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Fourier analysis used in analytic number theory

by

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In my previous papers [7], [8] and [6], I inadvertently used the second mean value theorem

$$\int_0^{a_1}\ldots\int_0^{a_n} f(t_1,\ldots,t_n)\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n \tag{1}$$

$$= f(a_1,\ldots,a_n)\int_{\xi_1}^{a_1}\ldots\int_{\xi_n}^{a_n}\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n$$

$$0\leqslant \exists\,\xi_j\leqslant a_j \quad (1\leqslant j\leqslant n),$$

for some positive non-decreasing function of many variables being

$$\chi_\lambda(t) = \frac{\sin 2\pi\lambda t}{\pi t}, \tag{2}$$

unaware until recently, that I could not succeed in proving (1) effectively. This theorem was used to deduce the Dirichlet integral theorem

$$\lim_{\substack{\lambda_j\to\infty\\(1\leqslant j\leqslant n)}}\int_0^{a_1}\ldots\int_0^{a_n}\{f(t_1+x_1,\ldots,t_n+x_n)- \tag{3}$$

$$-f(x_1,\ldots,x_n)\}\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n = 0,$$

or something like this. Hence arises the next question, namely for what sort of functions the satisfactory result (3) could be obtained.

In this note I shall introduce some new theorems which seem to be useful for investigating analytic number theory, though some extensions will be expected for these results (see [1], [6]).

Let

$$\xi = \varrho_1x_1+\ldots+\varrho_nx_n,\qquad \eta=\omega_1y_1+\ldots+\omega_ny_n,\qquad \mu=\omega_1t_1+\ldots+\omega_nt_n.$$

Writing

$$J(\mu) = \int_X E(-\mu\xi)\Big(\int_{P[1]} E(\xi\eta^k)\,dy\Big)^s dx,$$

Siegel proved, for $s \geqslant k+2$,

$$J(\mu) = |d|^{(1-s)/2} \prod_{l=1}^{r_1} F(\mu^{(l)}) \prod_{m=r_1+1}^{r_1+r_2} H(\mu^{(m)}) \tag{4}$$

by means of the Fourier transformation (see [4]). My method introduced in [8] was wrong and I could not follow this line in approach.

Finally I wish to offer my sincere apology and also express my thanks to Professor Siegel, who has given me fruitful sources of Fourier analysis.

1. Preliminary lemmas. Let there be an integrable function $\varphi(u_1, \ldots, u_n)$ over the interval

$$\{(u_1, \ldots, u_n);\ 0 \leqslant u_j \leqslant A_j\ (1 \leqslant j \leqslant n)\},$$

and let us represent $f(t_1, \ldots, t_n)$ by the integral

$$\int_0^{t_1} \ldots \int_0^{t_n} \varphi(u_1, \ldots, u_n)\, du_1 \ldots du_n \tag{5}$$

in the above interval, substituting u for t.

First of all I shall prove the following

LEMMA 1. *Let*

$$0 \leqslant b_j \leqslant b_j + x_j \leqslant c_j + x_j \leqslant A_j \quad (1 \leqslant j \leqslant n).$$

The function represented by (5) *satisfies the inequality*

$$\Big|\int_{b_1}^{c_1} \ldots \int_{b_n}^{c_n} \{f(t_1+x_1, \ldots, t_n+x_n) - f(x_1, \ldots, x_n)\}\chi_{\lambda_1}(t_1) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n\Big| \leqslant A \cdot B, \tag{6}$$

