$. Let $d = \min S_i$ and $D = \max S_i$. Since the sets A_j are σ-free, we must have

$$nd \in A_k, \qquad nD \in A_q, \qquad k < q. \tag{7}$$

By (i), S_i contains a number $g \in G$. Thus $ng \in A$ and, by (a), (5), and (7),

$$ng = nd\frac{g}{d} \leqslant x_k g_t < x_{k+1}\frac{\varepsilon}{3}.$$

Now (6) implies $ng \in A_k$. So we have

$$x_k \frac{\varepsilon}{3} < ng \leqslant x_k, \qquad \frac{x_k}{m} < n \leqslant x_k.$$

Therefore, $nD \in B_k$, which contradicts the definition of A_q.

3. Logarithmic densities of σ-free sets. For a homogeneous system σ the natural density $d(\sigma)$ need not exist. Example 2 below shows that even for a finitely generated system $\underline{d}(\sigma)$ may be less than $\overline{d}(\sigma)$. More uniform results are obtained by considering logarithmic densities. We introduce the following logarithmic notions in analogy to the corresponding terms on natural density.

The logarithmic counting function of $A \subset N$ is $A^*(n) = \sum\limits_{a\leqslant n} 1/a$ (summation over $a \in A$). The limit $\delta(A) = \lim\limits_{n\to\infty} A^*(n)/\log n$, if it exists, is called the logarithmic density of A. The lower and upper logarithmic densities $\underline{\delta}(A)$ and $\overline{\delta}(A)$ are defined by the liminf and limsup of the same expression. Let

$$\lambda_\sigma(n) = \max\{A^*(n):\ A\ \sigma\text{-free}\},$$

$$\underline{\lambda}_\sigma = \liminf_{n\to\infty} \frac{\lambda_\sigma(n)}{n}, \qquad \overline{\lambda}_\sigma = \limsup_{n\to\infty} \frac{\lambda_\sigma(n)}{n}.$$

If $\underline{\lambda}_\sigma = \overline{\lambda}_\sigma$ put $\underline{\lambda}_\sigma = \overline{\lambda}_\sigma = \lambda_\sigma$. Define

$$\underline{\delta}(\sigma) = \sup\{\underline{\delta}(A):\ A\ \sigma\text{-free}\}, \qquad \overline{\delta}(\sigma) = \sup\{\overline{\delta}(A):\ A\ \sigma\text{-free}\}.$$

If $\underline{\delta}(\sigma) = \overline{\delta}(\sigma)$ denote the common value by $\delta(\sigma)$.

We believe that on very general conditions for a homogeneous system $\delta(\sigma)$ and λ_σ exist and coincide.

THEOREM 3. *Suppose that the homogeneous system σ is generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. Let $a_1, a_2, \ldots, a_k$ be coprime integers greater than 1 and*

$$U = \{u\colon u = a_1^{r_1} a_2^{r_2} \ldots a_k^{r_k},\ r_i \in \{0\} \cup N\}.$$

If $S_i \subset U$ for each i then λ_σ exists and

$$\lambda_\sigma(n) = \lambda_\sigma \log n + O(\log\log n).$$

Furthermore, $\delta(\sigma)$ exists and $\underline{d}(\sigma) = \delta(\sigma) = \lambda_\sigma$. If $k = 1$ there is a σ-free set A satisfying

$$A^*(n) = \lambda_\sigma(n) \quad \text{and} \quad A(n) = \lambda_\sigma n + O(\log n).$$

Proof. This proof is similar to that of Theorem 1. If we denote by V the sequence of positive integers which are not a multiple of any of the numbers a_j then

(8) $$V^*(n) = d(V)\log n + O(1), \quad \text{where} \quad d(V) = \prod_{j=1}^{k}\left(1 - \frac{1}{a_j}\right).$$

Define

$$\lambda_\sigma^U(n) = \max\{A^*(n)\colon A \subset U, A\ \sigma\text{-free}\}.$$

