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A conjecture of Erdös on continued fractions

by

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1. For $0 < x \leqslant 1$ let $[a_1(x), a_2(x), \ldots]$ be the continued fraction expansion of x. Write

$$L_N(x) = \max_{1 \leqslant n \leqslant N} a_n(x).$$

About ten years ago Professor Erdös [4] conjectured that for almost all x

$$\liminf_{N\to\infty} N^{-1} L_N(x) \log\log N = 1.$$

Apart from the value of the constant I shall give a proof of Erdös' conjecture.

THEOREM 1. *For almost all* x

$$\liminf_{N\to\infty} N^{-1} L_N(x) \log\log N = 1/\log 2.$$

By modifying methods developed by Barndorff-Nielsen [1] for similar problems concerning independent identically distributed random variables we get the following refinement of Theorem 1.

THEOREM 2. *Let* ψ_N *be nonincreasing such that* $\psi_N N$ *is nondecreasing. Then*

$$L_N(x) \leqslant \psi_N N/\log 2$$

finitely often or infinitely often for almost all x *according as*

$$\sum \exp(-1/\psi_n) n^{-1} \log\log n$$

converges or diverges.

COROLLARY. *Let* $k \geqslant 2$ *be integer. Then the inequality*

$$N^{-1} L_N(x) \big(\log_2 N + 2\log_3 N + \ldots + (1+\delta)\log_k N\big) \leqslant 1/\log 2$$

has finitely many or infinitely many solutions for almost all x *according as* $\delta > 0$ *or* $\delta \leqslant 0$. *If* $k = 3$ *then* $1+\delta$ *is to be replaced by* $2+\delta$. *Here* $\log_k$ *denotes the* k*-fold iterated logarithm.*

Theorem 1 follows from the corollary if we put $k = 2$.

There is no analogous result to Theorem 1 with a finite nonzero limes superior. This follows from the well-known

THEOREM 3. *Let $\varphi(n)$ be a positive nondecreasing sequence. Then for almost all x the inequality*

$$L_N(x) \geqslant \varphi(N) \tag{1}$$

has finitely many or infinitely many solutions in integers N according as the series

$$\sum 1/\varphi(n) \tag{2}$$

converges or diverges.

COROLLARY. *Let $\varphi(n)$ be as in Theorem 3. Then for almost all x*

$$\limsup_{N\to\infty} L_N(x)/\varphi(N) \tag{3}$$

is either 0 *or* ∞.

Theorem 3 is an easy consequence of Bernstein's theorem on continued fractions. Indeed, if $\sup\varphi(n) < \infty$, then (2) diverges and hence by Bernstein's theorem $a_n(x) \geqslant \varphi(n)$ holds infinitely often for almost all x. If, however, $\varphi(n)\uparrow\infty$ then, as is easy to see, (1) holds finitely often or infinitely often according as the inequality $a_n(x) \geqslant \varphi(n)$ holds finitely often or infinitely often which, in turn, by Bernstein's theorem holds almost everywhere according as the series (2) converges or diverges.

For the proof of the corollary we distinguish the cases where the series (2) converges or diverges. If (2) converges we choose a monotone sequence $\tau(n)$ tending to ∞ but so slowly that still $\sum\tau(n)/\varphi(n) < \infty$. Then according to Theorem 3 the inequality $L_N(x) \geqslant \varphi(N)/\tau(N)$ holds only a finite number of times for almost all x. Hence (3) vanishes almost everywhere. If, on the other hand, (2) diverges we pick a monotone sequence $\tau(n)$ tending to 0 such that $\sum\tau(n)/\varphi(n) = \infty$. Then $L_N(x) \geqslant \varphi(N)/\tau(N)$ has infinitely many solutions for almost all x and thus (3) is infinite almost everywhere.

Theorems 1, 2, 3 and their corollaries improve upon results of Galambos [6], [7]. For the proof of Theorem 1 we use Theorem 4 below which also strengthens a result of Galambos [5]. Except for the application of Theorem 4 the proof of Theorem 1 is different from the one given in [7].

