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On properties of systems of arithmetic sequences

by

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1. Introduction and results. For integral a, n with $0 \leqslant a < n$ the symbol $a(n)$ means the set of all integers of the form $a+sn$, where s is an integer. Such a set will be called an *arithmetic sequence with modulus n*.

A system

(1) $$S = \{a_j(n_j)\}, \quad j = 1, 2, \ldots, k;\ 1 < n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$$

of arithmetic sequences is called a *covering* (CS) if every integer belongs to at least one sequence $a_j(n_j)$; the system is *disjoint* if the sets $a_j(n_j)$ are mutually disjoint. A disjoint covering system will be denoted DCS (see [1]).

A CS is said to be *regular* if none of its subsystems is a CS (see [3]); any DCS is a regular CS. The sequence $a_j(n_j)$ is *essential* in (1) if there exists some integer belonging to $a_j(n_j)$ but to no other sequence of (1). In a regular CS all the sequences are essential.

The aim of this article is to show some properties of systems of arithmetic sequences. We are concerned with properties proved for the case of disjoint systems in some previous articles.

THEOREM 1. *Let* (1) *be a covering system and* $a_r(n_r)$ *an essential sequence of* (1). *If*

$$n_r = \prod_{i=1}^{s} p_i^{\alpha_i} \quad (\textit{the standard form})$$

then

$$k \geqslant 1 + \sum_{i=1}^{s} \alpha_i (p_i - 1).$$

The validity of an analogous result for the case of DCS has been conjectured in [4] and proved in [6]. Obviously, if (1) is a DCS or a regular CS then any sequence is essential and Theorem 1 can be applied to any modulus.

In [7] the statement of Theorem 1 was proved in the case where the sequence $a_r(n_r)$ is disjoint with all the remaining sequences.

It can easily be shown that for a non-essential sequence the statement of Theorem 1 can be false.

The author has conjectured the following stronger statement (see [8]): *If* (1) *is a DCS and* $N = [n_1, n_2, \ldots, n_k]$ *is the least common multiple of* $n_1, n_2, \ldots, n_k$, *then from* $N = \prod p_t^{\alpha_t}$ *the inequality*

$$k \geqslant 1 + \sum_{t=1}^{s} \varphi_t(p_t - 1)$$

follows. This has been proved in [2], where coverings of groups by cosets are studied. It is possible that for the case of a non-disjoint but regular system the same holds.

If S is a system of arithmetic sequences then $f_S(n)$ means the number of sequences of S to which the integer n belongs; $f_S(n)$ is called the *covering function of* S. It is easy to verify that $f_S(n)$ is a periodic function with a period which is a divisor of $N = [n_1, n_2, \ldots, n_k]$.

The following theorem generalizes a result of S. K. Stein.

THEOREM 2. *Let* $S = \{a_j(n_j)\}$, $j = 1, 2, \ldots, k$, $n_1 < n_2 < \ldots < n_k$; $T = \{b_j(m_j)\}$, $j = 1, 2, \ldots, t$, $m_1 < m_2 < \ldots < m_t$. *Let* M *be the least common multiple of numbers* $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_t$. *If*

(2) $$f_S(v) = f_T(v) \quad \text{for all } v = 0, 1, \ldots, M-1,$$

then

(3) $$S = T,$$

i.e. $t = k$, $a_j = b_j$, $n_j = m_j$ *for* $j = 1, 2, \ldots, k$.

In [5] Stein proved that if S and T are disjoint systems — both with distinct moduli — then

(4) $$\bigcup_{j=1}^{k} a_j(n_j) = \bigcup_{j=1}^{t} b_j(m_j)$$

implies equality (3). However, if the systems S and T are disjoint then equations (2) and (4) are equivalent (since $f_S(v)$ and $f_T(v)$ are periodic functions with a period which is a divisor of M) and hence Theorem 2 implies Theorem 5.1 of [5].

From Theorem 2 we get some interesting consequences.