$\chi_\lambda(t)$ *being defined as* (2) *and* A, B *being defined as follows:*

$$\begin{aligned} A = {} & \Big|\int_0^{x_1+b_1} \ldots \int_0^{x_n+b_n} - \int_0^{x_1} \ldots \int_0^{x_n} \varphi(u_1, \ldots, u_n)\, du_1 \ldots du_n\Big| + \\ & + \Big\{\int_{x_1+b_1}^{x_1+c_1} \int_0^{x_2+b_2} \ldots \int_0^{x_n+b_n} + \ldots + \int_0^{x_1+b_1} \ldots \int_0^{x_{n-1}+b_{n-1}} \int_{x_n+b_n}^{x_n+c_n} + \\ & + \int_{x_1+b_1}^{x_1+c_1} \int_{x_2+b_2}^{x_2+c_2} \int_0^{x_3+b_3} \ldots \int_0^{x_n+b_n} + \ldots + \int_0^{x_1+b_1} \int_0^{x_2+b_2} \ldots \int_{x_{n-1}+b_{n-1}}^{x_{n-1}+c_{n-1}} \int_{x_n+b_n}^{x_n+c_n} + \\ & + \ldots + \int_{x_1+b_1}^{x_1+c_1} \int_{x_2+b_2}^{x_2+c_2} \ldots \int_{x_n+b_n}^{x_n+c_n} |\varphi(u_1, \ldots, u_n)|\, du_1 \ldots du_n\Big\}, \end{aligned}$$

and

$$B = \operatorname*{Max}_{\substack{b_j \leqslant \xi_j \leqslant c_j \\ (1 \leqslant j \leqslant n)}} \Big|\int_{\xi_1}^{c_1} \int_{\xi_2}^{c_2} \ldots \int_{\xi_n}^{c_n} \chi_{\lambda_1}(t_1)\chi_{\lambda_2}(t_2) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n\Big|.$$

To prove (6), we divide the interval $[b_j, c_j]$ by points

$$b_j = \xi_{j0} < \xi_{j1} < \ldots < \xi_{jk_j} = c_j \quad (1 \leqslant j \leqslant n),$$

we denote by Δ the subdivision

$$I = \bigcup_{j_1=0}^{k_1-1} \ldots \bigcup_{j_n=0}^{k_n-1} I_{j_1 \ldots j_n},$$

where

$$I = \{(t_1, \ldots, t_n);\ b_j \leqslant t_j < c_j\ (1 \leqslant j \leqslant n)\},$$

$$I_{j_1 \ldots j_n} = \{(t_1, \ldots, t_n);\ \xi_{1j_1} \leqslant t_1 < \xi_{1,j_1+1}, \ldots, \xi_{nj_n} \leqslant t_n < \xi_{n,j_n+1}\},$$

and we write

$$\delta(\Delta) = \operatorname*{Max}_{\substack{0 \leqslant s \leqslant k_j - 1 \\ (1 \leqslant j \leqslant n)}} (\xi_{j,s+1} - \xi_{js}).$$

Now we introduce the step function

$$f_\Delta(t_1+x_1, \ldots, t_n+x_n), \quad (t_1, \ldots, t_n) \in I,$$

such that its value is

$$f(\xi_{1j_1}+x_1, \ldots, \xi_{nj_n}+x_n)$$

when $(t_1, \ldots, t_n)$ belongs to $I_{j_1 \ldots j_n}$. Consequently,

$$\begin{aligned} & \int_{b_1}^{c_1} \ldots \int_{b_n}^{c_n} \{f_\Delta(t_1+x_1, \ldots, t_n+x_n) - f(x_1, \ldots, x_n)\}\chi_{\lambda_1}(t_1) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n \\ & = \sum_\Delta \{f(\xi_{1j_1}+x_1, \ldots, \xi_{nj_n}+x_n) - f(x_1, \ldots, x_n)\} \times \\ & \qquad \times \int_{\xi_{1j_1}}^{\xi_{1,j_1+1}} \ldots \int_{\xi_{nj_n}}^{\xi_{n,j_n+1}} \chi_{\lambda_1}(t_1) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n. \end{aligned} \tag{7}$$

From

$$\begin{aligned} & \int_{\xi_{1j_1}}^{\xi_{1,j_1+1}} \ldots \int_{\xi_{nj_n}}^{\xi_{n,j_n+1}} \chi_{\lambda_1}(t_1) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n \\ & = \int_{\xi_{1j_1}}^{c_1} \int_{\xi_{2j_2}}^{c_2} \ldots \int_{\xi_{n-1,j_{n-1}}}^{c_{n-1}} \int_{\xi_{nj_n}}^{c_n} - \int_{\xi_{1,j_1+1}}^{c_1} \int_{\xi_{2j_2}}^{c_2} \ldots \int_{\xi_{nj_n}}^{c_n} - \ldots - \int_{\xi_{1j_1}}^{c_1} \int_{\xi_{2j_2}}^{c_2} \ldots \int_{\xi_{n,j_n+1}}^{c_n} + \\ & \quad + \int_{\xi_{1,j_1+1}}^{c_1} \int_{\xi_{2,j_2+1}}^{c_2} \ldots \int_{\xi_{nj_n}}^{c_n} + \ldots + \int_{\xi_{1j_1}}^{c_1} \ldots \int_{\xi_{n-1,j_{n-1}+1}}^{c_{n-1}} \int_{\xi_{n,j_n+1}}^{c_n} + \\ & \quad + \ldots + (-1)^n \int_{\xi_{1,j_1+1}}^{c_1} \ldots \int_{\xi_{n,j_n+1}}^{c_n}, \end{aligned}$$