In dealing with continued fractions it is more convenient to use the Gaussian measure P instead of the Lebesgue measure λ. P is defined on the Lebesgue measurable sets E by

$$P(E) = \frac{1}{\log 2}\int_E \frac{dx}{1+x}.$$

THEOREM 4. *For any $\delta < 1$ and $y > 0$*

$$P\{x\colon\ L_N(x) < Ny/\log 2\} = \exp(-1/y) + O\big(\exp(-(\log N)^\delta)\big)$$

where the constant implied by O depends only, perhaps, on δ.

With an error term of the form $o(1)$ Theorem 4 is due to Galambos [5] who subsequently [6] showed that his result remains valid if P is replaced by any probability measure on $[0, 1]$ absolutely continuous with respect to λ. (In particular for λ itself.) If we replace P by probability measures having a "smooth" density then Theorem 4 itself remains valid; in particular, Theorem 4 continues to hold if we replace P by λ. Since we shall not need these facts we omit their proof.

As already indicated above I shall give a direct proof of Theorem 1. The proof of Theorem 2 will only be sketched in order to avoid lengthy repetitions of parts of Barndorff-Nielsen's paper [1]. The proof of Theorem 4 also will be only sketched since it consists only of a modification of Galambos' paper [5].

We remark in passing that Barndorff-Nielsen's Theorems 1 and 2 of [1] remain valid for sequences of random variables satisfying a uniform mixing condition

$$|P(AB) - P(A)P(B)| \leqslant \varphi(k)P(A)P(B)$$

for all $A \in M_1^t$ and $B \in M_{t+k}^\infty$. Here M_a^b denotes the σ-field generated by the random variables X_n $(a \leqslant n \leqslant b)$ and $\varphi(k)\downarrow 0$.

2. Lemmas on continued fractions.

The shift transformation T associated with the continued fraction expansion is defined by $Tx = 1/x \bmod 1$. T is called a shift since $a_{n+1}(x) = a_n(Tx) = a_1(T^n x)$ for all positive integers n. T maps the unit interval onto itself and preserves the Gaussian measure P, i.e., $P(T^{-1}E) = P(E)$ for any L-measurable set E. This explains the importance of P in investigations dealing with continued fractions. Lemma 1 just proves this point.

LEMMA 1. *For all positive integers n and w*

$$P\{x\colon\ a_n(x) \geqslant w\} = P\{x\colon\ a_1(x) \geqslant w\} = \log(1+1/w)/\log 2 = p(w)\ \ (say).$$

The functions $a_1(x), a_2(x), \ldots$ considered as random variables on $[0, 1]$ are not independent. However, they satisfy the following mixing condition.

LEMMA 2. *Let M_{uv} be the smallest σ-algebra with respect to which $a_n(x)$, $u \leqslant n \leqslant v$, are measurable. Then for any sets $A \in M_{1t}$ and $B \in M_{t+n\infty}$ we have*

$$|P(AB) - P(A)P(B)| \leqslant cq^n P(A)P(B) \tag{4}$$

where $0 < q < 1$ and $c \geqslant 1$ are numerical constants.

In other words if A is a set defined only in terms of $a_1(x), \ldots, a_t(x)$ and B is a set defined only in terms of $a_{t+n}(x), a_{t+n+1}(x), \ldots$ then (4) holds.

Lemmas 1 and 2 are well-known (see e.g. [2], pp. 40–50, or [9]). We also need

LEMMA 3 ([8]). *Let $(E_n, n \geqslant 1)$ be a sequence of measurable sets in a probability space. Denote by $A(N, x)$ the number of integers $n \leqslant N$ such that $x \in E_n$, i.e. $A(N, x) = \sum_{n \leqslant N} 1_{E_n}(x)$. Put*

$$\varphi(N) = \sum_{n \leqslant N} |E_n|.$$

($|E|$ denotes the measure of E). Suppose that there exists a convergent series $\sum c_k$ with $c_k \geqslant 0$ such that for all integers $n > m$ we have

$$(5) \qquad |E_n \cap E_m| \leqslant |E_n|\,|E_m| + |E_n| c_{n-m}.$$

Then for any $\varepsilon > 0$

$$(6) \qquad A(N, x) = \varphi(N) + O\left(\varphi^{1/2}(N) \log^{3/2+\varepsilon} \varphi(N)\right)$$

for almost all x.

