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Large values of Dirichlet polynomials

by

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Dedicated to Professor
Carl Ludwig Siegel

1. Statement of results. A Dirichlet polynomial is a finite Dirichlet series. It posesses a mean value on any abscissa $\sigma = a$ ($s = \sigma + it$ denotes a complex variable). Montgomery [6] considers the question: how often is the Dirichlet polynomial large in a bounded region. 'Large' here means that its modulus exceeds some bound V greater than the mean value. This paper improves Montgomery's results for certain ranges of V. The technique used is to feed the Halász inequality (Montgomery's basic tool) back into itself by means of an approximate functional equation for L-functions. Iterating from Montgomery's bound (1.6) below gives Theorems 1 and 2; analogous results may be obtained from other bounds. However, the iteration never gives results as good as would hold on the Lindelöf hypothesis.

There is an application to zero-density theorems, which we sketch. The 'Q^2T' density hypothesis for L-functions is established in a weak form for $\sigma > 5/6$, and the density hypothesis for $\zeta(s)$ for $\sigma > 189/230$.

We use the ideas of Jutila's proof [5] of the density hypotheses for $\sigma > 7/8$. Forti and Viola [1] have also improved Jutila's results: $\sigma > 0.841\ldots$ for L-functions, $\sigma > 0.805\ldots$ for $\zeta(s)$. In a later paper we will combine the iteration step with Forti and Viola's method.

We consider Dirichlet polynomials of the form

$$F(s, \chi) = \sum_{N+1}^{2N} a(m)\chi(m)m^{-s}, \tag{1.1}$$

where $\chi(m)$ is a Dirichlet character. A general Dirichlet polynomial can be broken into sums of the form (1.1). We are concerned with estimates for the sum

$$E = \sum_{\substack{q\leqslant Q\\ q\equiv 0 \,(\mathrm{mod}\, q_0)}} \sideset{}{^*}\sum_{\chi \bmod q} \sum_{r=1}^{R(\chi)} |F(s(r,\chi),\chi)| \tag{1.2}$$

where $s(r,\chi) = \sigma(r,\chi) + it(r,\chi)$ is a complex number satisfying

$$0 \leqslant \sigma(r,\chi) \leqslant l^{-1} \tag{1.3}$$

where

$$l = \log NQ^2T, \tag{1.4}$$

and

$$1 \leqslant |t(r_1,\chi_1) - t(r_2,\chi_2)| \leqslant T \tag{1.5}$$

the lower bound applying only for $\chi_1 = \chi_2$ and $r_1 \neq r_2$. The asterisk indicates a sum over proper characters. Let

$$D = Q^2T/q_0, \tag{1.6}$$

$$G = \sum_{N+1}^{2N} |a(m)|^2, \tag{1.7}$$

$$R = \sum_{\substack{q \leqslant Q \\ q \equiv 0 \,(\mathrm{mod}\, q_0)}} \ \sideset{}{^*}\sum_{\chi \bmod q} R(\chi). \tag{1.8}$$

For each triple R, M, D of real numbers greater than or equal to 1, let $B(R, M, D)$ be the least positive real number for which

$$E \leqslant G^{1/2} B(R, M, D) \tag{1.9}$$

holds for every choice of q_0, Q, T, of $N \leqslant M$, of coefficients $a(m)$ and sets of points $s(r,\chi)$ satisfying the relations above. It is easy to see that

$$B(R, N, D) \geqslant \max(R^{1/2}N^{1/2}, R). \tag{1.10}$$

The method of Halász, as developed by Montgomery [6] gives the bound

$$B(R, N, D) \ll R^{1/2}N^{1/2}\log l + RD^{1/4}l; \tag{1.11}$$

the $\ll$ notation indicating an inequality with a suppressed absolute constant. The factor $\log l$ in the first term can be omitted if $Q = 1$. If the Lindelöf hypothesis for L-functions is true, we can replace $D^{1/4}$ in (1.11) by $D^{\varepsilon}N^{1/4}$ for any $\varepsilon > 0$.

Our method establishes a lemma.

LEMMA. *Let k be any positive integer. Then*

$$\begin{aligned} &B(R, N, D) \\ &\ll R^{1/2}N^{1/2}\log l + RN^{1/4}l^2 + k^{k/2}R^{1-1/2k}\Delta^{1/2}\{B(R, D^k/N^k, dDD)\}^{1/2k} l^{2+k/4}, \end{aligned} \tag{1.12}$$

where

$$\Delta = \max_{\substack{q \leqslant Q \\ q \equiv 0 \,(\mathrm{mod}\, q_0)}} d_3(q), \tag{1.13}$$

$d_3(q)$ denoting as usual the number of ways of writing q as a product of three positive integers.

The third term in (1.12) is not defined if $N > D$; but in this case the first two terms are sufficient by (1.11).