(7) can be represented by

$$(8)\quad \sum_{\Delta}\int_{\xi_{1j_1}}^{c_1}\cdots\int_{\xi_{nj_n}}^{c_n}\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n\{f(\xi_{1j_1}+x_1,\ldots,\xi_{nj_n}+x_n)-$$
$$-f(\xi_{1,j_1-1}+x_1,\ldots,\xi_{nj_n}+x_n)-\ldots-f(\xi_{1j_1}+x_1,\ldots,\xi_{n,j_n-1}+x_n)+$$
$$+f(\xi_{1,j_1-1}+x_1,\ \xi_{2,j_2-1}+x_2,\ldots,\xi_{nj_n}+x_n)+\ldots+$$
$$+f(\xi_{1j_1}+x_1,\ldots,\xi_{n-1,j_{n-1}-1}+x_{n-1},\ \xi_{n,j_n-1}+x_n)-$$
$$-\ldots+(-1)^n f(\xi_{1,j_1-1}+x_1,\ldots,\xi_{n,j_n-1}+x_n)\},$$

by using the extended abel transformation. The expression (8) shows only the standard term when $0<j_1,\ldots,0<j_n$. From the assumption (5), (8) becomes

$$\sum_{\Delta}\int_{\xi_{1j_1}}^{c_1}\cdots\int_{\xi_{nj_n}}^{c_n}\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n\times$$
$$\times\int_{\xi_{1,j_1-1}+x_1}^{\xi_{1j_1}+x_1}\cdots\int_{\xi_{n,j_n-1}+x_n}^{\xi_{nj_n}+x_n}\varphi(u_1,\ldots,u_n)\,du_1\ldots du_n.$$

Hence we get the desired result (6) by making $\delta(\Delta)\to 0$.

We often use the following relation to perform induction:

Lemma 2.

$$(9)\quad f(t_1,\ldots,t_n)=f(0,t_2,\ldots,t_n)+\ldots+f(t_1,\ldots,t_{n-1},0)-$$
$$-f(0,0,t_3,\ldots,t_n)-\ldots-f(t_1,\ldots,t_{n-2},0,0)+$$
$$+\ldots+(-1)^{n-1}f(0,\ldots,0)+$$
$$+\int_0^{t_1}\cdots\int_0^{t_n}\frac{\partial^n}{\partial u_1\ldots\partial u_n}f(u_1,\ldots,u_n)\,du_1\ldots du_n.$$

2. On the Fourier series. Let $f(t_1,\ldots,t_n)$ be of period 1 for each variable and integrable over the unit cube

$$I=\{(t_1,\ldots,t_n);\ 0\leqslant t_j\leqslant 1\ \ (1\leqslant j\leqslant n)\}.$$

According to Hecke [3], the Fourier coefficients of $f(t_1,\ldots,t_n)$ are defined by

$$a(m_1,\ldots,m_n)=\int\cdots\int_I f(t_1,\ldots,t_n)\,e^{-2\pi i(m_1t_1+\ldots+m_nt_n)}\,dt_1\ldots dt_n.$$

The formal infinite sum

$$\lim_{\substack{l_j\to\infty\\(1\leqslant j\leqslant n)}} S(l_1,\ldots,l_n)$$

is termed the Fourier series of $f(x_1,\ldots,x_n)$, where

$$S(l_1,\ldots,l_n)=\sum_{|m_1|\leqslant l_1}\cdots\sum_{|m_n|\leqslant l_n}a(m_1,\ldots,m_n)\,e^{2\pi i(m_1x_1+\ldots+m_nx_n)}.$$

