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On twin almost primes

by

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Dedicated to the memory of my teacher, Giovanni Ricci

1. Introduction and results. Let p, P_k denote respectively a prime and an almost prime with at most k factors. We are interested here in counting solutions of the equation $P_k+2 = p$, attaching suitable weights depending on the prime factors of P_k.

Let $\Lambda_k = \Lambda_k(n)$ be the generalized von Mangoldt function

$$\Lambda_k = \mu * L^k, \tag{1.1}$$

k integral $\geqslant 1$, where μ denotes the Möbius function, L denotes the arithmetical function $\log n$, and $*$ denotes the Dirichlet convolution. Clearly $\Lambda_1 = \Lambda$, the von Mangoldt function, and it is easily shown that

$$\Lambda_k = \Lambda_{k-1}L + \Lambda_{k-1}*\Lambda, \tag{1.2}$$

therefore

$$\Lambda_2 = \Lambda L + \Lambda*\Lambda,$$

$$\Lambda_3 = \Lambda L^2 + 3\Lambda L*\Lambda + \Lambda*\Lambda*\Lambda,$$

and so on. An easy induction on k now shows that

$\Lambda_k(n) = 0$ *if n has more than k prime factors* and thus Λ_k can be taken as a weighting function for k-almost primes. Thus the natural sum to study is

$$\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n), \tag{1.3}$$

and our purpose in this paper is to show that for large k the sum (1.3) is quite near to the expected asymptotic value. We shall also obtain the asymptotic behaviour of (1.3) for $k \geqslant 2$, but assuming the still unproved Halberstam–Richert conjecture on the distribution of primes in arithmetic progressions.

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Our results are as follows.

THEOREM 1. *Let $k \geqslant 1$. For $x > x_0(k)$ we have*

$$\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n) = 2H\{k+O(k^{4/3}2^{-k/3})\}x(\log x)^{k-1}, \tag{1.4}$$

where $H = \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$ *and where the constant implied in $O(\ldots)$ is absolute.*

Remark. Since one conjectures the asymptotic formula

$$\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n) \sim 2Hkx(\log x)^{k-1} \tag{1.5}$$

for $k \geqslant 1$, the inequality (1.4) shows that for large k the value of the sum is very near the expected asymptotic value.

For the proof of Theorem 1 we need a result on distribution of primes in arithmetic progressions:

$$\sum_{q<x^{\theta-\varepsilon}} \max_{(a,q)=1} \max_{y\leqslant x} \left|\psi(y;q,a)-\frac{y}{\varphi(q)}\right| \ll \frac{x}{(\log x)^A} \tag{1.6}$$

for every fixed $\varepsilon > 0$ and every large A. The Bombieri–Vinogradov Theorem shows that (1.6) holds with $\theta = \frac{1}{2}$, and Theorem 1 depends on this. If we assume:

HALBERSTAM–RICHERT CONJECTURE: The inequality (1.6) holds with $\theta = 1$,

then we shall prove

THEOREM 2. *If the Halberstam–Richert Conjecture holds we have for $k \geqslant 2$ the asymptotic formula*

$$\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n) \sim 2Hkx(\log x)^{k-1}. \tag{1.7}$$

However, our proof of (1.7) for $k \geqslant 2$ fails in the most interesting case $k = 1$, and it is unlikely to get a non-trivial lower bound for $\sum_{n\leqslant x} \Lambda(n+2)\Lambda(n)$ without using some new idea.

In the last section of this paper we shall point out some consequences of Theorem 2 and some generalizations.

Finally I want to thank Professor A. Selberg of the Institute for Advanced Study for some useful conversations on this subject.

2. The auxiliary sum. The main idea in the proof is to evaluate not the original sum (1.3) but the modified sum

$$\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n)\Big(\sum_{\substack{d|n\\ d<z}} \lambda_d\Big)^2 \tag{2.1}$$

where the λ_d are parameters to be chosen according to the general rules of Selberg's sieve. In particular, we put the following restrictions on the coefficients λ_d: $\lambda_1 = 1$, $|\lambda_d| \leqslant 1$, $\lambda_d = 0$ if d is not square-free, $\lambda_d = 0$ if $d \geqslant z$.

The following estimate shows that the sum (2.1) is near to the sum (1.3) if z is sufficiently small.

LEMMA 1. *If $z < x^{1/20k}$ we have the inequality*

$$\Big|\sum_{n\leqslant x} \Lambda(n+2)\Lambda_k(n)\Big[1-\Big(\sum_{\substack{d|n\\ d<z}}\lambda_d\Big)^2\Big]\Big| \ll k^2x(\log x)^{k-2}(\log z), \tag{2.2}$$

where the constant implied in $\ll$ is absolute.

Proof. Let $\mathcal{N}_j$, $j = 0, 1, \ldots, k$, be the set of integers $n \leqslant x$ with exactly j prime factors $\leqslant z$. We have then

$$\Big(\sum_{\substack{d|n\\ d<z}}\lambda_d\Big)^2 = 1 \quad \text{if} \quad n \in \mathcal{N}_0,$$

$$\Big(\sum_{\substack{d|n\\ d<z}}\lambda_d\Big)^2 \leqslant 4^j \quad \text{if} \quad n \in \mathcal{N}_j,$$

therefore the left-hand side of (2.2) is majorized by

$$\sum_{j=1}^{k} 4^j \sum_{\mathcal{N}_j} \Lambda(n+2)\Lambda_k(n). \tag{2.3}$$

From the definition (1.1) of Λ_k we find that if $(m_1, m_2) = 1$ then

$$\Lambda_k(m_1m_2) = \sum_{h=0}^{k}\binom{k}{h}\Lambda_h(m_1)\Lambda_{k-h}(m_2). \tag{2.4}$$