Repeated use of (1.12) gives

$$\begin{aligned} &B(R, N, D) \\ &\ll R^{1/2}N^{1/2}\log l + \sum_{i=1}^{n} K_{i,n} R^{1-\alpha_{i,n}} N^{\beta_{i,n}} D^{\gamma_{i,n}} \Delta l^{\varepsilon_{i,n}} + K_{1,n} R N^{B_n} D^{\Gamma_n} \Delta l^{E_n}, \end{aligned} \tag{1.14}$$

where the exponents are given by the recurrence relations

$$\alpha_{i+1,j+1} = \tfrac{1}{2}\alpha_{i,j}\varkappa_{j+1}, \tag{1.15}$$

$$\beta_{i+1,j+1} = \tfrac{1}{4} - \tfrac{1}{2}\beta_{i,j}, \tag{1.16}$$

$$\gamma_{i+1,j+1} = \tfrac{1}{2}\beta_{i,j} + \tfrac{1}{2}\gamma_{i,j}\varkappa_{j+1}, \tag{1.17}$$

$$\varepsilon_{i+1,j+1} = \tfrac{1}{2}\varepsilon_{i,j}\varkappa_{j+1} + \tfrac{1}{4}k_{j+1} + 2, \tag{1.18}$$

with $(0, B_{j+1}, \Gamma_{j+1}, E_{j+1})$ being derived from $(0, B_j, \Gamma_j, E_j)$ by the same four relations. The initial values are

$$(\alpha_{0,j}, \beta_{0,j}, \gamma_{0,j}, \varepsilon_{0,j}) = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 1), \tag{1.19}$$

$$(0, B_0, \Gamma_0, E_0) = (0, 0, \tfrac{1}{4}, 1), \tag{1.20}$$

so that (1.11) corresponds to $n = 0$. The numbers $k_1, \ldots, k_n$ are arbitrary positive integers, k_1 chosen in the last iteration performed, k_2 in the penultimate, and so on. Moreover

$$\varkappa_j = 1/k_j, \tag{1.21}$$

$$K_{j,n} = \prod_{i=j}^{n} k_i^{\frac{1}{2}k_i\varkappa_{i+1}\cdots\varkappa_n}. \tag{1.22}$$

Here again the factor $\log l$ in the first term of (1.12) and (1.14) is not necessary if $Q = 1$.

We deduce a result about large values of $F(s,\chi)$.

THEOREM 1. *If*

$$|F(s,\chi)| \geqslant V \tag{1.23}$$

for each term in the sum (1.2), and if

$$V > c_1 K_{1,n} G^{1/2} \Delta N^{B_n} D^{\Gamma_n} l^{E_n}, \tag{1.24}$$

c_1 being a certain absolute constant, then

$$R \leqslant c_2 GNV^{-2}\log^2 l + \sum_{1}^{n} \{c_2 n K_{i,n} G^{1/2} V^{-1} \Delta N^{\beta_{i,n}} D^{\gamma_{i,n}} l^{\varepsilon_{i,n}}\}^{1/\alpha_{i,n}}, \tag{1.25}$$

where c_2 is another absolute constant.

If $Q = 1$, χ is unity and we write $F(s)$ for $F(s, \chi)$. The device in [4] of dividing the range (1.5) for s into subintervals of length T_0, where T_0 satisfies (1.24), gives the following inequality.

THEOREM 2. *If in the sum* (1.2) $Q = 1$ *and*

$$|F(s)| \geqslant V \tag{1.26}$$

for each term, then

$$\begin{aligned} R \leqslant\; & c_3 GNV^{-2} + c_3 GNV^{-2} T\{K_{1,n} G^{1/2} V^{-1} N^{B_n} l^{E_n}\}^{1/\Gamma_n} + \\ & + \sum_1^n \{c_3 n K_{i,n} G^{1/2} V^{-1} N^{\beta_{i,n}} l^{\varepsilon_{i,n}}\}^{1/\alpha_{i,n}} \{K_{1,n} G^{1/2} V^{-1} N^{B_n} l^{E_n}\}^{-\gamma_{i,n}/\alpha_{i,n}\Gamma_n} + \\ & + \sum_1^n \{c_3 n K_{i,n} G^{1/2} V^{-1} N^{\beta_{i,n}} l^{\varepsilon_{i,n}}\}^{1/\alpha_{i,n}} \{K_{1,n} G^{1/2} V^{-1} N^{B_n} l^{E_n}\}^{(\alpha_{i,n}-\gamma_{i,n})/\alpha_{i,n}\Gamma_n} T, \end{aligned} \tag{1.27}$$

where c_3 is an absolute constant.

2. Halász's inequality. Halász's inequality gives

$$|E|^2 \leqslant GR \sum_{\chi_1, r_1} \sum_{\chi_2, r_2} \left|H\big(\sigma(r_1, \chi_1) + \sigma(r_2, \chi_2) + it(r_1, \chi_1) - it(r_2, \chi_2), \chi_1 \bar{\chi}_2\big)\right|, \tag{2.1}$$

where the conditions on χ and r are those in the sum (1.2), and

$$H(s, \chi) = \sum_{m=1}^{\infty} b(m) \chi(m) m^{-s}, \tag{2.2}$$

where $b(m)$ can be taken as any positive real numbers with

$$b(m) \geqslant 1 \quad \text{for} \quad N < m \leqslant 2N. \tag{2.3}$$

We construct the coefficients $b(m)$ as follows. Let

$$J(\omega) = \frac{\pi^2}{2\omega(\omega - \pi i)(\omega + \pi i)} = \frac{1}{2\omega} - \frac{1}{4(\omega + \pi i)} - \frac{1}{4(\omega - \pi i)}. \tag{2.4}$$