By (3), the limit $\lim_{n\to\infty} \lambda_\sigma^U(n) = a$ exists and

(9) $$\lambda_\sigma^U(n) = a + O\left(\frac{\log^k n}{n}\right).$$

Now we have, by (8) and (9),

$$\lambda_\sigma(n) = \sum_{v\in V} \frac{1}{v} \lambda_\sigma^U\left(\frac{n}{v}\right),$$

(10) $$\lambda_\sigma(n) = ad(V)\log n + O\left(\sum_{v\leqslant n} \frac{1}{v} \frac{\left(\log(n/v)\right)^k}{n/v}\right),$$

where $v \in V$. Let $m = \log^2 n$ and $f(y) = \dfrac{\log^k y}{y}$. Assuming that $f(y)$ is strictly decreasing for $y \geqslant m$, we obtain, by (8),

$$\left(\sum_{v\leqslant n/m} + \sum_{n/m<v\leqslant n}\right) \frac{1}{v} f\left(\frac{n}{v}\right) = O\big(f(m)\log n + \log m\big).$$

Hence, by (10),

(11) $$\lambda_\sigma(n) = \lambda_\sigma \log n + O(\log\log n), \quad \text{where} \quad \lambda_\sigma = ad(V).$$

To prove $\underline{d}(\sigma) = \delta(\sigma) = \lambda_\sigma$, we construct a σ-free set A with natural density greater than $\lambda_\sigma - \varepsilon$ $(0 < \varepsilon < \lambda_\sigma)$. Let U' be a finite σ-free subset of U satisfying

$$\sum_{u\in U'} \frac{1}{u} > a - \varepsilon.$$

If $A = U'V$ then A is σ-free and

$$A(n) = \sum_{u\in U'} V\left(\frac{n}{u}\right) = nd(V) \sum_{u\in U'} \frac{1}{u} + O(1).$$

Therefore, $d(A)$ exists and $d(A) > (a - \varepsilon)d(V) > \lambda_\sigma - \varepsilon$.

If $k = 1$ let $a = a_1$ and $U_j = \{a^0, a^1, \ldots, a^j\}$, $U_{-1} = \emptyset$. Define $S \subset U$ by the following property:

$a^j \in S$ if and only if $(S \cap U_{j-1}) \cup \{a^j\}$ is σ-free $(j = 0, 1, \ldots)$.

Let $S_j = S \cap U_j$. We prove by induction that S_j is the only σ-free set in U_j with $S_j^*(a^j) = \lambda_\sigma^U(a^j)$. This is certainly true for $j = 0$. Let it be true for $j-1$ $(j \geqslant 1)$. Suppose now that M is a σ-free subset of U_j, $M \cap U_{j-1} \neq S_{j-1}$, then

$$M^*(a^j) \leqslant \lambda_\sigma^U(a^{j-1}) - \frac{1}{a^{j-1}} + \frac{1}{a^j} < S_j^*(a^j).$$

Hence $S^*(a^j) = \lambda_\sigma^U(a^j)$ for $j = 0, 1, \ldots$ If $A = SV$ then A is σ-free, $A^*(n) = \lambda_\sigma(n)$, and

$$A(n) = \sum_{s\in S} V\left(\frac{n}{s}\right) = \lambda_\sigma n + O(\log n), \quad \text{where} \quad \lambda_\sigma = d(V) \sum_{s\in S} \frac{1}{s}.$$

LEMMA 5. *Let the homogeneous system σ be generated by $\sigma_0 = \{S_1, S_2, \ldots\}$. Let $G = \{g_1, g_2, \ldots\}$ be a sequence of positive integers, $G_j = \{g_j, g_{j+1}, \ldots\}$, and σ_j the homogeneous system generated by $\{S\colon S \in \sigma_0, S \cap NG_j = \emptyset\}$. Suppose*

(i) $\lim_{j\to\infty} \delta(NG_j) = 0$,

(ii) *$\delta(\sigma_j)$ and λ_{σ_j} exist and $\delta(\sigma_j) = \lambda_{\sigma_j}$ for each $j \in N$.*

