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On the class numbers of certain quadratic extensions

by

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1. Recently Armitage [1] has given an example of an algebraic number field K whose Dedekind zeta function $\zeta_K(s)$ possesses a zero at $s = \frac{1}{2}$. Serre [3] has given another such example, obtained from $\boldsymbol{Q}(\sqrt{5}, \sqrt{41})$, by adjoining $x^{1/2}$ where $x = (5+\sqrt{5})(41+\sqrt{205})$. We note in particular that this field is totally real.

These examples provide motivation for the following results.

THEOREM. *Let K be an algebraic number field of degree n for which $\zeta_K(\frac{1}{2}) = 0$. Let F be a quadratic extension of K having discriminant d and Dedekind zeta function $\zeta_F(s) = \zeta_K(s)L(s,\chi)$. Let $\varepsilon > 0$ be arbitrary. Then,*

$$L(1,\chi) > c_1|d|^{-1/2}(\log|d|)^{2-\varepsilon},$$

where c_1 (as all future c_i) denotes a positive computable constant depending (at most) on ε and K.

COROLLARY. *Assume in addition that K is totally real and that F is a totally imaginary quadratic extension of K. Then, if h denotes the class number of F,*

$$h > c_2(\log|d|)^{2-\varepsilon}.$$

2. We shall have need of the following facts:

Under the assumptions of the theorem we have:

(A) $L(s,\chi)$, being an abelian L-series, is an entire function.

(B) $$\sum_{\substack{\mathfrak{a}\in K\\ N_K\mathfrak{a}\leqslant X}} \frac{1}{N_K\mathfrak{a}} \sim c_3 \log X.$$

Here $\mathfrak{a}$ denotes an integral ideal and $N_K\mathfrak{a}$ denotes its norm (over $\boldsymbol{Q}$).

(C) For fixed σ, with $\frac{1}{2} \geqslant \sigma > 0$,

$$|\zeta_F(-\sigma+it)| < c_4|d|^{\frac{1}{2}+\sigma}|t|^{1+n+2n\sigma},$$

where c_4 is dependent on σ. (See [2], Theorem 4.)

Under the additional assumptions of the corollary,

(D) $h > c_5 L(1, \chi)|d|^{1/2}$.
(See [4], p. 224.)

3. Proofs. Let $f(s) = \prod_{j=0}^{n+2}(s+j)$. We consider the contour integral

$$I = \frac{1}{2\pi i}\int_{(2)} \zeta_F(s+\tfrac{1}{2})\frac{X^s}{f(s)}\,ds.$$

$$I = \frac{1}{(n+2)!}\sum_{\substack{\mathfrak{a}\in F\\ N_F\mathfrak{a}\leqslant X}} \frac{1}{(N_F\mathfrak{a})^{1/2}}\left(1-\frac{N_F\mathfrak{a}}{X}\right)^{n+2}.$$

Since F is a quadratic extension of K, every ideal $\mathfrak{a}$ of K is also an ideal of F with $N_F\mathfrak{a} = (N_K\mathfrak{a})^2$.

Therefore,

$$I \geqslant c_6 \sum_{\substack{\mathfrak{a}\in K\\ N_K\mathfrak{a}\leqslant X^{1/2}}} \frac{1}{N_K\mathfrak{a}}\left(1-\frac{(N_K\mathfrak{a})^2}{X}\right)^{n+2} \geqslant c_7\log X \quad \text{(by (B))}.$$

On the other hand, from (A) it follows that $\zeta_F(\frac{1}{2}) = 0$. Because of this and (C), we find, on shifting the line of integration to $\sigma = -\frac{1}{2}-\frac{1}{5}\varepsilon$,

$$I \leqslant c_8 L(1,\chi)X^{1/2} + \left(O\left(\frac{|d|}{X}\right)^{\frac{1}{2}+\frac{\varepsilon}{5}}\right),$$

provided that we assume (without loss of generality) that $\varepsilon < \dfrac{1}{2n}$. Choosing $X = \dfrac{|d|}{(\log|d|)^{2(1-\varepsilon)}}$, we get the theorem. Combining this with (D), the corollary follows immediately.

4. Remarks. It would of course be of great interest to pull back the information of these results to the imaginary quadratic extensions of $\boldsymbol{Q}$. Unfortunately, this does not seem an easy task.

It seems that it should be possible to improve the exponent of $\log|d|$ in the results. It follows from the functional equation that the order of the zero at $s = \frac{1}{2}$ is even and hence $\geqslant 2$. Given that the zero is of order r, one could try to choose $f(s)$ to be a multiple of s^r and achieve the exponent $2r-\varepsilon$. If $r > n+2$ then the choice $f(s) = s^r$ is sufficient to this purpose and the proof goes through. If, however, $r \leqslant n+2$ then other factors are necessary to make the integral at $-\frac{1}{2}-\frac{1}{5}\varepsilon$ converge, and it is not obvious how to introduce these factors in a way that will count the ideals with non-negative weights.

I am pleased to record my debt to D. M. Goldfeld for several informative discussions on class number problems.

References

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