Then

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} J(\omega) x^{-\omega} d\omega = \begin{cases} 0 & \text{if} \quad x \geqslant 1, \\ \frac{1}{2}(1 - \cos \pi \log x) & \text{if} \quad x \leqslant 1. \end{cases} \tag{2.5}$$

Defining the integral function $K(\omega)$ by

$$K(\omega) = (e^{4\omega} + e^{3\omega} - e^{\omega} - 1) J(\omega), \tag{2.6}$$

we have

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(\omega) L(s + \omega, \chi) U^{\omega} d\omega = \sum_1^{\infty} \frac{b(m, U) \chi(m)}{m^s}, \tag{2.7}$$

where

$$b(m, U) = \begin{cases} 0 & \text{if} \quad m \leqslant U \text{ or } m \geqslant e^4 U, \\ 1 & \text{if} \quad eU \leqslant m \leqslant e^3 U, \\ \frac{1}{2}(1 - \cos \pi \log m/U) & \text{if} \quad U \leqslant m \leqslant eU \text{ or } e^3 U \leqslant m \leqslant e^4 U. \end{cases} \tag{2.8}$$

For

$$eU \leqslant N \leqslant \tfrac{1}{2} e^3 U \tag{2.9}$$

these coefficients satisfy (2.3). We have (2.1) with

$$H(s, \chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(\omega) L(s + \omega, \chi) U^{\omega} d\omega. \tag{2.10}$$

3. The transformation of the sum. We treat

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(\omega) L(s + \omega, \chi) U^{\omega} d\omega \tag{3.1}$$

where

$$s = \sigma(r_1, \chi_1) + \sigma(r_2, \chi_2) + it(r_1, \chi_1) - it(r_2, \chi_2), \tag{3.2}$$

$$\chi = \chi_1 \bar{\chi}_2, \tag{3.3}$$

and χ is a character $\bmod\, q = [q_1, q_2]$ induced by some character ψ proper to a modulus f. Let

$$M(s) = \sum_{d|q} \mu(d) \psi(d) d^{-s}, \tag{3.4}$$

so that

$$L(s, \chi) = M(s) L(s, \psi). \tag{3.5}$$

Since $K(\omega)$ is an integral function, the integrand in (3.1) has no pole unless χ is principal, when $L(s, \psi)$ is $\zeta(s)$ and there is a pole at $s + \omega = 1$ with residue

$$K(1 - s) M(1 - s) U^{1-s} \ll \frac{U\sigma(q)}{q(|t| + 1)^3} \ll \frac{U \log l}{(|t| + 1)^3}. \tag{3.6}$$

Here $\sigma(q)$ denotes the sum of the divisors. One treatment is to take the integral back to the line $\operatorname{Re}\omega = 0$, where it is

$$\ll \delta(q) f^{1/2} |t|^{1/2} \log^2(f|t| + e), \tag{3.7}$$

where

$$\delta(q) = \sum_{\substack{d|q \\ (d, f) = 1}} 1. \tag{3.8}$$

We use (3.7) when

(3.9) $$f|t| \leqslant U,$$

and the estimate (3.7) becomes

(3.10) $$\ll \delta(q)\, U^{1/2} t^3.$$

If (3.9) does not hold, we transform the contour integral (3.1) as in the proof of the approximate functional equation in [3]. Since $K(\omega)$ is an integral function we may change the line of integration to $\mathrm{Re}\,\omega = -\frac{1}{2}-\sigma$, and then change the variable of integration to $u = 1-s-\omega$, giving

(3.11) $$H(s,\chi) = \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} U^{1-s-u} K(1-s-u)\, M(1-u)\, L(1-u,\psi)\, du,$$

and we substitute for $L(1-u,\psi)$ according to the functional equation

(3.12) $$L(1-u,\psi) = G(u,\overline{\psi})\, L(u,\overline{\psi}),$$

where

(3.13) $$G(u) = G(u,\overline{\psi}) = \frac{i^a}{\tau(\overline{\psi})}\,\frac{f^{1/2}}{\pi^{1/2}}\,\frac{\Gamma\left(\frac{1}{2}(u+a)\right)}{\Gamma\left(\frac{1}{2}(1-u+a)\right)},$$

a similar formula holding for $\zeta(s)$. Here $a = 0$ or 1 as $\psi(-1)$ is 1 or -1. If $u = \lambda+i\tau$, λ and τ real, with $\lambda^2 \ll |\tau|$, Stirling's formula gives

(3.14) $$|G(u,\overline{\psi})| \sim (f|\tau|/2\pi)^{\lambda-1/2}.$$

We divide the range for t into intervals

(3.15) $$T_1 \leqslant |t| \leqslant e^{1/2} T_1,$$

and for f into

(3.16) $$F \leqslant f \leqslant e^{1/2} F.$$

Let

(3.17) $$X_0 = \tfrac{1}{2} F T_1/\pi U,$$

and

(3.18) $$X_a = e^{-a} X_0,$$

(3.19) $$Y_a = e^a U.$$

We write

(3.20) $$M(1-u)\, L(1-u,\overline{\psi}) = \varphi_1(u,\overline{\psi}) + \varphi_2(u,\overline{\psi})$$

where

(3.21) $$\varphi_1(u,\overline{\psi}) = \varphi_1(X,u;\overline{\psi},q) = \sum_{d|q}\mu(d)\psi(d)\, d^{u-1}\sum_{m\leqslant dX}\overline{\psi}(m)\, m^{-u},$$

(3.22) $$\varphi_2(u,\overline{\psi}) = \varphi_2(X,u;\overline{\psi},q) = \sum_{d|q}\mu(d)\psi(d)\, d^{u-1}\sum_{m> dX}\overline{\psi}(m)\, m^{-u},$$

and X is X_0, X_1, X_3 or X_4 according to which term $e^{a\omega}$ we consider in the decomposition (2.6) of $K(\omega)$.