Then $\delta(\sigma)$ and λ_σ exist and $\delta(\sigma) = \lambda_\sigma = \lim_{j\to\infty} \delta(\sigma_j)$. If, in addition to (i) and (ii), $\lim_{j\to\infty} \overline{d}(NG_j) = 0$ and $\underline{d}(\sigma_j) = \delta(\sigma_j) = \lambda_{\sigma_j}$ for each $j \in N$ then $\underline{d}(\sigma) = \delta(\sigma) = \lambda_\sigma$.

Proof. Since $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma$, the limit $\lim_{j\to\infty} \lambda_{\sigma_j} = \lambda$ exists and $\overline{\lambda}_\sigma \leqslant \lambda$. Let $\varepsilon_j > 0$ and $\lim_{j\to\infty} \varepsilon_j = 0$ $(j \in N)$. By (ii), there is a σ_j-free set A_j with $\underline{\delta}(A_j) > \lambda_{\sigma_j} - \varepsilon_j$. The set $A_j' = A_j \urcorner NG_j$ is σ-free and $\underline{\delta}(A_j') > \lambda_{\sigma_j} - \varepsilon_j - \delta(NG_j)$. For $j \to \infty$ follows $\underline{\delta}(\sigma) \geqslant \lambda$. Hence $\delta(\sigma)$ and λ_σ exist and $\delta(\sigma) = \lambda_\sigma = \lambda$.

If $\underline{d}(\sigma_j) = \delta(\sigma_j) = \lambda_{\sigma_j}$ and $\lim_{j\to\infty} \overline{d}(NG_j) = 0$ then we may demand $\underline{d}(A_j) > \lambda_{\sigma_j} - \varepsilon_j$. Now we have $\underline{d}(A_j') > \lambda_{\sigma_j} - \varepsilon_j - \overline{d}(NG_j)$, and for $j \to \infty$ follows the final part of Lemma 5.

If $A = \{a_1, a_2, \ldots\}$ is a sequence of positive integers let $A_j = \{a_j, a_{j+1}, \ldots\}$ and $\bar{A}_j = A \urcorner A_j$. It has been proved by Davenport and Erdös ([3], p. 258) that the logarithmic density $\delta(NA)$ exists and

$$\underline{d}(NA) = \delta(NA) = \lim_{j\to\infty} d(N\bar{A}_j).$$

Note that

(12) $$\lim_{j\to\infty} \delta(NA_j \urcorner N\bar{A}_j) = 0.$$

Lemma 6. *Suppose that the homogeneous system σ' is generated by $\sigma'_0 = \{S_1, S_2, \ldots, S_q\}$. Then for any homogeneous subsystem $\sigma \subset \sigma'$ the densities $\delta(\sigma)$ and λ_σ exist and coincide.*

Proof. Any homogeneous subsystem σ of σ' is of the form

$$\sigma = \{S\colon S = a_{ik}S_i,\ 1 \leqslant i \leqslant q,\ 1 \leqslant k < \infty\}, \qquad a_{ik} \in N,\ a_{i1} < a_{i2} < \ldots$$

Let

$$A_i = \{a_{i1}, a_{i2}, \ldots\}, \qquad A_{ij} = \{a_{ij}, a_{i,j+1}, \ldots\}, \qquad \bar{A}_{ij} = A_i \urcorner A_{ij}.$$

According to (12), for $\varepsilon > 0$ the number j can be chosen so large that

(13) $$\delta(NA_{ij} \urcorner N\bar{A}_{ij}) < \varepsilon/q \quad \text{for each } i = 1, \ldots, q.$$