Now every integer $n \in \mathcal{N}_j$ is written uniquely in the form $n = dm$, where $(d, m) = 1$ and $d = p_1^{a_1}\ldots p_j^{a_j}$ with $p_i \leqslant z$, $a_i \geqslant 1$ and m has no prime factor $\leqslant z$. Since $\Lambda_h(d) = 0$ if $h < j$, using (2.4) we deduce the inequality

$$\sum_{\mathcal{N}_j} \Lambda(n+2)\Lambda_k(n) \leqslant \sum_{h=j}^{k}\binom{k}{h}\sum_{d\leqslant x}{}' \Lambda_h(d)\sum_{m\leqslant x/d}\Lambda(dm+2)\Lambda_{k-h}(m), \tag{2.5}$$

where $\sum'$ is restricted to integers $d = p_1^{a_1}\ldots p_j^{a_k}$ with $2 < p_i \leqslant z$, $a_i \geqslant 1$.

The next step is to give a rough estimate of the sum

$$\sum_{m\leqslant x/d}\Lambda(dm+2)\Lambda_{k-h}(m),$$

valid uniformly in d for $d \leqslant x$.

Let

$$\Phi(d)=\prod_{\substack{p\mid d\\ p>2}}\frac{p-1}{p-2}. \tag{2.6}$$

We shall prove that for odd d we have

$$\sum_{m\leqslant\xi}{}' \Lambda(dm+2)\Lambda_k(m)\leqslant c_1 k\Phi(d)\,\xi(\log\xi+c_2)^{k-2}\log(\xi d+2) \tag{2.7}$$

where $\sum'$ denotes a sum over odd integers and c_1, c_2 are absolute constants.

A simple application of Selberg's sieve shows that (2.7) holds if $k=1$. Also using the identity $\Lambda_{k+1}=\Lambda_k L+\Lambda_k*\Lambda$ one checks easily that (2.7) holds in case $k=2$. Suppose (2.7) proved for k. We have

$$\begin{aligned}\sum_{m\leqslant\xi}{}' \Lambda(dm+2)\Lambda_{k+1}(m)\\ \leqslant\sum_{m\leqslant\xi}{}' \Lambda(dm+2)\Lambda_k(m)(\log\xi)+\sum_{\delta\leqslant\xi}\Lambda(\delta)\sum_{m\leqslant\xi/\delta}{}' \Lambda(d\delta m+2)\Lambda_k(m).\end{aligned} \tag{2.8}$$

If we denote by A_k a constant such that (2.7) holds with A_k in place of $c_1 k$, the right-hand side of (2.8) is majorized by

$$\begin{aligned}&A_k\Phi(d)\,\xi(\log\xi+c_2)^{k-2}\log\xi\log(\xi d+2)+\\ &\quad+A_k\Phi(d)\,\xi\log(\xi d+2)\sum_{\delta\leqslant\xi}{}' \frac{\Lambda(\delta)}{\delta}\Phi(\delta)\Big(\log\frac{\xi}{\delta}+c_2\Big)^{k-2}.\end{aligned} \tag{2.9}$$

A simple argument by partial summation shows that if $k\geqslant 2$ we have

$$\sum_{\delta\leqslant\xi}{}' \frac{\Lambda(\delta)}{\delta}\Phi(\delta)\Big(\log\frac{\xi}{\delta}+c_2\Big)^{k-2}\leqslant\frac{1}{k-1}(\log\xi+c_2)^{k-1}+10(\log\xi+c_2)^{k-2},$$

whence combining this inequality with (2.9) we get, assuming $c_2\geqslant 10$, the bound

$$A_{k+1}\leqslant A_k\frac{k}{k-1}$$

and finally

$$A_{k+1}\leqslant A_2 k,$$

and (2.7) follows.

By (2.7) and (2.5) we obtain

$$\begin{aligned}\sum_{\mathcal{N}_j}\Lambda(n+2)\Lambda_k(n)\\ \leqslant k\sum_{h=j}^{k-1}\binom{k}{h}\sum_{d\leqslant x}{}' \Lambda_h(d)\Phi(d)\frac{x}{d}(\log x+c_2)^{k-h-1}+\sum_{d\leqslant x}{}' \Lambda_k(d)\Lambda(d+2).\end{aligned}$$

An easy estimate shows that

$$\sum_{d\leqslant x}{}' \frac{\Lambda_h(d)}{d}\Phi(d)\ll j^h(\log z)^h,$$

since d is restricted to integers $p_1^{a_1}\dots p_j^{a_j}$ with $2<p_i\leqslant x$, $a_i\geqslant 1$. Hence

$$\begin{aligned}\sum_{\mathcal{N}_j}\Lambda(n+2)\Lambda_k(n)&\ll kx\sum_{h=j}^{k-1}\binom{k}{h}(j\log z)^h(\log x+c_2)^{k-h-1}\\ &\ll\frac{kx}{\log x}\binom{k}{j}(j\log z)^j(\log x+c_2+\log z)^{k-j}\end{aligned}$$

and finally the expression (2.3) is majorized by

$$(2.3)\ll\frac{kx}{\log x}\sum_{j=1}^{k}4^j\binom{k}{j}(j\log z)^j(\log x+c_2+j\log z)^{k-j}.$$

Using the inequality $\log z<\dfrac{1}{20k}\log x$, and

$$\binom{k}{j}j^j<(ek)^j j,\qquad \Big(1+\frac{j}{20k}\Big)^{k-j}<e^{j/20}$$

we obtain

$$(2.3)\ll k^2x(\log x)^{k-2}(\log z)\sum_{j=1}^{k}j\Big(\frac{4e^{21/20}}{20}\Big)^j\ll k^2x(\log x)^{k-2}(\log z),$$

Q.E.D.