Four of the eight terms resulting are of the form

(3.23) $$\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty} Y_a^{1-s-u} J(1-s-u)\, G(u,\overline{\psi})\, \varphi_2(u,\overline{\psi})\, du.$$

We move the line of integration to $\mathrm{Re}\,u = 2$. The integral (3.23) is in fact an error term. In the other four we take the line of integration back to $\mathrm{Re}\,u = -3/4$ and consider

(3.24) $$\frac{1}{2\pi i}\int_{-3/4-i\infty}^{-3/4+i\infty} Y_a^{1-s-u} J(1-s-u)\, G(u,\overline{\psi})\, \varphi_1(u,\overline{\psi})\, du.$$

This integral also becomes an error term; but in moving the contour we have acquired residues

(3.25) $$\tfrac{1}{2}G(1-s)\varphi_1(1-s,\overline{\psi}) - \tfrac{1}{4}G(1-s+\pi i)\varphi_1(1-s+\pi i,\overline{\psi})\, Y_a^{-\pi i} - \\ - \tfrac{1}{4}G(1-s-\pi i)\varphi_1(1-s-\pi i,\overline{\psi})\, Y_a^{\pi i}$$

from the poles of $J(1-s-u)$. As happened with $b(m)$, for $m \leqslant dX_4$ the residues from the four sets of poles cancel, giving

(3.26) $$\sum_{d|q}\mu(d)\psi(d)\, d^{-s}\sum_a\sum_{dX_4<m\leqslant dX_a}\{\tfrac{1}{2}G(1-s)\overline{\psi}(m)\, m^{-1+s} - \\ -\tfrac{1}{4}G(1-s+\pi i)\overline{\psi}(m)\, m^{-1+s-\pi i}\, Y_a^{-\pi i} - \tfrac{1}{4}G(1-s-\pi i)\overline{\psi}(m)\, m^{-1+s+\pi i}\, Y_a^{\pi i}\},$$

where a takes the value 0,1 or 3. If ψ is unity there is a pole of $G(u)$ at $u = 0$, and an extra residue

(3.27) $$J(1-s)\, Y_a^{1-s}\varphi_1(0,1) \ll (1+|t|)^{-3}\delta(q) X_a Y_a \ll \delta(q)(1+|t|)^{-2}.$$

4. Separating the variables. Our next task is to write

(4.1) $$\psi(m) = \psi_1(m)\overline{\psi}_2(m),$$

where ψ_1 is a proper character mod f_1 and ψ_2 is a character mod f_2, where

(4.2) $$h_1 = q/f,$$

(4.3) $$f_1 = q_1/h_1,$$

(4.4) $$h = (q_1, q_2) = q_1 q_2/q,$$

(4.5) $$f_2 = q_2/h = q/q_1.$$

For each character ϱ we write

$$\varrho = \prod_p \varrho[p], \tag{4.6}$$

where $\varrho[p]$ is a character modulo some power of p. Let p^{a_1} and p^{a_2} be the highest powers of p occurring in q_1 and q_2, and $a = \max(a_1, a_2)$. If $\chi[p]$ is proper mod p^a we put $\psi_1[p] = \chi_1[p]$, $\psi_2[p] = \chi_2[p]$. If $a_1 \neq a_2$ $\chi[p]$ must be proper mod p^a, since the ratio of two characters mod p^{a-1} cannot be proper mod p^a. Only if $a_1 = a_2$ can $\psi[p]$ be a proper character mod p^b with $b < a$. In this case we define $\psi_1[p] = \psi[p]$, $\psi_2[p]$ to be the principal character mod p. The formula (4.6) now defines ψ_1 and ψ_2. Noting that

$$h_1 = \prod_p p^{a_p - b_p}, \tag{4.7}$$

where the notation is as above with suffices p, we see that ψ_1 is proper mod f_1 and that ψ_2 is a character mod f_2 (induced by a character whose modulus is the largest divisor of f_2 that is prime to h_1).

In (2.1) we have a sum over χ_1 and r_1. We have already divided the ranges by (3.15) and (3.16), and split up the sums into parts corresponding to the divisors d of q_2 in (3.4) and subsequent equations. The condition $d \mid q_2$ in (3.4) can be replaced by $d \mid h_1$, since $\psi(d)$ is zero unless $(d, f) = 1$. We subdivide again according to the value of h_1, for h_1 and χ_2 together determine ψ_2. We replace the condition (3.16) by

$$F_1 < f_1 \leqslant e^{1/4} F_1, \quad f_1 \equiv 0 \ (\mathrm{mod}\, f_0), \tag{4.8}$$

where

$$f_0 = q_0/(q_0, h_1) \tag{4.9}$$

and

$$e^{-1/4} H < h \leqslant H, \tag{4.10}$$

where

$$H = F_1 q_2 / F \tag{4.11}$$

and F_1 and H are powers of $e^{1/4}$, giving a finer subdivision of the sum.