Theorem 1. *Let a function $f(t_1,\ldots,t_n)$ defined as (5) over the unit cube I be of period 1 for each variable. Then the Fourier series of $f(x_1,\ldots,x_n)$ converges to $f(x_1,\ldots,x_n)$.*

Proof. From the definition, it follows that

$$(10)\quad S(l_1,\ldots,l_n)$$
$$=\sum_{|m_1|\leqslant l_1}\cdots\sum_{|m_n|\leqslant l_n}\int\cdots\int_I f(t_1,\ldots,t_n)\,e^{2\pi i(m_1(x_1-t_1)+\ldots+m_n(x_n-t_n))}\,dt_1\ldots dt_n$$
$$=\sum_{|m_1|\leqslant l_1}\cdots\sum_{|m_n|\leqslant l_n}\int_{-1/2}^{1/2}\cdots\int f(x_1+t_1,\ldots,x_n+t_n)\,e^{2\pi i(m_1t_1+\ldots+m_nt_n)}\,dt_1\ldots dt_n.$$

Noting that

$$(11)\quad \sum_{|m|\leqslant l}e^{2\pi imt}=\frac{\sin(2l+1)\pi t}{\sin\pi t},\qquad \int_{-1/2}^{1/2}\frac{\sin(2l+1)\pi t}{\sin\pi t}\,dt=1,$$

we have

$$S(l_1,\ldots,l_n)-f(x_1,\ldots,x_n)$$
$$=\int_{-1/2}^{1/2}\cdots\int\{f(t_1+x_1,\ldots,t_n+x_n)-f(x_1,\ldots,x_n)\}\tau_{l_1}(t_1)\ldots\tau_{l_n}(t_n)\,dt_1\ldots dt_n,$$

where

$$\tau_l(t)=\frac{\sin(2l+1)\pi t}{\sin\pi t}.$$

It suffices to prove

$$(12)\quad \lim_{\substack{l_j\to\infty\\(1\leqslant j\leqslant n)}}\int_0^{1/2}\cdots\int\{f(x_1+t_1,\ldots,x_n+t_n)-f(x_1,\ldots,x_n)\}\tau_{l_1}(t_1)\ldots\tau_{l_n}(t_n)\,dt_1\ldots dt_n$$
$$=0.$$

To prove (12), we divide the integral in the following way:

$$(13)\quad \int_0^{1/2}\cdots\int_0^{1/2}=\int_0^{c_1}\cdots\int_0^{c_n}+\int_{c_1}^{1/2}\int_0^{1/2}\cdots\int_0^{1/2}+\ldots+\int_0^{1/2}\cdots\int_0^{1/2}\int_{c_n}^{1/2}-$$
$$-\int_{c_1}^{1/2}\int_{c_2}^{1/2}\int_0^{1/2}\cdots\int_0^{1/2}-\ldots-\int_0^{1/2}\int_0^{1/2}\cdots\int_{c_{n-1}}^{1/2}\int_{c_n}^{1/2}+$$
$$+\ldots+(-1)^{n-1}\int_{c_1}^{1/2}\int_{c_2}^{1/2}\cdots\int_{c_n}^{1/2},\quad 0<c_j<1/2\ \ (1\leqslant j\leqslant n).$$

We know from the second mean value theorem that

$$\int_{\xi}^{1/2} \tau_l(t)\,dt = \int_{\xi}^{1/2} \frac{\pi t}{\sin \pi t}\chi_l(t)\,dt = \frac{\pi}{2}\int_{\eta}^{1/2} \chi_l(t)\,dt$$

where $0 \leqslant \xi \leqslant \eta \leqslant 1/2$. Noting this relation, we now apply Lemma 1 to each term of the right-hand side of (13), replacing $\chi(t)$ by $\tau(t)$. Firstly, by taking c_j sufficiently small, we can make

$$\int_0^{c_1}\int_0^{c_2}\cdots\int_0^{c_n}$$

sufficiently small in modulus, since A can do so in (6). Secondly, by taking l_j sufficiently large, we can make other terms of the right-hand side of (13) sufficiently small in modulus, since B can do so in (6).