Now we have

$$\begin{aligned}\sum_{n\leqslant x}\Lambda(n+2)\Lambda_k(n)\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2\\ =\sum_{n\leqslant x}\Lambda(n+2)\sum_{\substack{d\mid n\\ d<y}}\mu(d)\Big(\log\frac{n}{d}\Big)^k\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2+\\ +\sum_{n\leqslant x}\Lambda(n+2)\sum_{\substack{d\mid n\\ d\leqslant n/y}}\mu\Big(\frac{n}{d}\Big)(\log d)^k\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2=\Sigma_1+\Sigma_2.\end{aligned} \tag{2.10}$$

The sum Σ_1 is estimated using (1.6), provided $yz^2<x^{\theta-\varepsilon}$. The sum Σ_2 instead is estimated trivially by

$$|\Sigma_2|\leqslant\Big(\log\frac{x}{y}\Big)^k\sum_{n\leqslant x}\Lambda(n+2)\Big(\sum_{\substack{d\mid n\\ d\leqslant x/y}}1\Big)\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2 \tag{2.11}$$

and the last sum is estimated using Selberg's sieve.

3. Estimation of Σ_1. In this section we prove the estimate:

LEMMA 2. *We have for $z < x^{1/20k}$ and $yz^2 < x^{\theta-\varepsilon}$, $y > z^2x^{\varepsilon}$, the inequality*

$$|\Sigma_1 - 2Hkx(\log x)^{k-1}| \ll k^2x(\log x)^{k-2}(\log z). \tag{3.1}$$

Proof. A standard application of (1.6) shows that

$$\Sigma_1 \sim x\sum_{j=0}^{k}(-1)^{k-j}\frac{k!}{j!}\sum_{d<y}{}'\sum_{\nu_1,\nu_2<z}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}\mu(d)\left(\log\frac{x}{d}\right)^j}{\varphi([d,\nu_1,\nu_2])} \tag{3.2}$$

where $\sum'$ denotes a sum over odd integers and $[a, b, \ldots, r]$ denotes the least common multiple of $a, b, \ldots, r$. The asymptotic relation (3.2) is understood with an error term $\ll \dfrac{x}{(\log x)^A}$ for every fixed positive A.

Let $\varphi_2(r) = \prod_{p|r}(p-2)$ and note that for square-free r we have

$$\varphi(r) = \sum_{d|r}\varphi_2(d). \tag{3.3}$$

The inner double sum in the right-hand side of (3.2) is

$$\sum_{d}{}'\sum_{\nu_1,\nu_2}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x}{d}\right)^j\varphi\big((d,[\nu_1,\nu_2])\big)$$

and by (3.3) it is easily transformed into

$$\sum_{\nu_1,\nu_2}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\sum_{r|[\nu_1,\nu_2]}\varphi_2(r)\sum_{d\equiv 0\,(\mathrm{mod}\,r)}{}'\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x}{d}\right)^j. \tag{3.4}$$

Now

$$\sum_{d\equiv 0\,(\mathrm{mod}\,r)}{}'\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x}{d}\right)^j = \frac{\mu(r)}{\varphi(r)}\sum_{\substack{d\leqslant y/r\\(d,r)=1}}{}'\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x/r}{d}\right)^j \tag{3.5}$$

and, since $y/r \geqslant y/z^2 > x^{\varepsilon}$ it is easily seen that

$$\sum_{\substack{d\leqslant y/r\\(d,r)=1}}\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x/r}{d}\right)^j = \sum_{\substack{d=1\\(d,r)=1}}^{\infty}\frac{\mu(d)}{\varphi(d)}\left(\log\frac{x/r}{d}\right)^j + O\big((\log x)^{-A}\big) \tag{3.6}$$

for every fixed positive A, uniformly for $r \leqslant z^2$. If we define

$$c_h(r) = (-1)^h\sum_{\substack{d=1\\(d,r)=1}}^{\infty}\frac{\mu(d)}{\varphi(d)}(\log d)^h \tag{3.7}$$

we obtain

$$\Sigma_1 \sim x\sum_{j=0}^{k}(-1)^{k-j}\frac{k!}{j!}\times \tag{3.8}$$
$$\times\sum_{\nu_1,\nu_2<z}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\sum_{h=0}^{j}\binom{j}{h}\sum_{r|[\nu_1,\nu_2]}\mu(r)\frac{\varphi_2(r)}{\varphi(r)}c_h(r)\left(\log\frac{x}{r}\right)^{j-h}.$$

It is easily shown that

$$c_0(r) = 0, \qquad c_1(r) = 2H\frac{\varphi(r)}{\varphi_2(r)} \tag{3.9}$$

and we shall prove that

$$\left|\sum_{\nu_1,\nu_2}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\sum_{r|[\nu_1,\nu_2]}\mu(r)\frac{\varphi_2(r)}{\varphi(r)}c_h(r)\left(\log\frac{x}{r}\right)^a\right| \ll (\log x)^a \tag{3.10}$$

for every $a, h \geqslant 0$. Using (3.9), (3.10) the asymptotic formula (3.8) becomes

$$\Sigma_1 = 2Hkx\sum_{\nu_1,\nu_2<z}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\sum_{r|[\nu_1,\nu_2]}\mu(r)\left(\log\frac{x}{r}\right)^{k-1} + O\big(x(\log x)^{k-2}\big). \tag{3.11}$$

