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Absolutely continuous distribution functions of additive functions $f(p) = (\log p)^{-a}$, $a > 0$

by

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Introduction. The following question was raised by Erdös in a private communication to the author. Suppose f is a real-valued additive arithmetic function such that, for all primes p,

$$f(p) = (\log p)^{-a} \tag{1}$$

for some $a > 0$. For which values of $a > 0$ does f have an absolutely continuous distribution? In the same communication Erdös pointed out that f has absolutely continuous distribution for $a = 1$.

In this paper it is shown that f has an absolutely continuous distribution if $0 < a < 2$.

Notations and definitions. An arithmetic function f is said to be *additive* if

$$f(mn) = f(m) + f(n)$$

whenever $(m, n) = 1$. f is called *strongly additive* if, in addition, f satisfies $f(p^k) = f(p)$ for all primes p and for all positive integers k.

A real-valued arithmetic function h is said to *have a distribution* if there exists a distribution function G such that the density of $\{m \geqslant 1: h(m) \leqslant c\}$ exists and equals $G(c)$, whenever c is a continuity point of G.

Throughout this paper we let p denote a prime number.

The result.

Theorem. *Let f be a real-valued additive arithmetic function satisfying for all primes p,*

$$f(p) = (\log p)^{-a}$$

for some $0 < a < 2$. Then f has an absolutely continuous distribution.

Proof. That f has a distribution follows from Erdös–Wintner theorem (see [2]). From the results of [1] it follows that the distribution of f is absolutely continuous if and only if the distribution of the corresponding

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strongly additive arithmetic function is absolutely continuous. So without loss of generality we assume that f is strongly additive. The characteristic function L of the distribution of f is given by (see [2])

$$L(t) = \prod_p \left(1-\frac{1}{p}\left(1-\exp\left(it(\log p)^{-a}\right)\right)\right).$$

We shall now show that L is square integrable; the theorem then follows from Plancherel's theorem.

Now let $t > 1$. We have

$$|L(2t)|^2 \leqslant \prod_p \left(1-\frac{4}{p}\left(1-\frac{1}{p}\right)\sin^2 t(\log p)^{-a}\right) \leqslant \prod_p \left(1-\frac{4}{p}\sin^2 t(\log p)^{-a}\right). \tag{2}$$

Let r be any positive integer. For $n < t^{r/(1+r)}$ and $0 < b < c < \pi$, define

$$E(n, t, b, c) = \{p\colon b < t(\log p)^{-a} - n\pi < c\}.$$

Clearly, $p \in E(n, t, b, c) \Rightarrow \log p > \left(t/(c+\pi t^{r/(1+r)})\right)^{1/a}$. We need the following estimate

$$\sum_{p\leqslant x}\frac{1}{p} = \log\log x + E + O\left(\exp(-d\sqrt{\log x})\right), \tag{3}$$

where E and d are constants. From (3) it follows that, for $n < t^{r/(1+r)}$

$$\sum_{p\in E(n,t,b,c)}\frac{1}{p} = \frac{1}{a}\left(\log\frac{t}{n\pi+b} - \log\frac{t}{n\pi+c}\right) + O(n^r t^{-r}) = \frac{c-b}{an\pi} + O(n^{-2}) + O(n^r t^{-r}).$$

Hence

$$\sum_{n<t^{r/(1+r)}}\ \sum_{p\in E(n,t,b,c)}\frac{1}{p} = \frac{c-b}{a\pi}\sum_{n<t^{r/(1+r)}}\frac{1}{n} + O(1) + O\left(t^{-r}\int_1^{t^{r/(1+r)}} x^r\,dx\right) = \frac{(c-b)r}{\pi a(r+1)}\log t + O(1). \tag{4}$$

Let k be any integer greater than 2. For $j = 1, 2, \ldots, 2k-1$, define

$$A(j,t) = \bigcup_{n<t^{r/(1+r)}}\left(E\left(n, t, \frac{j\pi}{4k}, \frac{(j+1)\pi}{4k}\right)\right) \cup \left(E\left(n, t, \pi\left(1-\frac{(j+1)}{4k}\right), \pi\left(1-\frac{j}{4k}\right)\right)\right).$$

Note that, for each fixed t and k, the sets $E\left(n, t, \frac{i\pi}{4k}, \frac{(i+1)\pi}{4k}\right)$, $n < t^{r/(r+1)}$, $i = 1, \ldots, 4k-1$, are disjoint sets of primes. By (2) and (4), we have

$$|L(2t)|^2 \leqslant \exp\left(-2\times 4\sum_{j=1}^{2k-1}\left(\sum_{p\in A(j,t)}\frac{1}{p}\right)\sin^2\frac{j\pi}{4k}\right) \leqslant \exp\left(-\frac{8r\log t}{4ka(r+1)}\sum_{j=1}^{2k-1}\sin^2\frac{j\pi}{4k}\right).$$

Since for $j = 1, \ldots, k-1$,

$$\sin^2\frac{j\pi}{4k} + \sin^2\frac{2k-j}{4k}\pi = 1 \quad\text{and}\quad \sin^2\frac{\pi}{4} = \frac{1}{2},$$

we have

$$|L(2t)|^2 \leqslant \exp\left(-\frac{2r(k-1)}{a(r+1)k}\log t - \frac{r}{ka(r+1)}\log t\right) = \exp\left(-\frac{r}{a(r+1)}\left(2-\frac{1}{k}\right)\log t\right). \tag{5}$$

Now we choose r and k such that $\frac{r}{a(r+1)}\left(2-\frac{1}{k}\right) = \delta > 1$. This is possible because $a < 2$. Thus by (5) we have

$$|L(t)|^2 \leqslant \exp(-\delta\log|t|) = |t|^{-\delta},$$

for some $\delta > 1$. Hence L is square integrable. This completes the proof of the theorem.

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