Suppose the numbers $n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$ are given. Then putting $a_j = 0, 1, \ldots, n_j - 1$ ($j = 1, 2, \ldots, k$) we can construct $P = n_1 \cdot n_2 \cdot \ldots \cdot n_k$ systems of arithmetic sequences having $n_1, n_2, \ldots, n_k$ as moduli. Denote these systems by

(5) $$S_1, S_2, \ldots, S_P.$$

If two systems from (5) have the same covering functions then (by Theorem 2) there exist at least two equal moduli. This is a generalization of the well-known fact (proved by Davenport, Mirsky, Newman and Rado — see [1]) that no DCS with distinct moduli exists.

(Indeed: if (1) is a DCS then it can easily be shown that the system

(6) $$b_j(n_j), \quad b_j \equiv a_j + 1 \pmod{n_j}, \quad j = 1, 2, \ldots, k$$

is a DCS again; hence systems (1) and (6) have the same covering function and thus the moduli n_j cannot be distinct.)

On the other hand, if $n_1 < n_2 < \ldots < n_k$, then systems (5) are distinct. (Indeed: if $u \neq v$ then S_u and S_v have — owing to Theorem 2 — different covering functions.)

2. Proofs of Theorems.

Proof of Theorem 1. Denote by N the least common multiple of $n_1, n_2, \ldots, n_k$. Suppose $N = p_t^{\beta_t} q_t$ ($t = 1, 2, \ldots, s$) with $(q_t, p_t) = 1$. The sequence $a_r(n_r)$ is essential in (1); therefore there exists an integer — say b_r — which belongs to $a_r(n_r)$ but to no other sequence of (1). Consider the numbers

(7) $$b_r + u_t p_t^{\gamma_t} q_t = A(t, u_t, \gamma_t),$$

where $t = 1, 2, \ldots, s$; $u_t = 1, 2, \ldots, p_t - 1$; $\gamma_t = 0, 1, \ldots, \alpha_t - 1$.

There exist

$$\sum_{t=1}^{s} \alpha_t(p_t - 1)$$

numbers of the form (7) and they are distinct. Now we shall show some properties of numbers (7).

(a) Numbers (7) do not belong to $a_r(n_r)$.

Indeed: If $x \in a_r(n_r)$ then $x - b_r$ is divisible by n_r and hence it is divisible by $p_t^{\alpha_t}$ for any $t = 1, 2, \ldots, s$. On the other hand, $A(t, u_t, \gamma_t) - b_r = u_t p_t^{\gamma_t} q_t$ is not divisible by $p_t^{\alpha_t}$ because $\gamma_t < \alpha_t$ and $(u_t q_t, p_t) = 1$.

(b) If $A(t, u_t, \gamma_t) \in a_j(n_j)$ then $p_t^{\gamma_t+1}$ divides n_j.

Suppose the contrary holds and write $n_j = K p_t^{\omega_t}$, where $(K, p_t) = 1$ and $\omega_t \leqslant \gamma_t$. However, n_j is then a divisor of

$$u_t p_t^{\gamma_t} q_t = A(t, u_t, \gamma_t) - b_r$$

and hence $b_r \in a_j(n_j)$, which is a contradiction since by (a) we have $r \neq j$ and b_r has been chosen so that it does not belong to any other sequence of (1).

(c) If $\gamma_t \neq \gamma_t'$ or $u_t \neq u_t'$ then the numbers $x = A(t, u_t, \gamma_t)$ and $y = A(t, u_t', \gamma_t')$ belong to distinct sequences of (1).

We shall prove this indirectly. Suppose x and y belong to $a_z(n_z)$. Then

$$x = a_z + K n_z, \quad y = a_z + H n_z, \quad K, H \text{ integers}.$$

These equalities imply

$$(x-b_r)-(y-b_r) = Ln_z, \qquad L \text{ integer}. \tag{8}$$

If $\gamma_t > \gamma_t'$ then $p_t^{\gamma_t}$ divides $u_t p_t^{\gamma_t} q_t = x - b_r$ and by (b) it divides also n_z. However, the number $y-b_r = u_t' p_t^{\gamma_t'} q_t$ is not divisible by $p_t^{\gamma_t}$, which contradicts (8) (if $\gamma_t' > \gamma_t$ then we argue similarly).