5. The reflected sum. In order to use Hölder's inequality we write

$$\Big\{ \sum_{dX_4 < m \leqslant dX_a} m^{-1+s+b\pi i} \overline{\psi}(m) \Big\}^k = \sum_{d^k X_4^k < m \leqslant d^k X_a^{kK}} \overline{\psi}(m) g(m) m^{-1+s+b\pi i} \tag{5.1}$$

where b is 0, 1 or -1 and

$$|g(m)| \leqslant d_k(m), \tag{5.2}$$

the kth divisor function of m. The range of summation can be written as the union of at most $6k$ ranges $M_1 < m \leqslant M_2$, where $M_2 \leqslant 2M_1$. Since

$$M_2 \leqslant D^{k/2} \tag{5.3}$$

we have

$$\sum_{M_1+1}^{M_2} |g(m)|^2 m^{2\sigma-2} \ll (\log M_2)^{k^2-1} M_1^{2\sigma-1} \ll e^{2k} (kl)^{k^2-1} M_1^{-1}. \tag{5.4}$$

We deduce that

$$\sum_{f_1} \sum_{\psi_1} \sum_{s} \Big| \sum_{M_1 < m \leqslant M_2} \psi_1(m) \overline{\psi}_2(m) g(m) m^{-1+s+b\pi i} \Big| \tag{5.5}$$
$$\ll e^k (kl)^{k^2/2-1/2} M_1^{-1/2} B(R, M_1, F^2T/f_0)$$
$$\ll e^k (kl)^{k^2/2-1/2} \Big(\frac{dFT_1}{N}\Big)^{-k/2} B\Big(R, \Big(\frac{dFT_1}{N}\Big)^k, \frac{F_1^2 T}{f_0}\Big),$$

where the summation assumes h_1, ψ_2, q_2 to be fixed, f_1 to satisfy (4.8), h to satisfy (4.10) and s to be of the form (3.2) and to satisfy (3.15). We sum over $O(k)$ values of M_1 and apply Hölders inequality to get

$$\sum_{f_1} \sum_{\psi_1} \sum_{s} \Big| \sum_{dX_4 < m \leqslant dX_a} \psi_1(m) \overline{\psi}_2(m) m^{-1+s+b\pi i} \Big| \tag{5.6}$$
$$\ll (kl)^{k/2} R^{1-1/k} \Big(\frac{N}{dFT_1}\Big)^{1/2} \Big\{ B\Big(R, \Big(\frac{dFT_1}{N}\Big)^k, \frac{F_1^2 T}{f_0}\Big)\Big\}^{1/k}.$$

When (3.15), (4.8) and (4.10) hold we have

$$G(1-s+b\pi i) \ll F^{1/2} T_1^{1/2}. \tag{5.7}$$

After multiplying together the estimates (5.6) and (5.7) (which corresponds to taking G outside the sum at its maximum value) we sum the ranges (3.15) with T_1 running through powers of $e^{1/2}$ not exceeding $e^{1/2}T$, the ranges (4.8) with F_1 running through powers of $e^{1/4}$ not exceeding Q/h_1, and the ranges (4.10) with H running through powers of $e^{1/4}$ which exceed q_0. The total is

$$\ll (kl)^{k/2} l^3 R^{1-1/k} \Big(\frac{N}{d}\Big)^{1/2} \Big\{ B\Big(R, \Big(\frac{dq_2QT}{h_1 q_0 N}\Big)^k, \frac{Q^2T}{f_0 h_1^2}\Big)\Big\}^{1/k}. \tag{5.8}$$

Since $d \mid h_1$ and $f_0 h_1 \leqslant q_0$, we can sum again over pairs (d, h_1) with $d \mid h_1$, $h_1 \mid q_2$. We see that the terms (3.26) summed over all pairs $(\chi_1, s(r, \chi_1))$ are

$$\ll k^{k/2} \Delta R^{1-1/k} N^{1/2} \Big\{ B\Big(R, \Big(\frac{q_2QT}{q_0N}\Big)^k, \frac{Q^2T}{q_0}\Big)\Big\}^{1/k} l^{k/2+3} \tag{5.9}$$
$$\ll k^{k/2} \Delta R^{1-1/k} N^{1/2} \{B(R, D^k/N^k, D)\}^{1/k} l^{k/2+3}.$$

6. The contour integrals. After a minor change of variables we write the integral (3.23) as

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^{1-\sigma-u}J(1-\sigma-u)G(u+it,\overline{\psi})\varphi_2(X,u+it;\overline{\psi},q)\,du. \tag{6.1}$$