Denote by σ_j the homogeneous system generated by

$$\{S\colon S = a_{ik}S_i,\ 1 \leqslant i \leqslant q,\ 1 \leqslant k < j\}.$$

By Theorem 3, $\delta(\sigma_j)$ and λ_{σ_j} exist and $\delta(\sigma_j) = \lambda_{\sigma_j}$. Hence there is a σ_j-free set H_j with $\underline{\delta}(H_j) > \lambda_{\sigma_j} - \varepsilon$. If $t_i \in S_i$ the set

$$H'_j = H_j \urcorner \bigcup_{i=1}^{q} t_i(NA_{ij} \urcorner N\bar{A}_{ij})$$

is σ-free and, by (13),

(14) $$\underline{\delta}(\sigma) \geqslant \underline{\delta}(H'_j) > \lambda_{\sigma_j} - 2\varepsilon.$$

Since $\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma$ the limit $\lim_{j\to\infty} \lambda_{\sigma_j} = \lambda$ exists and $\bar{\lambda}_\sigma \leqslant \lambda$. Now, on letting $j \to \infty$ and $\varepsilon \to 0$ in (14), we see that $\delta(\sigma)$ and λ_σ exist and $\delta(\sigma) = \lambda_\sigma = \lambda$.

Finally, we are going to extend Lemma 6 by Lemma 5. Let $G = \{g_1, g_2, \ldots\}$ and $G_j = \{g_j, g_{j+1}, \ldots\}$. We shall say that G has property P if $\lim_{j\to\infty} \delta(NG_j) = 0$.

Theorem 4. *Suppose that the homogeneous system σ' is generated by $\sigma'_0 = \{S_1, S_2, \ldots\}$. Let G be a sequence with property P and $M_j = \bigcup\{S\colon S \in \sigma'_0,\ S \cap NG_j = \emptyset\}$. If each set M_j has property P, then for any homogeneous subsystem $\sigma \subset \sigma'$ the densities $\delta(\sigma)$ and λ_σ exist and coincide.*

Proof. If we denote by σ_j and σ'_j the homogeneous systems generated by

$$\{S\colon S \in \sigma,\ S \cap NG_j = \emptyset\} \quad \text{and} \quad \{S\colon S \in \sigma'_0,\ S \cap NG_j = \emptyset\}$$

then $\sigma_j \subset \sigma'_j$. Suppose

(a) $\delta(\sigma_j)$ and λ_{σ_j} exist and $\delta(\sigma_j) = \lambda_{\sigma_j}$ for each $j \in N$, then the existence of $\delta(\sigma)$ and λ_σ, and $\delta(\sigma) = \lambda_\sigma$ follow from Lemma 5. If each set M_j is finite, then (a) is true by Lemma 6.

(b) Theorem 4 is true, if each set M_j is finite. In the general case (a) follows from (b) applied to σ'_j.

4. Examples

Example 1. Let σ consist of the solutions in positive integers of the equation

(15) $$x_1 + x_2 + \ldots + x_{2k} = 2(y_1 + y_2 + \ldots + y_{2k}).$$

Clearly, the interval $(n/2, n]$ is σ-free. Therefore, $\underline{\tau}_\sigma \geqslant \frac{1}{2}$. We prove

$$d(\sigma) = 1/r, \quad \text{where} \quad r = \min\{z\colon z \in N,\ z \nmid 2k\}.$$

Obviously, the congruence class 1 modulo r is σ-free. Hence $\underline{d}(\sigma) \geqslant 1/r$. Let $A \subset N$ be σ-free. By equating some of the variables in (15) it follows that the equation

(16) $$x_1 + x_2 + \ldots + x_j = 2(y_1 + y_2 + \ldots + y_j)$$

has no solution in A, if j divides $2k$, thus especially for $j = 1, \ldots, r-1$. For $x_2 = y_1, x_3 = y_2, \ldots, x_j = y_{j-1}$ the last equation becomes

(17) $$x_1 = y_1 + \ldots + y_{j-1} + 2y_j \quad (j = 2, 3, \ldots, r-1).$$

By (16), $x_1 + x_2 = 2(y_1 + y_2)$ has no solution in A. For $x_1 = x_2$ this means that

(18) $$x_1 = y_1 + y_2$$

is also unsolvable in A. Let $a \in A$. Substituting $y_2 = y_3 = \ldots = y_j = a$ in (17) and $y_2 = a$ in (18) we see that none of the equations

$$x_1 = y_1 + ja \quad (j = 1, 2, \ldots, r-1)$$

has a solution in A. Hence $\bar{d}(A) \leqslant 1/r$.