The double sum on the right-hand side of (3.11) is equal to

(3.12)

$$(\log x)^{k-1} + \sum_{h=1}^{k-1}\binom{k-1}{h}\sum_{\nu_1,\nu_2<z}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\Lambda_h([\nu_1,\nu_2])\left(\log\frac{x}{[\nu_1,\nu_2]}\right)^{k-1-h}.$$

Since

$$\Big(\sum_{\nu|r}\lambda_\nu\Big)^2 \leqslant 4^h$$

if r has not more than h prime factors, we see that

$$\sum_{\nu_1,\nu_2<z}{}'\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{\varphi([\nu_1,\nu_2])}\Lambda_h([\nu_1,\nu_2])\left(\log\frac{x}{[\nu_1,\nu_2]}\right)^{k-1-h} \tag{3.13}$$
$$\leqslant 4^h\sum_{m<z^2}\mu^2(m)\frac{\Lambda_h(m)}{\varphi(m)}\left(\log\frac{x}{m}\right)^{k-1-h}.$$

We have also

$$\sum_{m\leqslant\xi}\frac{\mu^2(m)}{\varphi(m)}\Lambda_h(m) = \sum_{m\leqslant\xi}\frac{\mu^2(m)}{\varphi(m)}\Lambda_{h-1}(m)\log m +$$
$$+\sum_{m\leqslant\xi}\mu^2(m)\frac{\Lambda_{h-1}(m)}{\varphi(m)}\sum_{\delta\leqslant\xi/m}\frac{\mu^2(\delta)}{\varphi(\delta)}\Lambda(\delta)$$
$$\leqslant(\log\xi+10)\sum_{m\leqslant\xi}\frac{\mu^2(m)}{\varphi(m)}\Lambda_{h-1}(m)$$

because

$$\sum_{\delta \leqslant \xi/m} \frac{\mu^2(\delta)}{\varphi(\delta)} \Lambda(\delta) \leqslant \log \frac{\xi}{m} + 10,$$

therefore

$$(3.14) \qquad \sum_{m \leqslant \xi} \frac{\mu^2(m)}{\varphi(m)} \Lambda_h(m) \leqslant (\log \xi + 10)^h$$

for all ξ, h. Using now (3.13) and (3.14) we deduce that the sum in (3.12) is majorized by

$$\sum_{h=1}^{k-1} \binom{k-1}{h} 4^h (2\log z + 10)^h (\log x)^{k-1-h}$$
$$= (\log x + 8\log z + 40)^{k-1} - (\log x)^{k-1} \ll k(\log x)^{k-2}(\log z)$$

provided $z < x^{1/20k}$; inequality (3.1) follows from this. Thus in order to complete the proof of Lemma 2 we have to prove (3.10).

In the sum (3.10) we replace $\log \frac{x}{r}$ by $\log \frac{x}{n} + \log \frac{n}{r}$, where $n = [\nu_1, \nu_2]$ and note again that

$$\Big| \sum_{[\nu_1, \nu_2] = n} \lambda_{\nu_1} \lambda_{\nu_2} \Big| \leqslant d^2(n).$$

Hence in order to prove (3.10) it is sufficient to prove that

$$(3.15) \qquad \sum_{n \leqslant x}{}' \frac{\mu^2(n)}{\varphi(n)} d^2(n) \Big| \sum_{r|n} \mu(r) \frac{\varphi_2(r)}{\varphi(r)} c_h(r) \Big(\log \frac{n}{r}\Big)^a \Big| \ll (\log x)^a$$

for every $a, h \geqslant 0$.

We define the arithmetical function $b_h(r)$ by

$$b_h = \Big(\frac{\varphi_2}{\varphi} c_h\Big) * \mu$$

so that

$$(3.16) \qquad \frac{\varphi_2(r)}{\varphi(r)} c_h(r) = \sum_{\varrho|r} b_h(\varrho).$$

Replacing r by ϱs and n by ϱm we see that the left-hand side of (3.15) is

$$\sum_{n \leqslant x}{}' \frac{\mu^2(n)}{\varphi(n)} d^2(n) \Big| \sum_{r|n} \sum_{\varrho|r} b_h(\varrho) \mu(r) \Big(\log \frac{n}{r}\Big)^a \Big|$$
$$\leqslant \sum_{\varrho \leqslant x}{}' \frac{\mu^2(\varrho)}{\varphi(\varrho)} d^2(\varrho) |b_h(\varrho)| \sum_{m \leqslant x/\varrho} \frac{\mu^2(m)}{\varphi(m)} d^2(m) \Big| \sum_{s|m} \mu(s) \Big(\log \frac{m}{s}\Big)^a \Big|$$

$$= \sum_{\varrho \leqslant x}{}' \frac{\mu^2(\varrho)}{\varphi(\varrho)} d^2(\varrho) |b_h(\varrho)| \sum_{m \leqslant x/\varrho} \frac{\mu^2(m)}{\varphi(m)} d^2(m) \Lambda_a(m)$$
$$\ll (\log x)^a \sum_{\varrho \leqslant x}{}' \frac{\mu^2(\varrho)}{\varphi(\varrho)} d^2(\varrho) |b_h(\varrho)|$$

by (3.14). So it remains to prove that

$$(3.17) \qquad \sum_{\substack{\varrho=1 \\ \varrho \text{ odd}}}^{\infty} \frac{\mu^2(\varrho)}{\varphi(\varrho)} d^2(\varrho) |b_h(\varrho)| < +\infty.$$

We have

$$(3.18) \qquad \sum_{(d,r)=1}{}' \frac{\mu(d)}{\varphi(d)} d^{-s} = \frac{1}{\zeta(s+1)} F(s) P_r(s)$$

where $F(s)$ is given by an absolutely convergent Dirichlet series for $\sigma > -\frac{1}{2}$ and where

$$(3.19) \qquad P_r(s) = \prod_{\substack{p|r \\ p>2}} \Big(1 - \frac{1}{p-1} p^{-s}\Big)^{-1}.$$