Remark. If $\varphi(\infty) < \infty$ then (6) holds even without assuming (5). This is just the convergence part of the Borel Cantelli lemma.

3. Proof of Theorem 4. We change the proof given by Galambos [5] only insofar that we judiciously choose the parameters Z and m occurring there. The details are as follows. First we may assume that $1/y \leqslant (\log N)^{\delta}$ since the other case is easily reduced to this one. We choose

$$Z = [\log N / \log\log N].$$

Then relation (10) in [5] becomes

$$(7) \qquad \sum_{k \geqslant Z} S_k < \frac{1}{Z!} (4c/y)^Z \exp(4c/y) \ll N^{-a}$$

for some $a > 0$ using Stirling's formula. Next we choose $m = [\log^2 N]$. We observe that the number of terms in R_k does not exceed $ZmN^{k-1} \leqslant N^{k-1} \log^3 N$ since each term in R_k contains at least one pair of indices i_j, i_{j+1} satisfying $i_{j+1} - i_j < m$. Consequently the number of terms in S_k^* equals $\binom{N}{k} + O(N^{k-1} \log^3 N)$. Hence for $k \leqslant Z$

$$(8) \qquad R_k \ll N^{k-1} \log^3 N p^k(w) \cdot (2c)^k \ll N^{k-1} \log^3 N \left(\frac{4c}{Ny}\right)^k \ll N^{-a}$$

and

$$(9) \qquad S_k^* = \left(\binom{N}{k} + O(N^{k-1} \log^3 N)\right)\left(1 + O(N^{-1})\right)^k (Ny)^{-k} (1 + \theta c q^{\log^2 N})^k$$
$$= \frac{y^{-k}}{k!} + O(N^{-a}) \quad \text{where } |\theta| \leqslant 1.$$

The inequalities (7), (8) and (9) replace [5] (10), (13) and (14). The remaining changes are only minor.

4. Proof of Theorem 1. Since $\lambda(E)/\log 4 \leqslant P(E) \leqslant \lambda(E)/\log 2$ the measures P and λ are equivalent, i.e. they have the same sets of measure 0. Hence we are to show Theorem 1 for all x except a set of P-measure 0. For integers $M, N \geqslant 0$ we put

$$L(M, N, x) = \max_{M < n \leqslant M+N} a_n(x)$$

and

$$\psi(n) = n / (\log\log n \log 2).$$

Since the transformation T preserves P we have by Theorem 4 for any integer $k \geqslant k_0$

$$(10) \qquad \begin{aligned} P(E_k) :&= P\{x:\ L(k^{2k}, k^{2(k+1)}, x) \leqslant \psi(k^{2(k+1)})\} \\ &= P\{x:\ L(0, k^{2(k+1)}, x) \leqslant \psi(k^{2(k+1)})\} \\ &\geqslant \tfrac{1}{2} \exp(-\log\log k^{2(k+1)}) \geqslant \tfrac{1}{8} (k \log k)^{-1}. \end{aligned}$$

Now E_k depends only on $a_n(x)$ with $k^{2k} < n \leqslant k^{2(k+1)} + k^{2k}$. Hence by Lemma 2 we have for any pair $k < l$ of integers

$$|P(E_k \cap E_l) - P(E_k) P(E_l)| \leqslant c q^{l-k} P(E_k) P(E_l)$$

since $(k+1)^{2(k+1)} - k^{2(k+1)} - k^{2k} \geqslant 1$. Consequently, Lemma 3 implies that for almost all x the events E_k occur infinitely often since $\varphi(N) \gg \log\log N$ by (10). On the other hand, by Lemma 1

$$P(F_k) := P\{x:\ L(0, k^{2k}, x) \geqslant \psi(k^{2(k+1)})\} \leqslant \sum_{n \leqslant k^{2k}} P\{x:\ a_n(x) \geqslant \psi(k^{2(k+1)})\}$$
$$= k^{2k} p\left([\psi(k^{2(k+1)})]\right) \ll k^{2k} \log\log k^{2(k+1)} \cdot k^{-2(k+1)} \ll k^{-3/2}.$$