It would be interesting to know whether the logarithmic density $\delta(\sigma)$ exists for every homogeneous system defined by a linear equation.

Example 2. We construct a finitely generated homogeneous system σ with $\underline{d}(\sigma) < \bar{d}(\sigma)$. Suppose that a is a positive integer not equal to 1. Let σ consist of all 3-term geometric progressions of ratio a, a^2, a^3 or a^4. This system is generated by

$$\{1, a, a^2\}, \quad \{1, a^2, a^4\}, \quad \{1, a^3, a^6\}, \quad \{1, a^4, a^8\}.$$

We determine $\bar{d}(\sigma) = \tau_\sigma$ and $\underline{d}(\sigma) = \lambda_\sigma$ according to the considerations to Theorem 1 and Theorem 3. By (4) and (11), we have

(19) $$\bar{d}(\sigma) = d(V)\sum_{r\in R} 1/r, \qquad \underline{d}(\sigma) = a\,d(V),$$

where R is the set satisfying $R(n) = \tau_\sigma^U(n)$ and $\alpha = \lim_{n\to\infty} \lambda_\sigma^U(n)$. We determine $\lambda_\sigma^U(n) = S^*(n)$ as indicated in the final part of the proof of Theorem 3. Thus we obtain, by (19),

$$\bar{d}(\sigma) = \left(1-\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^4}+\frac{1}{a^8}+\ldots\right),$$

$$\underline{d}(\sigma) = \left(1-\frac{1}{a}\right)\left(1+\frac{1}{a}+\frac{1}{a^3}+\frac{1}{a^4}\right)\sum_{j=0}^{\infty}\frac{1}{a^{9j}} < \bar{d}(\sigma).$$

EXAMPLE 3. Denote by $C = \{c_1, c_2, \ldots\}$ the sequence of integers greater than 1, which are a product of at most k primes (multiple factors counted multiply). Define $S_j = \{1, c_j\}$, and let σ be the homogeneous system generated by $\sigma_0 = \{S_1, S_2, \ldots\}$.

By Lemma 1 and Lemma 2, τ_σ exists, and there is a σ-free set A with $\bar{d}(A) = \tau_\sigma$. Since $c_j \geqslant 2$ for each $j \in N$, we have $\tau_\sigma \geqslant \frac{1}{2}$. Let us prove

$$\delta(\sigma) = \lambda_\sigma = \frac{1}{k+1}. \tag{20}$$

Suppose that A is a σ-free set in $[1, n]$ satisfying $A^*(n) = \lambda_\sigma(n)$. We sketchily follow the words of Halberstam and Roth ([3], pp. 246–249) for a proof of Behrend's theorem on primitive sequences.

$$A^*(n) = \lambda_\sigma(n) = \frac{1}{n}\sum_{u\leqslant n} r(u) + O(1), \tag{21}$$

where $r(u)$ is the number of divisors of u belonging to A. Let u be a product of $s(u)$ primes. According to de Bruijn, Tengbergen, and Kruyswijk [1], the set of divisors of u can be completely divided into $\binom{s(u)}{[s(u)/2]}$ disjoint symmetrical chains. A symmetrical chain of m divisors cannot contain more than $\frac{m}{k+1}+1$ numbers of A. Therefore, if $d(u)$ is the number of divisors of u,

$$r(u) \leqslant \frac{d(u)}{k+1} + \binom{s(u)}{[s(u)/2]}$$

and, by (21),

$$\lambda_\sigma(n) \leqslant \frac{1}{n(k+1)}\sum_{u\leqslant n} d(u) + O\left(\frac{1}{n}\sum_{u\leqslant n}\frac{d(u)}{(s(u))^{1/2}}\right),$$

$$\lambda_\sigma(n) \leqslant \frac{\log n}{k+1} + O\left(\frac{\log n}{(\log\log n)^{1/2}}\right). \tag{22}$$