We assume (3.15), (4.8) and (4.10) to hold; then

$$G(u+it) \ll F^{3/2}(T_1+|\tau|)^{3/2}. \tag{6.2}$$

When we replace $G(u+it)$ in (6.1) by the bound (6.2), we can sum over ψ. In order to apply Hölder's inequality we write

$$\{\varphi_2(u+it,\psi)\}^k = \sum_{d^kX^k<m} \overline{\psi}(m)g(m)m^{-u-it}, \tag{6.3}$$

where the coefficients are not those of (5.1), but (5.2) holds for them. We divide the summation into ranges $M<m\leqslant 2M$, and we have

$$\sum_{M+1}^{2M}|g(m)|^2m^{-4} \ll M^{-3}(\log M)^{k^2-1}. \tag{6.4}$$

As in the last section we have

$$\sum_{f_1}\sum_{\psi_1}\sum_{s}\Big|\sum_{M+1}^{2M}\psi_1(m)\overline{\psi}_2(m)g(m)m^{-u-it}\Big| \ll (\log M)^{k^2/2-1/2}M^{-3/2}B(R,M,F_1^2T/f_0), \tag{6.5}$$

where the summation assumes h_1, ψ_2 and q_2 to be fixed, f_1 to satisfy (4.8), h to satisfy (4.10) and s to be of the form (3.2) and to satisfy (3.15). Clearly

$$\sum_{M}(\log M)^{k^2/2-1/2}M^{-3/2}B(R,M,F_1^2T/f_0) \ll (kl)^{k^2/2-1/2}(dX)^{-3k/2}B(R,d^kX^k,F_1^2T/f_0), \tag{6.6}$$

the sum being over a geometrical progression of values of M with first term d^kX^k. Hölder's inequality now gives

$$\sum_{f_1}\sum_{\psi_1}\sum_{s}|\varphi_2(u+it,\overline{\psi})| \ll (kl)^{k/2}R^{1-1/k}(dX)^{-3/2}\{B(R,d^kX^k,F_1^2T/f_0)\}^{1/k}, \tag{6.7}$$

a bound which is independent of u. Since

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}J(1-\sigma-u)(T_1+|\tau|)^{3/2}\,du \ll T_1^{3/2}, \tag{6.8}$$

the sum over f_1, ψ_1 and s of the integrals (6.1) is

$$\begin{aligned}&\ll Y^{-1}\Big(\frac{T_1}{dX}\Big)^{3/2}R^{1-1/k}\Big\{B\Big(R,d^kX^k,\frac{F_1^2T}{f_0}\Big)\Big\}^{1/k}(kl)^k\\ &\ll (kl)^kR^{1-1/k}N^{1/2}d^{-3/2}\Big\{B\Big(R,\Big(\frac{dFT_1}{N}\Big)^k,\frac{F_1^2T}{f_0}\Big)\Big\}^{1/k}.\end{aligned} \tag{6.9}$$

Comparison with (5.6) and (5.7) indicates that the sum over the contour integrals is majorized by the sum over the reflected sums, and the same upper bound (5.9) applies.

7. Proof of the results. Adding the estimates (3.7) for $f|t|\leqslant Ul$, (5.9) for the reflected sum and contour integrals and (3.6) and (3.7) for extra terms which occur when $\chi_1=\chi_2$, we have shown that

$$\begin{aligned}&\max_{\chi_1,r_1}\sum_{\chi_2,r_2}\big|H\big(\sigma(r_1,\chi_1)+\sigma(r_2,\chi_2)+it(r_1,\chi_1)-it(r_2,\chi_2),\chi_1\overline{\chi}_2\big)\big|\\ &\qquad\ll k^k\Delta R^{1-1/k}N^{1/2}\{B(R,D^k/N^k,D)\}^{1/k}l^{k/2+3}+N\log l+RN^{1/2}l^4,\end{aligned} \tag{7.1}$$

and substitution into (2.1) gives

$$\begin{aligned}E &\ll G^{1/2}R^{1/2}N^{1/2}\log l+G^{1/2}RN^{1/4}l^2+\\ &\quad+k^{k/2}G^{1/2}R^{1-1/2k}\Delta^{1/2}N^{1/4}\{B(R,D^k/N^k,D)\}^{1/2k}l^{k/4+2},\end{aligned} \tag{7.2}$$

from which the lemma (1.12) follows. The induction argument begins with (3.7), which implies

$$B(R,N,D) \ll R^{1/2}N^{1/2}\log l+RD^{1/4}l. \tag{7.3}$$

To prove Theorem 1 we need only check that the second term in (7.2) can be absorbed when $N<D$. This follows since

$$\begin{aligned}&N^{1/4-B_n/2}\Delta^{1/2-\varkappa/2}D^{B_n/2-\Gamma_n\varkappa/2}l^{E_n\varkappa/2+k/4+2}\\ &\qquad>\Big(\frac{D}{N}\Big)^{B_n/2}(D^{\Gamma_n}\Delta l^{E_n})^{\varkappa/2}N^{1/4}l^2>N^{1/4}l^2.\end{aligned} \tag{7.4}$$