THEOREM 2. *Let $f(t_1, \ldots, t_n)$ be a function such that*

$$\frac{\partial^{p_1+\ldots+p_n}}{\partial t_1^{p_1}\ldots\partial t_n^{p_n}} f(t_1, \ldots, t_n) \qquad (p_j = 0 \text{ or } 1)$$

are of period 1 for each variable and continuous over I. Then the Fourier series of $f(x_1, \ldots, x_n)$ converges to $f(x_1, \ldots, x_n)$.

Proof. Noting the relation (9) of Lemma 2, and using Theorem 1 we can deduce the desired result by induction.

THEOREM 3. *Let $f(t_1, \ldots, t_n)$ be integrable over the whole n-dimensional Euclidean space T and let*

$$\frac{\partial^{p_1+\ldots+p_n}}{\partial t_1^{p_1}\ldots\partial t_n^{p_n}} f(t_1, \ldots, t_n) \qquad (p_j = 0 \text{ or } 1)$$

be continuous over T. If the series

$$\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} \frac{\partial^{p_1+\ldots+p_n}}{\partial t_1^{p_1}\ldots\partial t_n^{p_n}} f(t_1+k_1, \ldots, t_n+k_n)$$

are all uniformly convergent over T, then

$$\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} f(x_1+k_1, \ldots, x_n+k_n) \tag{14}$$
$$= \lim_{\substack{l_j\to\infty\\(1\leqslant j\leqslant n)}} \sum_{|m_1|\leqslant l_1}\cdots\sum_{|m_n|\leqslant l_n} e^{2\pi i(m_1x_1+\ldots+m_nx_n)} \times$$
$$\times \int_T\cdots\int f(t_1, \ldots, t_n)\, e^{-2\pi i(m_1t_1+\ldots+m_nt_n)}\,dt_1\ldots dt_n.$$

Proof. We only note that

$$\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty}\int_I\cdots\int f(t_1+k_1, \ldots, t_n+k_n)\, e^{-2\pi i(m_1t_1+\ldots+m_nt_n)}\,dt_1\ldots dt_n$$
$$= \int_T\cdots\int f(t_1, \ldots, t_n)\, e^{-2\pi i(m_1t_1+\ldots+m_nt_n)}\,dt_1\ldots dt_n.$$

For the Poisson summation formula the following formulation, due to Bochner, is convenient to use (see [1], [9]), as well as Theorem 3.

THEOREM 4. *Let $f(t_1, \ldots, t_n)$ be continuous and integrable over the whole space T. If the series of the left-hand side of (14) is uniformly convergent over the unit cube I cited at the beginning of §2, then the formula (14) holds whenever the right-hand side of (14) is convergent.*

Here I shall prove the following Theorem 5 instead of Theorem 4, which can easily be deduced from the former, because such a formulation will probably give rise to further questions in Fourier analysis.

THEOREM 5. *Let $f(t_1, \ldots, t_n)$ be of period 1 for each variable and bounded and integrable over the unit cube I. If the Fourier series of $f(x_1, \ldots, x_n)$ is convergent, then it equals $f(x_1, \ldots, x_n)$ whenever the function is continuous at $(x_1, \ldots, x_n)$.*

Proof. Define

$$\sigma(p_1, \ldots, p_n) = \frac{1}{p_1\ldots p_n}\sum_{l_1=0}^{p_1-1}\cdots\sum_{l_n=0}^{p_n-1} S(l_1, \ldots, l_n).$$

Noting (11) and also

$$\sum_{l=0}^{p-1}\frac{\sin(2l+1)\pi u}{\sin\pi u} = \frac{\sin^2 p\pi u}{\sin^2\pi u}, \qquad \int_{-1/2}^{1/2}\frac{\sin^2 p\pi u}{\sin^2\pi u}\,du = p,$$

we have from (10)

$$\sigma(p_1, \ldots, p_n) - f(x_1, \ldots, x_n) = \int_{-1/2}^{1/2}\cdots\int\{f(t_1+x_1, \ldots, t_n+x_n) - f(x_1\ldots, x_n)\}\times$$
$$\times\frac{\sin^2 p_1\pi t_1}{p_1\sin^2\pi t_1}\cdots\frac{\sin^2 p_n\pi t_n}{p_n\sin^2\pi t_n}\,dt_1\ldots dt_n.$$