It follows that

$$(3.20) \qquad c_h(r) = \Big(\frac{d}{ds}\Big)^h \Big(\frac{F(s)P_r(s)}{\zeta(s+1)}\Big)_{s=0}$$
$$= P_r(0) \cdot \text{polynomial in } \Big(\frac{P_r'}{P_r}\Big)^{(\nu)}(0), \quad \nu < h.$$

But now

$$P_r(0) = \frac{\varphi(r)}{\varphi_2(r)}$$

and

$$\Big(\frac{P_r'}{P_r}\Big)^{(\nu)}(0) = \sum_{p|r} \frac{a_\nu(p)}{p-2}$$

where

$$a_\nu(p) \ll (\log p)^{\nu+1}.$$

Using (3.20) we deduce that

$$(3.21) \qquad \frac{\varphi_2(r)}{\varphi(r)} c_h(r) = \sum_{d|r} \frac{a_h(d)}{\varphi_2(d)}$$

where

$$(3.22) \qquad a_h(d) \ll d^\varepsilon$$

for every fixed $\varepsilon > 0$. But clearly Möbius' inversion formula gives

$$b_h(d) = \frac{a_h(d)}{\varphi_2(d)}$$

and (3.17) follows from (3.22), Q.E.D.

4. Estimation of Σ_2. In this section we prove the estimate:

LEMMA 3. *We have for $y > z^2x^{\alpha+\varepsilon}$, $x/y > z^2x^\varepsilon$, $0 < \alpha < 1$, and a suitable choice of the coefficients λ_d, the inequality*

$$|\Sigma_2| \ll \alpha^{-1}\frac{x}{(\log z)^2}\left(\log\frac{x}{y}\right)^{k+1}, \tag{4.1}$$

where the constant implied in $\ll$ is absolute.

Proof. In view of (2.11) it is sufficient to show

$$\sum_{n\leqslant x}\Lambda(n+2)\Big(\sum_{\substack{d\mid n\\ d\leqslant x/y}}1\Big)\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2 \ll \alpha^{-1}x\frac{\log(x/y)}{(\log z)^2}. \tag{4.2}$$

Following the general ideas of Selberg's sieve, we shall replace $\Lambda(n+2)$ in (4.2) by $(\log x)\big(\sum_{\substack{d\mid n+2\\ d<\zeta}}\lambda_d^*\big)^2$ where as usual

$$\lambda_1^* = 1,\quad |\lambda_d^*|\leqslant 1,\quad \lambda_d^* = 0 \quad \text{if } d \text{ is not square-free},$$
$$\lambda_d^* = 0 \quad \text{if} \quad d \geqslant \zeta.$$

Then we shall prove that for $y > (z\zeta)^2x^\varepsilon$, $x/y > z^2x^\varepsilon$ we have

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}}}\Big(\sum_{\substack{d\mid n\\ d\leqslant x/y}}1\Big)\Big(\sum_{\substack{d\mid n\\ d<z}}\lambda_d\Big)^2\Big(\sum_{\substack{d\mid n+2\\ d<\zeta}}\lambda_d^*\Big)^2 \ll x\frac{\log(x/y)}{(\log\zeta)(\log z)^2} \tag{4.3}$$

for a suitable choice of λ_d^*, λ_d.

Let us write

$$a_\delta = \sum_{[d,\nu_1,\nu_2]=\delta}{}' \lambda_{\nu_1}\lambda_{\nu_2} \tag{4.4}$$

where $\sum'$ is restricted to odd d, ν_1, ν_2 satisfying $d\leqslant x/y$, $\nu_1, \nu_2 < z$. The left-hand side of (4.4) becomes

$$\sum_{\delta<\frac{x}{y}z^2}a_\delta\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}\\ n\equiv 0 (\mathrm{mod}\,\delta)}}\Big(\sum_{\substack{d\mid n+2\\ d<\zeta}}\lambda_d^*\Big)^2. \tag{4.5}$$

We choose

$$\lambda_d^* = \mu(d)\,d\,\frac{\displaystyle\sum_{rd<\zeta}\frac{\mu^2(rd)}{\varphi(rd)}}{\displaystyle\sum_{r<\zeta}\frac{\mu^2(r)}{\varphi(r)}} \tag{4.6}$$

and write for simplicity

$$\Sigma_\zeta = \sum_{r<\zeta}\frac{\mu^2(r)}{\varphi(r)}. \tag{4.7}$$

With this choice of λ_d^* we have

$$\sum_{\substack{n\leqslant x\\ n\ \mathrm{odd}\\ n\equiv 0(\mathrm{mod}\,\delta)}}\Big(\sum_{\substack{d\mid n+2\\ d<\zeta}}\lambda_d^*\Big)^2 = \frac{x}{2\delta}\sum_{\substack{\nu_1,\nu_2<\zeta\\ (\nu_i,2\delta)=1}}\frac{\lambda_{\nu_1}^*\lambda_{\nu_2}^*}{[\nu_1,\nu_2]} + O(\zeta^{2+\varepsilon}), \tag{4.8}$$

and by our choice of λ_d^* the usual manipulations of the sum involving the λ^* show that

$$\sum_{\substack{\nu_1,\nu_2<\zeta\\ (\nu_i,2\delta)=1}}\frac{\lambda_{\nu_1}^*\lambda_{\nu_2}^*}{[\nu_1,\nu_2]} = (\Sigma_\zeta)^{-2}\sum_{\substack{r<\zeta\\ (r,2\delta)=1}}\frac{\mu^2(r)}{\varphi(r)}\Big(\sum_{\substack{n<\zeta/r\\ n\mid 2\delta}}\frac{\mu^2(n)}{\varphi(n)}\Big)^2. \tag{4.9}$$

