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On a question of P. Turán

by

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1. In a paper, recently published in this journal, P. Turán ([3], p. 136) showed:

Let $A_a(x, k, l)$ denote for fixed a with $\frac{1}{4} \leqslant a < \frac{1}{3}$ the number of positive integers $n \leqslant x$, which are $\equiv l \bmod k$, all of whose prime divisors are $\geqslant x^a$. Then with the notations

$$A_1(a) = \int_{1/3}^{(1-a)/2} \frac{\log \frac{r}{1-2r}}{r(1-r)}\, dr, \qquad A_2(a) = \int_{(1-a)/2}^{1-2a} \frac{\log \frac{1-a-r}{a}}{r(1-r)}\, dr$$

we have the following

THEOREM. *If for an arbitrary small $\delta > 0$, $k > k_0(\delta, a)$ and*

$$x \geqslant \exp\{\log^2 k \log\log k\}$$

the inequality

$$A_a(x, k, l) \leqslant 2\big(1 + A_1(a) + A_2(a) - \delta\big) \frac{x}{\varphi(k)\log x} \tag{1.1}$$

holds for all $l \bmod k$, then none of the functions $L(s, k, \chi)$ can vanish on the interval

$$1 - \frac{1}{(\log k \log\log k)^2} \leqslant \sigma \leqslant 1.$$

The special case $a = \frac{1}{4}$ of this theorem has already been communicated by P. Turán in a letter to E. Hecke in 1942. Since the function $A_a(x, k, l)$ can be estimated from above directly by means of sieve methods and these methods have been very much improved since then, Turán posed the question of whether applications of the sharpest known form of Selberg's sieve (W. B. Jurkat, H.-E. Richert [1]) really lead to estimates of the form (1.1).

In this paper it will be shown that the main term coming from an application of [1] is slightly greater than the right side of (1.1) without δ, i.e. that with the present state of knowledge nothing can be deduced about Siegel zeros from Selberg's sieve via the above theorem of Turán.

Prof. W. B. Jurkat told the author that he found a similar result after making his calculations for [1].

2. With the notations of [1] we have $A_\alpha(x, k, l) = B_{k,l}(x, x, x^\alpha)$ and we get from ([1], (7.1)) with $h = x/k$, $u = \left(\log\frac{x}{k}\right)\Big/\alpha\log x$:

(2.1) $$\frac{A_\alpha(x, k, l)}{hR_k(h^{1/u})} \leqslant F(u) + c\frac{\log\log 3k}{(\log h)^{1/14}},$$

where c is a positive constant, $F(u)$ the function defined in Section 3 below, and $R_k(z)$ the product

$$R_k(z) := \prod_{\substack{p<z\\ p\nmid k}}\left(1-\frac{1}{p}\right).$$

Since we have for $R_k(z)$ (see [2], p. 81)

$$R_k(z) \leqslant \prod_{p<z}\left(1-\frac{1}{p}\right)\prod_{p\mid k}\frac{p}{p-1} = \frac{k}{\varphi(k)}\,\frac{e^{-\gamma}}{\log z}\big(1+o(1)\big),$$

γ being Euler's constant, we get from (2.1)

$$A_\alpha(x, k, l) \leqslant \frac{1}{\varphi(k)}\cdot\frac{x}{\log x}\cdot\frac{e^{-\gamma}}{\alpha}\left\{F\left(\frac{\log(x/k)}{\alpha\log x}\right) + c\frac{\log\log 3k}{(\log(x/k))^{1/14}}\right\}.$$

Since $F(u)$ is monotonic decreasing (see [1], p. 227, (5.13)), our proposition can be reduced to

(2.2) $$\frac{e^{-\gamma}}{\alpha}F\left(\frac{1}{\alpha}\right) = 2\big(1+A_1(\alpha)+A_2(\alpha)\big) \qquad (\tfrac{1}{4} \leqslant \alpha \leqslant \tfrac{1}{3}).$$

3. To show (2.2) we first compute $F(u)$, which is defined by

(3.1) $$F(u) = e^{\gamma}\left\{\omega(u) + \frac{\varrho(u)}{u}\right\},$$

$\omega(u)$ and $\varrho(u)$ being the continuous solutions of

(3.2) $$\big(u\omega(u)\big)' = \omega(u-1), \qquad (u-1)\varrho'(u) = -\varrho(u-1) \qquad (u \geqslant 2)$$

with the values $\varrho(u) = 1$, $\omega(u) = 1/u$ for $0 < u \leqslant 2$.

Integrating (3.2) we get for $2 \leqslant u \leqslant 3$

$$\omega(u) = \frac{1}{u}\{1+\log(u-1)\}, \qquad \varrho(u) = 1-\log(u-1)$$

and for $3 \leqslant u \leqslant 4$

$$\omega(u) = \frac{1}{u}\left\{\int_3^u \frac{1}{v-1}\big(\log(v-2)+1\big)\,dv + 1 + \log 2\right\},$$

$$\varrho(u) = \int_3^u \frac{1}{v-1}\big(\log(v-2)-1\big)\,dv + 1 - \log 2,$$

so that we have by (3.1) for $3 \leqslant u \leqslant 4$:

$$F(u) = \frac{e^{\gamma}}{u}\left\{2\int_3^u \frac{1}{v-1}\log(v-2)\,dv + 2\right\} = \frac{2e^{\gamma}}{u}\left\{1 + A_3\left(\frac{1}{u}\right)\right\}, \qquad \text{say}.$$

Our proposition (2.2) now reduces to

$$A_1(\alpha) + A_2(\alpha) = A_3(\alpha).$$

But the last equality is valid since it is valid for $\alpha = \frac{1}{3}$ and the derivatives of the right hand and left hand sides agree for $\frac{1}{4} \leqslant \alpha \leqslant \frac{1}{3}$.

The author emphasizes that he does not believe he has established the theorem "With sieve methods one can never prove theorems about Siegel zeros". Moreover one should give, also in view of the possible improvements of Turán's theorem (see [3], p. 136), the following interpretation: Sieve methods are now developed to a point where even minor improvements can be of major interest.

References

[1] W. B. Jurkat and H.-E. Richert, *An improvement of Selberg's sieve methods I*, Acta Arith. 11 (1965), pp. 217–240.
[2] K. Prachar, *Primzahlverteilung*, Berlin–Göttingen–Heidelberg 1957.
[3] P. Turán, *Über die Siegel-Nullstelle der Dirichletschen Funktionen*, Acta Arith. 24 (1973), pp. 135–141.

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