We divide the right integral in the following way:

$$\int_{-\delta_1}^{\delta_1}\cdots\int_{-\delta_n}^{\delta_n} + \Big(\int_{-1/2}^{-\delta_1}+\int_{\delta_1}^{1/2}\Big)\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} + \ldots + \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}\Big(\int_{-1/2}^{-\delta_n}+\int_{\delta_n}^{1/2}\Big) -$$
$$-\Big(\int_{-1/2}^{-\delta_1}+\int_{\delta_1}^{1/2}\Big)\Big(\int_{-1/2}^{-\delta_2}+\int_{\delta_2}^{1/2}\Big)\int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2} - \ldots - \int_{-1/2}^{1/2}\cdots\int_{-1/2}^{1/2}\Big(\int_{-1/2}^{-\delta_{n-1}}+\int_{\delta_{n-1}}^{1/2}\Big)\times$$
$$\times\Big(\int_{-1/2}^{-\delta_n}+\int_{\delta_n}^{1/2}\Big) + \ldots + (-1)^{n-1}\Big(\int_{-1/2}^{-\delta_1}+\int_{\delta_1}^{1/2}\Big)\cdots\Big(\int_{-1/2}^{-\delta_n}+\int_{\delta_n}^{1/2}\Big).$$

For any given positive ε, we can take positive δ_j $(1\leqslant j\leqslant n)$ such that

$$|f(x_1+t_1,\ldots,x_n+t_n)-f(x_1,\ldots,x_n)|<\varepsilon$$

provided $|t_j|\leqslant\delta_j$. Since

$$\Big(\int_{-1/2}^{-\delta}+\int_{\delta}^{1/2}\Big)\frac{\sin^2 p\pi u}{p\sin^2\pi u}\,du\leqslant\frac{1}{p\sin^2\pi\delta}\leqslant\frac{1}{4p\delta^2}\qquad(0<\delta<1/2),$$

we get

$$|\sigma(p_1,\ldots,p_n)-f(x_1,\ldots,x_n)|<\varepsilon+2M\Big(\frac{1}{4p_1\delta_1^2}+\ldots+\frac{1}{4p_n\delta_n^2}\Big)+$$
$$+2M\Big(\frac{1}{4p_1\delta_1^2}\,\frac{1}{4p_2\delta_2^2}+\ldots+\frac{1}{4p_{n-1}\delta_{n-1}^2}\,\frac{1}{4p_n\delta_n^2}\Big)+$$
$$+\ldots+2M\frac{1}{4p_1\delta_1^2}\cdots\frac{1}{4p_n\delta_n^2}$$
$$=\varepsilon+2M\Big\{\Big(1+\frac{1}{4p_1\delta_1^2}\Big)\cdots\Big(1+\frac{1}{4p_n\delta_n^2}\Big)-1\Big\}<2\varepsilon$$

by letting $p_j\to\infty$ $(1\leqslant j\leqslant n)$, M being the maximum modulus of f in I.

Hence, if we could deduce from the assumption $S(l_1,\ldots,l_n)\to A\,(l_j\to\infty)$ that

$$\sigma(p_1,\ldots,p_n)\to A\qquad(p_j\to\infty),$$

we may conclude that

$$A=f(x_1,\ldots,x_n).$$

To prove this, we divide the sum

$$\sum_{k_1=0}^{p_1-1}\cdots\sum_{k_n=0}^{p_n-1}S(k_1,\ldots,k_n)$$

in the following way $(0<l<p_j\ (1\leqslant j\leqslant n))$:

$$\sum_{k_1=0}^{l-1}\cdots\sum_{k_n=0}^{l-1}+\sum_{k_1=l}^{p_1-1}\sum_{k_2=0}^{l-1}\cdots\sum_{k_n=0}^{l-1}+\ldots+\sum_{k_1=0}^{l-1}\cdots\sum_{k_{n-1}=0}^{l-1}\sum_{k_n=l}^{p_n-1}+$$
$$+\sum_{k_1=l}^{p_1-1}\sum_{k_2=l}^{p_2-1}\sum_{k_3=0}^{l-1}\cdots\sum_{k_n=0}^{l-1}+\ldots+\sum_{k_1=0}^{l-1}\sum_{k_2=0}^{l-1}\cdots\sum_{k_{n-1}=l}^{p_{n-1}-1}\sum_{k_n=l}^{p_n-1}+\ldots+\sum_{k_1=l}^{p_1-1}\cdots\sum_{k_n=l}^{p_n-1}.$$