Since $\Sigma_\zeta\sim\log\zeta$ and

$$\sum_\delta |a_\delta|\,\zeta^{2+\varepsilon} \ll \frac{x}{y}z^2\zeta^2x^\varepsilon \ll x^{1-\varepsilon}$$

by our assumption on y, we deduce from (4.5), (4.8), (4.9) that

$$(4.3)\sim x(\log\zeta)^{-2}\sum_{\delta<\frac{x}{y}z^2}a_\delta\frac{1}{2\delta}\sum_{\substack{r<\zeta\\ (r,2\delta)=1}}\frac{\mu^2(r)}{\varphi(r)}\Big(\sum_{\substack{n<\zeta/r\\ n\mid 2\delta}}\frac{\mu^2(n)}{\varphi(n)}\Big)^2. \tag{4.10}$$

We write the right-hand side of (4.10) in the form

$$\frac{x}{2(\log\zeta)^2}\sum_r\sum_{n_1}\sum_{n_2}\frac{\mu^2(r)}{\varphi(r)}\frac{\mu^2(n_1)}{\varphi(n_1)}\frac{\mu^2(n_2)}{\varphi(n_2)}\sum_\delta\frac{a_\delta}{\delta} \tag{4.11}$$

where

$$r<\zeta,\quad n_1, n_2 < \zeta/r,\quad r \text{ is odd},$$
$$\delta < \frac{x}{y}z^2,\quad 2\delta\equiv 0\ (\mathrm{mod}\,[n_1,n_2]),\quad (\delta, 2r) = 1. \tag{4.12}$$

Now a little thought discloses that $\sum_\delta a_\delta/\delta$ is a positive definite quadratic form in the λ_ν. In fact, we have

$$\sum_\delta\frac{a_\delta}{\delta} = \sum\frac{\lambda_{\nu_1}\lambda_{\nu_2}}{[d,\nu_1,\nu_2]}$$

and writing $L_m = \sum_{[\nu_1,\nu_2]=m}\lambda_{\nu_1}\lambda_{\nu_2}$, $\varrho = (d, [\nu_1,\nu_2])$, $[\nu_1,\nu_2] = n\varrho$, $d = t\varrho$, the previous sum becomes, since $[\nu_1,\nu_2]$ is square-free:

$$\sum_\varrho\frac{1}{\varrho}\sum\frac{L_{n\varrho}}{tn}$$

where the inner sum ranges over

$$(t, n) = 1, \quad (n, \varrho) = 1, \quad tn \leqslant \frac{x}{y} z^2/\varrho,$$
$$2\varrho tn \equiv 0 \,(\mathrm{mod}\,[n_1, n_2]), \quad (\varrho tn, r) = 1.$$

Collecting together terms with a given $tn = N$, the sum becomes a linear combination with positive coefficients, of sums of the type

$$\sum_{\substack{n|N \\ \left(n, \frac{N}{n}\right)=1 \\ (n,\varrho)=1}} L_{n\varrho} = \Big(\sum_{\substack{\nu|N_0 \\ (\nu,\varrho)=1}} \lambda_{\varrho\nu}\Big)^2$$

where N_0 is the square-free part of N.

But this means that the expression (4.11) can only increase if we increase the range for n_1, n_2 in (4.12). Hence in the right-hand side of (4.10) we can omit the condition $n < \zeta/r$ in the summation over n, and still get an upper bound. Using now

$$\sum_{n|2\delta} \frac{\mu^2(n)}{\varphi(n)} = \frac{2\delta}{\varphi(2\delta)}$$

we conclude that

$$(4.13) \qquad (4.3) \leqslant [1+o(1)] \frac{x}{(\log\zeta)^2} \sum_{\delta < \frac{x}{y}z^2} \frac{a_\delta}{\varphi(\delta)} \frac{2\delta}{\varphi(2\delta)} \sum_{\substack{r<\zeta \\ (r, 2\delta)=1}} \frac{\mu^2(r)}{\varphi(r)}.$$

Now we have the asymptotic formula

$$(4.14) \qquad \frac{2\delta}{\varphi(2\delta)} \sum_{\substack{r<\zeta \\ (r, 2\delta)=1}} \frac{\mu^2(r)}{\varphi(r)} = \log\zeta + A_0 + A_1 \sum_{p|2\delta} \frac{\log p}{p-1} + O(\delta^\varepsilon \zeta^{-1/4})$$

uniformly in δ, for two absolute constants A_0, A_1. Hence the right-hand side of (4.13) is majorized by

$$(4.15) \qquad [1+o(1)] \frac{x}{\log\zeta} \sum_{\delta < \frac{x}{y}z^2} \frac{a_\delta}{\varphi(\delta)} + O\Big(\frac{x}{(\log\zeta)^2} \sum_{\delta < \frac{x}{y}z^2} \frac{a_\delta}{\varphi(\delta)} \sum_{p|2\delta} \frac{\log p}{p-1}\Big)$$

provided $\zeta > x^\varepsilon$, which we may suppose.