On the other hand, if $A = \{a\colon a > 1, s(a) \equiv 1 \bmod (k+1)\}$ then A is

σ-free, and it follows as before that

$$A^*(n) \geqslant \frac{\log n}{k+1} + O\left(\frac{\log n}{(\log\log n)^{1/2}}\right). \tag{23}$$

By (22) and (23), we obtain (20). Note that the constants involved in the O-estimates of (22) and (23) can be chosen independent of k.

EXAMPLE 4. Let σ consist of all n-term geometric progressions ($n \geqslant 3$, rational ratio). Systems of this kind have been investigated by Rankin [4] and by Riddell [5]. The system σ is generated by

$$\sigma_0 = \{S\colon S = \{a^{n-1}, a^{n-2}b^1, a^{n-3}b^2, \ldots, b^{n-1}\},\ a < b,\ (a, b) = 1\}.$$

Let $G = \{1^{n-1}, 2^{n-1}, 3^{n-1}, \ldots\}$. Since $\sum_j 1/j^{n-1}$ converges, we have $\lim_{j\to\infty}\bar{d}(NG_j) = 0$. By Theorem 2, τ_σ exists, and from Lemma 1 follows the existence of a σ-free set A with $\bar{d}(A) = \tau_\sigma$. Lemma 5 and Theorem 3 ensure the existence of $\delta(\sigma)$ and λ_σ. Moreover, $\underline{d}(\sigma) = \delta(\sigma) = \lambda_\sigma$.

Suppose that $E \subset \{0\}\cup N$ is a set which does not contain an n-term arithmetic progression. Let A consist of those positive integers which have in their unique prime factorization only exponents belonging to E. Then A is σ-free, $d(A)$ exists, and

$$\underline{d}(\sigma) \geqslant d(A) = \prod_p\left\{\left(1-\frac{1}{p}\right)\sum_{r\in E}\frac{1}{p^r}\right\}, \tag{24}$$

where the product is taken over all primes. As in the proof of Theorem 3, it follows by induction that $\sum_{r\in E} 1/p^r$ is maximal if and only if E is identical with the set E_n defined by the following property:

$r \in E_n$ if and only if $(E_n \cap [0, r-1]) \cup \{r\}$ does not contain an n-term arithmetic progression ($r \in \{0\}\cup N$).

The estimates of Rankin and Riddell obtained by (24) can be improved for $n \geqslant 4$, because they use a set $E \neq E_n$. If n is a prime, then it follows from a paper of Scheid [6] that E_n consists of the nonnegative integers, which have no digit $n-1$, when they are expressed in the scale of n. In this case we have

$$\underline{d}(\sigma) \geqslant \prod_p\left\{\left(1-\frac{1}{p}\right)\prod_{k=0}^{\infty}\left(1+\frac{1}{p^{n^k}}+\frac{1}{p^{2n^k}}+\ldots+\frac{1}{p^{(n-2)n^k}}\right)\right\},$$

$$\underline{d}(\sigma) \geqslant \prod_p\left\{\left(1-\frac{1}{p}\right)\prod_{k=0}^{\infty}\frac{1-\dfrac{1}{p^{(n-1)n^k}}}{1-\dfrac{1}{p^{n^k}}}\right\} = \frac{1}{\zeta(n-1)}\prod_{k=1}^{\infty}\frac{\zeta(n^k)}{\zeta((n-1)n^k)}.$$