Here, for example,

$$\sum_{k_1=l}^{p_1-1}\sum_{k_2=0}^{l-1}\cdots\sum_{k_n=0}^{l-1}=\int_I\cdots\int f(t_1+x_1,\ldots,t_n+x_n)\times$$
$$\times\Big(\frac{\sin^2 p_1\pi t_1}{\sin^2\pi t_1}-\frac{\sin^2 l\pi t_1}{\sin^2\pi t_1}\Big)\frac{\sin^2 l\pi t_2}{\sin^2\pi t_2}\cdots\frac{\sin^2 l\pi t_n}{\sin^2\pi t_n}\,dt_1\ldots dt_n$$

is smaller than $M(p_1+l)l^{n-1}$ in modulus. Thus we obtain

$$\Big|\sum_{k_1=0}^{p_1-1}\cdots\sum_{k_n=0}^{p_n-1}S(k_1,\ldots,k_n)-\sum_{k_1=l}^{p_1-1}\cdots\sum_{k_n=l}^{p_n-1}S(k_1,\ldots,k_n)\Big|$$
$$\leqslant M\{l^n+l^{n-1}((l+p_1)+\ldots+(l+p_n))+\ldots+$$
$$+l((l+p_1)\ldots(l+p_{n-1})+\ldots+(l+p_2)\ldots(l+p_n))\}$$
$$=M\{(2l+p_1)\ldots(2l+p_n)-(l+p_1)\ldots(l+p_n)\}.$$

Consequently, if

$$|S(k_1,\ldots,k_n)-A|<\varepsilon$$

for $k_j>l$ $(1\leqslant j\leqslant n)$, then we have

$$A_1\leqslant\sigma(p_1,\ldots,p_n)\leqslant A_2,$$

where

$$A_1=(A-\varepsilon)(p_1-l)\ldots(p_n-l)-M\{(p_1+2l)\ldots(p_n+2l)-(p_1+l)\ldots(p_n+l)\},$$
$$A_2=(A+\varepsilon)(p_1-l)\ldots(p_n-l)+M\{(p_1+2l)\ldots(p_n+2l)-(p_1+l)\ldots(p_n+l)\}.$$

We can therefore complete the proof:

$$\lim_{\substack{p_j\to\infty\\(1\leqslant j\leqslant n)}}\frac{\sigma(p_1,\ldots,p_n)}{p_1\ldots p_n}=A.$$

3. On the Fourier integrals. Let $f(t_1,\ldots,t_n)$ be integrable over the whole space T. The integral

$$\hat f(x_1,\ldots,x_n)=\int_T\cdots\int f(t_1,\ldots,t_n)e^{-2\pi i(x_1t_1+\ldots+x_nt_n)}\,dt_1\ldots dt_n$$

is termed the Fourier transform of $f(x_1,\ldots,x_n)$. As is well known, $\hat f(x_1,\ldots,x_n)$ is bounded and continuous over the whole space.

Noting

$$\int_{-\lambda}^{\lambda}e^{2\pi iut}\,du=\frac{\sin2\pi\lambda t}{\pi t}=\chi_\lambda(t),$$

we have

$$\int_{-\lambda_1}^{\lambda_1}\cdots\int_{-\lambda_n}^{\lambda_n}\hat f(u_1,\ldots,u_n)e^{2\pi i(u_1x_1+\ldots+u_nx_n)}\,du_1\ldots du_n$$
$$=\int_T\cdots\int f(t_1,\ldots,t_n)\Big\{\int_{-\lambda_1}^{\lambda_1}\cdots\int_{-\lambda_n}^{\lambda_n}e^{2\pi i\{u_1(x_1-t_1)+\ldots+u_n(x_n-t_n)\}}\,du_1\ldots du_n\Big\}dt_1\ldots dt_n$$
$$=\int_T\cdots\int f(t_1+x_1,\ldots,t_n+x_n)\chi_{\lambda_1}(t_1)\ldots\chi_{\lambda_n}(t_n)\,dt_1\ldots dt_n$$