8. The classification of zeros. We count the number $N(\alpha,T,\chi)$ of zeros $\varrho=\beta+i\gamma$ of $L(s,\chi)$ in the rectangles

$$\alpha\leqslant\beta\leqslant\alpha+l^{-1},\qquad T\leqslant|\gamma|\leqslant 2T \tag{8.1}$$

where $\alpha>1/2$ and T will be assumed sufficiently large. Let X be a large integer to be chosen, and let

$$M(s,\chi) = \sum_{m\leqslant X}\mu(m)\chi(m)m^{-s} \tag{8.2}$$

be a partial sum for the reciprocal of $L(s,\chi)$. We have

$$(8.3)\qquad \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} L(\varrho+\omega,\chi)M(\varrho+\omega,\chi)Y^{\omega}\Gamma(\omega)\,d\omega = e^{-1/Y}+\sum_{m>X} b(m)\chi(m)m^{-\varrho}e^{-m/Y},$$

where Y is a large integer to be chosen and

$$(8.4)\qquad b(m) = \sum_{d|m,\, d\leqslant X}\mu(d);$$

the integrand in (8.3) is regular at $\omega = 0$ if ϱ is a zero of $L(s,\chi)$, but there is a pole at $\omega = 1-\varrho$ if $\chi(m) = 1$, of residue

$$(8.5)\qquad M(1,1)\,Y^{1-\varrho}\Gamma(1-\varrho),$$

which is less than $1/10$ in modulus if T is sufficiently large, if

$$(8.6)\qquad \log X \leqslant \log Y \leqslant 2l$$

and if

$$(8.7)\qquad |\gamma| \geqslant c_1 l,$$

where c_1 is an absolute constant. We note that there is an absolute constant c_2 such that

$$(8.8)\qquad \Big|\sum_{m>c_2 lY} b(m)\chi(m)m^{-\varrho}e^{-m/Y}\Big| < 1/10.$$

Splitting the range $X < m < c_2 lY$ into $O(l)$ intervals $I(n)$, the division between $I(n)$ and $I(n+1)$ being at $2^n Y$, we find that all zeros of $L(s,\chi)$ satisfying (8.1) fall into at least one of the following classes.

Class (i, n): zeros ϱ for which

$$(8.9)\qquad \Big|\sum_{m\in I(n)} b(m)\chi(m)m^{-\varrho}e^{-m/Y}\Big| > (6l)^{-1}.$$

Class (ii, n): zeros ϱ for which

$$(8.10)\qquad \max_{|\gamma-t|\leqslant 2^n} |L(\tfrac12+it,\chi)M(\tfrac12+it,\chi)| > c_3 2^n Y^{\alpha-1/2},$$

where c_3 is a suitable absolute constant. The existence of c_1, c_2 and c_3 follows as in [3].

We note that if zeros of $\zeta(s)$ alone are considered, and if

$$(8.11)\qquad X^{-1/2}Y^{\alpha-1/2} > c_4 T^{6/37}\log T$$

for a suitable c_4, then class (ii) is empty by Haneke's Theorem [2].

The following lemma is a consequence of Montgomery's work [6].

LEMMA 2. *For each proper character χ whose modulus does not exceed Q let the sequence $t_1,\dots,t_R$ of $R = R(\chi)$ real numbers satisfy $-T\leqslant t_r\leqslant T$ for each r, and $t_{r+1}-t_r\geqslant 1$ for $r = 1,\dots,R-1$. Then*

$$(8.12)\qquad \sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi \bmod q}\ \sum_{r=1}^{R(\chi)} |L(\tfrac12+it_r,\chi)|^4 \ll Q^2T\log^5 Q^2T.$$

To each class (ii, n) zero corresponds a point t at which the maximum in (8.10) occurs. Consider a subsequence of the points t arising in this way, chosen to satisfy the conditions of Lemma 2 whilst numbering at least $c_5(2^n l)^{-1}$ times the number of class (ii, n) zeros. By (8.12) the subsequence has at most

$$(8.13)\qquad \ll Q^2Tl^5(2^nX^{-1/2}Y^{\alpha-1/2})^{-4}$$

members. Multiplying by $O(2^n l)$ and summing over n, we see that there are at most

$$(8.14)\qquad \ll Q^2TX^2Y^{2-4\alpha}l^6$$

class (ii) zeros. If

$$(8.15)\qquad XQT^{1/2}\leqslant Y,$$

the number of class (ii) zeros is thus

$$(8.16)\qquad \ll Y^{4-4\alpha}l^6.$$

9. Zeros of $L(s,\chi)$. We use Theorem 1 to estimate the number of class (i, n) zeros. To avoid conflicts of notation we drop the suffix n in $a_{i,n}$, B_n etc., and write σ for the abscissa α of (8.1). For simplicity we take $q_0 = 1$. We use Jutila's idea and raise the sum in (8.9) to the ath power, where the positive integer a is chosen as follows. Let $Y_1, Y_2, \dots$, be a decreasing sequence of positive numbers (which we choose in (9.11) below) with $Y_1\geqslant Y$. The upper limit of the sum (8.9) is 2^nY. If n is positive $a = 2$. If n is negative or zero, a is chosen by

$$(9.1)\qquad Y_a < 2^nY \leqslant Y_{a-1}.$$

The ath power of the sum in (8.9) can be split into a sums of the form

$$(9.2)\qquad \sum_{2^{u-1}Y^a<m\leqslant 2^uY^a} B(m)\chi(m)m^{-\varrho} = F(\varrho-\sigma,\chi).$$