Now suppose that σ' consists of all 3-term geometric progressions with integral ratio. We wish to show $\underline{d}(\sigma') < \overline{d}(\sigma')$. Let U be the sequence of positive integers which have no prime divisor different from 2 or 3. Denote by σ^* the system of those progressions in σ' which have ratio in U. It is not difficult to check that

$$\max\Big\{\sum_{u\in A}\frac{1}{u} : A\ \sigma^*\text{-free},\ A\subset\{2^{r_1}3^{r_2}: r_1, r_2 = 0, 1, 2\}\Big\} = 2.$$

Thus

$$a = \lim_{n\to\infty}\lambda_{\sigma^*}^{U}(n) \leqslant 2\sum_{j=0}^{\infty}\frac{1}{2^{3j}}\sum_{k=0}^{\infty}\frac{1}{3^{3k}} = \frac{8}{7}\,\frac{27}{13}$$

and, by (11),

$$\underline{d}(\sigma') \leqslant \underline{d}(\sigma^*) \leqslant \Big(1-\frac{1}{2}\Big)\Big(1-\frac{1}{3}\Big)\frac{8}{7}\,\frac{27}{13} = \frac{72}{91} = 0.791\ldots$$

On the other hand the set

$$\Big(\frac{n}{32}, \frac{n}{27}\Big)\cup\Big(\frac{n}{24}, \frac{n}{12}\Big)\cup\Big(\frac{n}{9}, \frac{n}{8}\Big)\cup\Big(\frac{n}{4}, n\Big)$$

is σ'-free in $[1, n]$. Hence

$$\overline{d}(\sigma') = \tau_{\sigma'} \geqslant \frac{5}{864}+\frac{1}{24}+\frac{1}{72}+\frac{3}{4} = \frac{701}{864} = 0.811\ldots$$

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Some remarks on Goldbach's problem

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In this paper we shall prove by a modification of Chen's work ([3]) that every sufficiently large even integer x is written as a sum of a prime and a natural number which has at most one prime factor less than $x^{1089/2089}$.

1. Let x be a large even integer. Let $G_2(x)$ be the number of primes $p \leqslant x$ such that $x-p$ has at most two prime factors. Chen ([3]) has proved that

$$(1)\qquad G_2(x) \geqslant \frac{0.67\,xC_x}{(\log x)^2},\quad\text{where}\quad C_x = \prod_{\substack{p|x\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big).$$

In fact, if we put $G_2(x, I)$ — the number of primes $p \leqslant x$ such that $x-p$ is a prime or $x-p = p_1p_2$ with primes p_1 and p_2 satisfying $p_1 \notin I$ and $p_1 \leqslant p_2$, for a subset I of $(1, x^{1/2}]$, he has proved that $G_2(x, (1, x^{1/10}]) \geqslant 0.67xC_x/(\log x)^2$. (Further Halberstam [6] or [7] has shown that 0.67 can be replaced by 0.689.) Now we wish to maximize $I \subset (1, x^{1/2}]$ such that $G_2(x, I) \geqslant AxC_x/(\log x)^2$, where A is some positive absolute constant. To study this we use the following mean value theorem which is similar to Bombieri's one.

LEMMA 0. *Assume that $M+N \ll x^{1/2}$. For an arbitrarily large constant A, there exist positive constants $B = B(A)$ and $E = E(A)$ such that if $M \gg (\log x)^E$, and $b(m) \ll (\log x)^C$ with some positive constant C for any m in $M < m \leqslant M+N$, then*

$$\sum_{d\leqslant x^{1/2}/(\log x)^B}\ \max_{(a,d)=1}\ \max_{(M+N)^{1+\theta}<y\leqslant x}\Big|\sum_{\substack{m=M+1\\ (m,d)=1}}^{M+N} b(m)\Big(\sum_{\substack{n\leqslant y/m\\ n\equiv am^*(d)}}\Lambda(n) - \frac{1}{\varphi(d)}\cdot\frac{y}{m}\Big)\Big|$$
$$\ll x/(\log x)^A,$$

where θ is an arbitrarily given positive number, $n \equiv am^*(d)$ means $n \equiv am^*$ (mod d), and m^* satisfies $mm^* \equiv 1$ (d).