by Fubini's theorem. Replacing b_j $(1 \leqslant j \leqslant n)$ by 0 and letting c_j $(1 \leqslant j \leqslant n)$ to be ∞ in (6), we can infer

$$\lim_{\substack{\lambda_j\to\infty \\ (1\leqslant j\leqslant n)}} \int_0^\infty \cdots \int_0^\infty \{f(t_1+x_1, \ldots, t_n+x_n)-f(x_1, \ldots, x_n)\} \times \times \chi_{\lambda_1}(t_1) \ldots \chi_{\lambda_n}(t_n)\, dt_1 \ldots dt_n = 0$$

for the function defined as (5). Thus we get

THEOREM 6. *Let $f(t_1, \ldots, t_n)$ be a function defined as* (5) *over the whole space T. Then the Fourier inversion formula*

$$(15)\qquad \lim_{\substack{\lambda_j\to\infty \\ (1\leqslant j\leqslant n)}} \int_{-\lambda_1}^{\lambda_1} \cdots \int_{-\lambda_n}^{\lambda_n} \hat{f}(t_1, \ldots, t_n)\, e^{2\pi i(x_1t_1+\ldots+x_nt_n)}\, dt_1 \ldots dt_n = f(x_1, \ldots, x_n)$$

holds.

Noting the relation (9), we also have

THEOREM 7. *Let $f(t_1, \ldots, t_n)$ be a function such that*

$$\frac{\partial^{p_1+\ldots+p_n}}{\partial t_1^{p_1} \ldots \partial t_n^{p_n}} f(t_1, \ldots, t_n) \qquad (p_j = 0 \text{ or } 1)$$

are continuous and integrable over T. Then the Fourier inversion formula (15) *also holds.*

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The Diophantine equation $y^2 = Dx^4+1$, II

by

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Ljunggren [4] has shown by a deep and complicated method that the equation of the title, where D is a positive integer not a square, has at most two solutions in positive integers x, y. Elementary methods have been employed for special values of D which specify the solutions more closely ([1], [3]).

Conditions of a simple type have been found under which there are no such solutions ([2], [5], [6]). We prove

THEOREM. *Let $D \neq 2$ be an integer such that $u^2-Dv^2 = 2\varepsilon$ has solutions where $\varepsilon = \pm 1$. Then neither $y^2 = Dx^4+1$ nor $y^2 = 4Dx^4+1$ has a solution with $x > 0$ unless at least one of the equations $X^4-DY^4 = 2\varepsilon$ or $4X^4-DY^4 = 2\varepsilon$ has solutions.*

We shall require the following result, which is due to Nagell ([7], Theorems 8, 11).

LEMMA. *Let $D > 2$ be an integer, not a square. Then*

(i) *if the equation $u^2-Dv^2 = 2$ has solutions in integers u, v then there is exactly one class of solutions, which is therefore ambiguous; if $\alpha = U+VD^{1/2}$ is the fundamental solution, then $\frac{1}{2}\alpha^2$ is the fundamental solution of $u^2-Dv^2 = 1$;*

(ii) *ditto for the equation $u^2-Dv^2 = -2$;*

(iii) *at most one of the equations $u^2-Dv^2 = -1, 2$ and -2 has solutions in integers.*

Proof of the theorem. Let $\alpha = U+VD^{1/2}$ be the fundamental solution of $u^2-Dv^2 = 2\varepsilon$ and let $\beta = \frac{1}{2}\alpha^2$. If either $y^2 = Dx^4+1$ or $y^2 = 4Dx^4+1$ has any solution with $x > 0$, let x be the smallest positive integer which provides a solution of either of them. Then

$$y+x^2D^{1/2} \text{ or } y+2x^2D^{1/2} = \beta^n, \qquad n \geqslant 1,$$

i.e.

$$x^2 \text{ or } 2x^2 = \frac{\beta^n-\beta'^n}{2D^{1/2}}.$$