Thus by the convergence part of Lemma 3 (which in fact is the convergence part of the Borel–Cantelli lemma) for almost all x the events F_k occur only a finite numbers of times. Hence the events

$$E_k - F_k = \{x:\ L(0, k^{2k} + k^{2(k+1)}, x) \leqslant \psi(k^{2(k+1)})\}$$

occur infinitely often for almost all x. But this implies that the events

$$L(0, k^{2(k+1)}, x) \leqslant \psi(k^{2(k+1)})$$

occur infinitely often for almost all x. Consequently,

$$(11) \qquad \liminf_{N \to \infty} N^{-1} L_N(x) \log\log N \log 2 \leqslant 1 \quad \text{a.e.}$$

This proves half of Theorem 1.

We now prove the opposite inequality. Let $r > 1$. Again by Theorem 4

$$P(G_k) := P\{x:\ L(0, [r^k], x) \leqslant r^{-2} \psi([r^{k+1}]) \ll \exp(-r \log\log r^k) \ll k^{-r}.$$

Since $\sum k^{-r} < \infty$ the convergence part of Lemma 3 implies that the events G_k occur only a finite number of times for almost all x, or

$$L(0, [r^k], x) > r^{-2}\psi([r^{k+1}])$$

for all $k \geqslant k_0(x, r)$. Now let $N \geqslant N_0(x, r)$ be given. Define k by $[r^k] \leqslant N < r^{k+1}$. Since $L(0, [r^k], x) \leqslant L_N(x)$ and $\psi(N) \leqslant \psi([r^{k+1}])$ we conclude that for almost all x and all $N \geqslant N_0$

$$L_N(x) > r^{-2}\psi(N).$$

Since $r > 1$ was arbitrary we obtain

$$\liminf_{N\to\infty} N^{-1}L_N(x)\log\log N \log 2 \geqslant 1 \qquad \text{a.e.}$$

This together with (11) proves Theorem 1.

5. Proof of Theorem 2. The following lemma corresponds to [1], Lemma 4.

LEMMA 4. *Without loss of generality we may assume that*

$$(2\log\log n)^{-1} \leqslant \psi_n \leqslant 2/\log\log n. \tag{12}$$

Proof. Suppose that Theorem 2 has been proved for sequences ψ_n satisfying (12). To any nonincreasing sequence ψ_n such that $n\psi_n$ is nondecreasing we define a sequence

$$\psi_n^1 = \begin{cases} 2/\log\log n & \text{if} \quad \psi_n > 2/\log\log n, \\ \psi_n & \text{if} \quad (2\log\log n)^{-1} \leqslant \psi_n \leqslant 2/\log\log n, \\ (2\log\log n)^{-1} & \text{if} \quad \psi_n < (2\log\log n)^{-1}. \end{cases}$$

Then the series

$$\sum \exp(-1/\psi_n^1)n^{-1}\log\log n \tag{13}$$

and

$$\sum \exp(-1/\psi_n)n^{-1}\log\log n \tag{14}$$

converge or diverge simultaneously. Indeed if $\psi_n > \psi_n^1$ for infinitely many n, say $n_1, n_2, \ldots, n_k, \ldots$ then the n_k-th partial sums of (13) and (14) are not less than

$$\exp(-\tfrac{1}{2}\log\log n_k)\sum_{n=81}^{n_k} n^{-1} \to \infty.$$

On the other hand, since

$$\sum_{n=3}^{\infty} \exp(-2\log\log n)n^{-1}\log\log n < \infty$$

the terms with $\psi_n < \psi_n^1$ cannot influence the simultaneous convergence of (13) and (14).

The remainder of the proof is the same as in [1].