If $\gamma_t = \gamma_t'$ but $u_t \neq u_t'$ then owing to (b) the potence $p_t^{\gamma_t+1}$ divides n_z but does not divide the number

$$(x-b_r)-(y-b_r) = (u_t-u_t')p_t^{\gamma_t} q_t,$$

which contradicts (8) again.

(d) If $t \neq v$ then $x = A(t; u_t, \gamma_t)$ and $y = A(v, u_v, \gamma_v)$ belong to distict sequences of (1).

Suppose this does not hold and x, y belong to $a_z(n_z)$. Then we get (8) again. However, $y-b_r$ is divisible by q_v; thus $p_t^{\gamma_t+1}$ is also a divisor of $y-b_r$; by (b) $p_t^{\gamma_t+1}$ is also a divisor of n_z. This contradicts (8) again, because $x-b_r = u_t p_t^{\gamma_t} q_t$ is not divisible by $p_t^{\gamma_t+1}$.

Thus according to (c) and (d), for different triples t, u_t, γ_t the numbers $A(t, u_t, \gamma_t)$ belong to distinct sequences of (1). We have said above that there exist

$$\sum_{t=1}^{s} a_t(p_t-1)$$

such triples. From (a) it follows that no number from (7) belongs to $a_r(n_r)$ and hence the proof is finished.

Proof of Theorem 2. According to (2) for arbitrary complex z we have

$$\sum_{v=0}^{M-1} f_S(v) z^v = \sum_{v=0}^{M-1} f_T(v) z^v.$$

Reordering both sides, we obtain:

$$\sum_{j=1}^{k} (z^{a_j} + z^{a_j+n_j} + \ldots + z^{a_j+M-n_j}) = \sum_{j=1}^{t} (z^{b_j} + z^{b_j+m_j} + \ldots + z^{b_j+M-m_j}).$$

Now, if $z \neq \exp\left(\frac{2\pi i}{M} g\right)$, g being an integer, then the last equality implies

$$\sum_{j=1}^{k} z^{a_j} \frac{z^M-1}{z^{n_j}-1} = \sum_{j=1}^{t} z^{b_j} \frac{z^M-1}{z^{m_j}-1}. \tag{9}$$

Suppose $k > t$ and h is the smallest integer for which

$$a_{k-h}(n_{k-h}) \neq b_{t-h}(m_{t-h}). \tag{10}$$

Then from (9) (dividing by z^M-1) we have

$$\sum_{j=1}^{k-h} \frac{z^{a_j}}{z^{n_j}-1} = \sum_{j=1}^{t-h} \frac{z^{b_j}}{z^{m_j}-1}. \tag{11}$$

We can suppose $n_{k-h} \geqslant m_{t-h}$ (in the opposite case we argue similarly). Let z approach the number $\exp\left(\frac{2\pi i}{n_{k-h}}\right)$. If $n_{k-h} > m_{t-h}$ then from (11) we have

$$z^{a_{k-h}} \to 0,$$

which is impossible. If $n_{k-h} = m_{t-h}$ then (by (11))

$$z^{a_{k-h}} - z^{b_{t-h}} \to 0;$$

hence

$$\exp\left(\frac{2\pi i}{n_{k-h}} a_{k-h}\right) = \exp\left(\frac{2\pi i}{n_{k-h}} b_{t-h}\right),$$

which is possible only if $a_{k-h} = b_{t-h}$. Thus $a_{k-h}(n_{k-h}) = b_{t-h}(m_{t-h})$, which contradicts (10). Hence for all $h = 0, 1, \ldots, t-1$ we have

$$a_{k-h}(n_{k-h}) = b_{t-h}(m_{t-h}),$$

i.e. all the sequences of T also belong to S. However, S cannot then contain any other sequences (see (2)) and thus $S = T$ and the proof is finished.

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