For the main term in (4.15) we use the identity (see [1] for a proof)

$$(4.16) \qquad \sum_{\delta < \frac{x}{y}z^2} \frac{a_\delta}{\varphi(\delta)} = \sum_{m<z}{}' \sum_{\substack{d<x/y \\ (m,d)=1}}{}' \frac{\mu^2(m)}{\varphi_2(m)} \frac{1}{\varphi(d)} \Big(\sum_{\substack{r|d \\ r<z/m}}{}' \mu(r)\zeta_{mr}\Big)^2$$

where $\sum'$ is a sum over odd numbers and where ζ_r is expressed by means of the λ_ν by the formulas

$$(4.17) \qquad \zeta_r = \mu(r)\varphi_2(r) \sum_{\nu<z/r}{}' \frac{\lambda_{\nu r}}{\varphi(\nu r)},$$

$$(4.18) \qquad \lambda_\nu = \mu(\nu)\varphi(\nu) \sum_{r<z/\nu}{}' \frac{\mu^2(r\nu)}{\varphi_2(r\nu)} \zeta_{r\nu}.$$

For n odd and square-free we define

$$(4.19) \qquad g(n) = \prod_{p|n} \Big(p + \frac{1}{p} - 1\Big),$$

$$(4.20) \qquad \Psi(n) = \prod_{p|n} \frac{(p-1)^2}{p^2-p+1}$$

and let $g_1 = g*\mu$, so that $g_1(n) = \prod_{p|n} \Big(p + \frac{1}{p} - 2\Big)$.

We have

$$(4.21) \qquad \sum_{\substack{d\leqslant x/y \\ (d, m)=1}}{}' \frac{1}{\varphi(d)} \Big(\sum_{\substack{r|d \\ r<z/m}}{}' \mu(r)\zeta_{mr}\Big)^2$$
$$= \sum_{\substack{r_1,r_2<z/m \\ (r_i,m)=1}}{}' \mu(r_1)\mu(r_2)\zeta_{mr_1}\zeta_{mr_2} \sum_{\substack{d\leqslant x/y \\ (d, m)=1 \\ d\equiv 0(\mathrm{mod}[r_1,r_2])}}{}' \frac{1}{\varphi(d)}$$

and a standard calculation shows that if $(n, m) = 1$ and n is square-free, then for certain constants A_0, A_1:

$$(4.22) \qquad \sum_{\substack{d\leqslant\xi \\ (d, m)=1 \\ d\equiv 0(\mathrm{mod}\,n)}}{}' \frac{1}{\varphi(d)}$$
$$= A_0 \frac{\Psi(m)}{g(n)} \Big(\log\frac{\xi}{n} + A_1 + \sum_{p|m} \alpha_p + \sum_{p|n} \beta_p\Big) + O\Big(\frac{1}{g(n)} (mn)^\varepsilon \Big(\frac{n}{\xi}\Big)^{1/4}\Big)$$

where

$$\alpha_p = A_2 \frac{p^2 \log p}{(p-1)(p^2-p+1)}, \quad \beta_p = \alpha_p + A_3 \frac{\log p}{p-1}.$$

We substitute (4.22) in the right-hand side of (4.21), taking $n = [r_1, r_2]$. Since we assume $x/y > z^2x^{\varepsilon}$, the contribution of the error term in (4.22) is negligible, and we remain with

$$(4.23)\qquad \sum_{\substack{r_1,r_2<z/m\\(r_i,m)=1}}{}' \mu(r_1)\mu(r_2)\frac{\zeta_{mr_1}\zeta_{mr_2}}{g([r_1,r_2])}A_0\Psi(m)\times$$
$$\times\Big\{\log\frac{x/y}{[r_1,r_2]}+c_1+\sum_{p|m}\alpha_p+\sum_{p|[r_1,r_2]}\beta_p\Big\}.$$

Now we choose

$$(4.24)\qquad \zeta_r = A^{-1}g(r)\sum_{\nu<z/r}{}'\frac{\mu^2(r\nu)}{g_1(r\nu)}$$

where the constant A is taken in such a way that

$$(4.25)\qquad \lambda_1 = \sum_{r<z}{}'\frac{\mu^2(r)}{\varphi_2(r)}\zeta_r = 1.$$

Note that

$$(4.26)\qquad A\sim c_2(\log z)^2$$

for some constant c_2, and that the conditions on λ_ν are verified.

In the evaluation of (4.23) the main term comes from $\log\dfrac{x/y}{[r_1,r_2]}$, the contribution given by the other terms in $\{\ldots\}$ being in fact of a lower order of magnitude.

We follow here a method of Selberg [5], pp. 50–57. Define

$$(4.27)\qquad g'(n) = \sum_{d|N}\mu(d)g\Big(\frac{n}{d}\Big)\log\frac{n}{d},$$

so that

$$(4.28)\qquad g(n)\log n = \sum_{d|n}g'(d).$$

We perform the same transformations as in [5], pp. 56–57, except for having g_1 in place of φ, and get

$$(4.29)\qquad \sum_{\substack{r_1,r_2<z/m\\(r_i,m)=1}}{}'\mu(r_1)\mu(r_2)\frac{\zeta_{mr_1}\zeta_{mr_2}}{g([r_1,r_2])}\log\frac{x/y}{[r_1,r_2]}$$
$$= A^{-2}\mu^2(m)\frac{g^2(m)}{g_1^2(m)}\Big\{\log\frac{x}{y}\sum_{\substack{r<z/m\\(r,m)=1}}{}'\frac{\mu^2(r)}{g_1(r)}+\sum_{\substack{r<z/m\\(r,m)=1}}{}'\frac{\mu^2(r)}{g_1^2(r)}g'(r)\Big\}.$$

We have also for square-free r

$$(4.30)\qquad g'(r) = g_1(r)\Big\{\log r+\sum_{p|r}\frac{\log p}{g_1(p)}\Big\}$$

whence it follows that

$$(4.31)\qquad \sum_{\substack{r<z/m\\(r,m)=1}}{}'\frac{\mu^2(r)}{g_1^2(r)}g'(r)\sim\sum_{\substack{r<z/m\\(r,m)=1}}{}'\frac{\mu^2(r)}{g_1(r)}\log r,$$

and in any case

$$(4.32)\qquad (4.29)\ll\frac{g^2(m)}{g_1^2(m)}\Big(\log\frac{x}{y}\Big)\Big(\log z\Big)^{-3}.$$

A similar estimate, which need not be given here in detail, shows that the contribution to (4.23) of the terms involving $\sum_{p|[r_1,r_2]}\beta_p$ is of a lower order of magnitude.