As became evident from the proof of Lemma 4 our situation is somewhat simpler than the one considered in [1], at least in one respect, because the distribution function $F(t)$ of the partial quotients a_n is explicitly known $F(t) = 1 - p([t])$. On the other hand, since in [1] the random variables are assumed to be independent, the distribution function of the maximum of the first N random variables is simply $(F(t))^N$. But we already have proved Theorem 4 which gives an estimate of the distribution function of L_N. By Theorem 4

$$\begin{aligned} P(L_N(x) \leqslant N\psi_N/\log 2) &= \exp(-1/\psi_N) + O(\exp(-(\log N)^{\delta})) \\ &= \exp(-1/\psi_N)(1+o(1)) \end{aligned}$$

in view of Lemma 4. Consequently, we can replace the factors $(F(\lambda_n))^n$ in [1], pp. 388–392 simply by $\exp(-1/\psi_n)$ without affecting the convergence properties of the series under consideration. The proof of Theorem 2 is entirely parallel to the proof of [1], Theorem 1, pp. 388–392. We only have to take precautions at these steps where the independence of the random variables is used. In most cases Lemma 2 will take care of that. For example Kolmogorov's zero-one law continues to hold for mixing sequences of random variables (see [3]). The only place which requires a slight modification is the estimate of S_1 defined in [1], p. 388. Choose k so that $1 + cq^k < e^{1/8}$ where c and q are the constants occurring in Lemma 2. Using the notation of [1] we have for $n \geqslant n_0$

$$\begin{aligned} S_1 &= P(E_{m_n} \cap E_{m_{n+1}}) \\ &\leqslant P\{E_{m_n} \cap (a_r(x) \leqslant \psi_{m_{n+1}} m_{n+1}/\log 2,\ m_n + k \leqslant r \leqslant m_{n+1})\} \\ &\leqslant P(E_{m_n})P(x\colon\ L(0, m_{n+1} - m_n - k, x) \leqslant \psi_{m_{n+1}} m_{n+1}/\log 2)(1 + cq^k) \\ &\leqslant P(E_{m_n})e^{-1/4} \end{aligned}$$

by Lemma 2 and Theorem 4. The other modifications are only of a routine nature.

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On composite n for which $\varphi(n)\,|\,n-1$

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§ 1. Introduction. In [4], D. H. Lehmer asked if there are any composite natural numbers n for which $\varphi(n)\,|\,n-1$, where φ is Euler's function. This is still an unanswered question, many people feeling it is as difficult as the odd perfect number problem. There have been partial results however, such as: if such an n exists then n is divisible by at least 11 distinct primes, and if $3\,|\,n$, then $n > 5.5\cdot 10^{570}$ and n is divisible by at least 212 distinct primes (Lieuwens [5]).

If A is an arbitrary set of positive integers, then we denote by $N(A, x)$ the number of members of A which do not exceed x. Let F denote the set of composite n for which $\varphi(n)\,|\,n-1$. In [6] we proved

$$N(F, x) = O\left[x\exp\left(-c_1(\log x\log\log x)^{1/2}\right)\right] \tag{1}$$

for some $c_1 > 0$. If $n\in F$, then $a^{n-1}\equiv 1(\mathrm{mod}\,n)$ for every a with $(a, n) = 1$, that is, n is a Carmichael number (also called an absolute pseudoprime). Hence a result of Knödel [3] dealing with Carmichael numbers also implies (1). However, a result of Erdös [1], also dealing with Carmichael numbers, gives the better estimate

$$N(F, x) = O\left[x\exp(-c_2\log x\log\log\log x/\log\log x)\right]$$

for some $c_2 > 0$. In the present note, borrowing somewhat the methods of Knödel and Erdös, we prove

$$N(F, x) = O\left(x^{2/3}(\log\log x)^{1/3}\right). \tag{2}$$

In fact we prove a more general theorem for which (2) is a special case. Indeed, in [6] we considered the sets

$$F(a) = \{n:\ n\equiv a(\mathrm{mod}\,\varphi(n))\},$$
$$F'(a) = \{n\in F(a):\ n\neq pa \text{ for each prime } p\nmid a\},$$

where a is an arbitrary integer. We prove that for any a,

$$N(F'(a), x) = O\left(x^{2/3}(\log\log x)^{1/3}\right). \tag{3}$$