Thus we have proved, using (4.16), (4.21) and (4.32) that

$$\sum_{\delta<\frac{x}{y}z^2}\frac{a_\delta}{\varphi(\delta)}\ll\Big(\log\frac{x}{y}\Big)\Big(\log z\Big)^{-3}\sum_{m<z}\frac{\mu^2(m)}{\varphi_2(m)}\frac{g^2(m)}{g_1^2(m)}\Psi(m)\ll\frac{\Big(\log\dfrac{x}{y}\Big)}{(\log z)^2}.$$

With rather similar calculations, one shows that the $O(\ldots)$ term in (4.15) is of lower order of magnitude than the main term. This proves (4.3), with λ_d, λ_d^* determined by the equations (4.6), (4.18), (4.24), (4.25), Q.E.D.

5. Proofs of Theorems 1 and 2. By Lemmas 1, 2, 3 we obtain for

$$(5.1)\qquad y>z^2x^{\alpha+\varepsilon},\quad yz^2<x^{\theta-\varepsilon},\quad z<x^{1/20k}$$

the result

$$\Big|\sum_{n\leqslant x}\Lambda(n+2)\Lambda_k(n)-2Hkx(\log x)^{k-1}\Big|$$
$$\ll k^2x(\log x)^{k-2}(\log z)+\alpha^{-1}x\frac{\Big(\log\dfrac{x}{y}\Big)^{k+1}}{(\log z)^2}.$$

If $\theta=\frac{1}{2}$, we can choose $y=x^{\theta-\varepsilon}z^{-2}$, $\alpha=\frac{1}{4}$, $z=x^{\beta}$ with $\beta<1/20k$. The error term is then, after division by $x(\log x)^{k-1}$:

$$\ll k^2\beta+\Big(\frac{1}{2}+\frac{1}{10k}\Big)^{k+1}\beta^{-2}\ll k^2\beta+2^{-k}\beta^{-2}$$

and we can choose

$$\beta = k^{-2/3}(\sqrt[3]{2})^{-k};$$

this proves Theorem 1.

If $\theta = 1$, we can choose

$$y = x^{1-\varepsilon}z^{-2}, \quad \alpha = \tfrac{1}{2}, \quad z = x^{\beta}, \quad \beta < 1/20k$$

and get the error term

$$\ll [k^2\beta + (2\beta)^{k-1}]x(\log x)^{k-1}.$$

Since $\beta > 0$ is at our disposal, we get Theorem 2.

The technique worked out here can obviously be generalized to many other similar situations. In particular, one can work with functions different from Λ_k, but with a similar structure. A particularly interesting choice is

$$\Lambda_2^{(h,k)} = \Lambda L^{h+k+1} + (h+k+1)\binom{h+k}{h}(\Lambda L^h) * (\Lambda L^k) \tag{5.2}$$

which leads to an asymptotic formula

$$\sum_{n \leqslant x} \Lambda(n+2)\Lambda_2^{(h,k)}(n) \sim 4Hx(\log x)^{h+k+1} \tag{5.3}$$

on the assumption of the Halberstam–Richert conjecture. It is easy to see that (5.3) determines asymptotically the number of solutions of the equation

$$p-2 = p_1p_2, \quad p \leqslant x,$$

counted with the weight

$$\left(\frac{\log p_1}{\log x}\right)^{h+1}\left(\frac{\log p_2}{\log x}\right)^{k+1},$$

in terms of the number of solutions of the equation $p-2 = p_1$, $p \leqslant x$. Since we can approximate a function $f(\xi, \eta)$ by polynomials in ξ, η, it is easy to see that (5.3) for all $h, k \geqslant 0$ determines asymptotically the number of solutions of $p-2 = p_1p_2$, $x^{\alpha_1} < p_1 < x^{\beta_1}$, $x^{\alpha_2} < p_2 < x^{\beta_2}$, $p \leqslant x$ for every $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $0 < \alpha_1 < \beta_1 < 1$, $0 < \alpha_2 < \beta_2 < 1$, $\alpha_1 + \alpha_2 < 1 < \beta_1 + \beta_2$, in terms of an asymptotic part and the distribution of twin-primes up to x.

One can then show that assuming the Halberstam–Richert conjecture the following holds: if the equation $p-2 = p_1$, $p \leqslant x$ has only $o\left(\frac{x}{(\log x)^2}\right)$ solutions, then the equation

$$p-2 = p_1 \ldots p_k, \quad p_i > x^{\varepsilon}, \quad p \leqslant x$$

has only $o\left(\frac{x}{(\log x)^2}\right)$ solutions for every $\varepsilon > 0$ and every fixed *odd* k.

Instead, if k is *even*, the previous equation will have about twice the expected number of solutions. This situation is unlikely to be true for large k, thus giving support to the twin-primes conjecture.

It would be of interest to see how well this technique could be used to prove the existence of solutions of $p-2 = p_1 \ldots p_k$, with k not too large. The work of Buchštab [3], Halberstam, Jurkat and Richert [4] Uchiyama [6] shows that $k \leqslant 3$. A self-contained simple proof with $k \leqslant 4$ can be found in [1].

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