If (8.9) holds, one of these a sums exceeds $a^{-1}(6l)^{-a}$ in modulus. We apply Theorem 1 with

$$(9.3)\qquad a(m) = B(m)m^{-\sigma},$$

$$(9.4)\qquad N = 2^{u-1}Y^a < 2^{an}Y^a,$$

$$(9.5)\qquad G\leqslant (2^nY)^{a(1-2\sigma)}e^{-a2^n}(3al)^{4a^2},$$

$$(9.6)\qquad V = a^{-1}(6l)^{-a},$$

the estimate for G having been obtained in [4]. We apply Theorem 1 to a subset of the class (i, n) zeros, containing a proportion $\geqslant l^{-2}$ of them, chosen in such a way that two zeros of the same L-function differ by at least unity in their imaginary part. The number of class (i, n) zeros is now estimated by (1.25) as

$$\ll \frac{GNl^3}{V^2}+\sum_i\left\{\frac{G^{1/2}\Delta N^{\beta_i}}{V}(Q^2T)^{\gamma_i}l^{\varepsilon_i}\right\}^{1/\alpha_i}l^2, \tag{9.7}$$

providing that

$$V \geqslant K_1 G^{1/2}\Delta N^{B}(Q^2T)^{\Gamma}l^{E}. \tag{9.8}$$

If (9.8) holds and if

$$\sigma > \tfrac{1}{2}+\beta_i \tag{9.9}$$

for each i, the first term in (9.7) increases and the other terms decrease as N increases. Hence the expression in (9.7) is

$$\ll Y_{a-1}^{(2-2\sigma)a}a^2(6l)^{2a}(3al)^{4a^2}l^3+ + \sum_i\{a(6l)^a(3al)^{2a^2}\Delta Y_a^{(1/2+\beta_i-\sigma)a}D^{\gamma_i}l^{\varepsilon_i}\}^{1/\alpha_i}l^2, \tag{9.10}$$

in the notation (1.6). We now state the definition of Y_a: that

$$Y_a^{2(a+1)(1-\sigma)}(a+1)^2(6l)^{2(a+1)}\big(3(a+1)l\big)^{4(a+1)^2}l = \sum_i\{a(6l)^a(3al)^{2a^2}\Delta Y_a^{(1/2+\beta_i-\sigma)a}D^{\gamma_i}l^{\varepsilon_i}\}^{1/\alpha_i}. \tag{9.11}$$

When $a=2$ we replace Y_1 in (9.10) by Y. This is possible even for positive n, because of the negative exponentials in the coefficients $B(m)$.

Suppose now

$$X \geqslant D^{\delta} \tag{9.12}$$

for some fixed $\delta>0$. Then the a which occur are $O(|\log\delta|)$ and for D sufficiently large, both Y_a and the expression (9.11) are decreasing functions of a. Summing over n, we find that there are

$$\ll Y^{4-4\sigma}l^{24}+\sum_i Y_2^{2(1/2+\beta_i-\sigma)/\alpha_i}D^{\gamma_i/\alpha_i}l^{3+(10+\varepsilon_i)/\alpha_i} \tag{9.13}$$

class (i) zeros.

Let l^{A} denote a fixed power of l, which may be different in different formulae. From (9.11)

$$Y_2 = D^{\frac{\gamma_i}{(2-6\alpha_i)\sigma-(1+2\beta_i-6\alpha_i)}}l^{A}, \tag{9.14}$$

where we take the value of i for which the term is maximum. If

$$Y_2^{2\sigma-1-2B} \geqslant D^{\Gamma}l^{A}, \tag{9.15}$$

then (9.8) will hold at each application of Theorem 1. If

$$(2\gamma_i-2\Gamma+6\alpha_i\Gamma)\sigma > \gamma_i-\Gamma+2\gamma_iB-2\beta_i\Gamma+6\alpha_i\Gamma \tag{9.16}$$

then (9.15) holds for D sufficiently large. The expression in (9.13) is

$$\ll Y^{4-4\sigma}l^{A}+Y_2^{6-6\sigma}l^{A} \ll D^{\frac{6\gamma_i(1-\sigma)}{(2-6\alpha_i)\sigma-(1+2\beta_i-6\alpha_i)}}, \tag{9.17}$$

where we have assumed that

$$Y \leqslant Y_2^{3/2}; \tag{9.18}$$

we must check that (9.18) is consistent with (8.15) and (9.12). Clearly (9.17) absorbs (8.16), so that (9.17) is our upper bound for the number of zeros of L-functions formed with proper characters to moduli not exceeding Q in the region (8.1).

If (8.15) and (9.18) are contradictory, we determine Y by (8.15) and make Y_a the maximum of $Y^{2/(a+1)}$ and the value given by (9.11). To satisfy (9.8) we require

$$Y_2^{2(\sigma-1/2-B)} \geqslant D^{\Gamma}l^{A}, \tag{9.19}$$

which is possible for

$$\sigma > 1/2+B+3\Gamma/(2+4\delta) \tag{9.20}$$

and D sufficiently large. In place of (9.17) we have the upper bound

$$\ll Y^{4-4\sigma}l^{A} \ll D^{(2+4\delta)(1-\sigma)}l^{A}. \tag{9.21}$$

At this stage we can replace the conditions (8.1) by

$$\sigma \leqslant \beta, \quad |\gamma| \leqslant T, \tag{9.22}$$

the effect of this being only to increase the exponent A by two.

With $k=3$ we have

$$\begin{aligned}(\alpha_{01},\beta_{01},\gamma_{01}) &= (1/2,1/2,0),\\ (\alpha_{11},\beta_{11},\gamma_{11}) &= (1/12,0,1/4),\\ (0,B_1,\Gamma_1) &= (0,1/4,1/24),\end{aligned} \tag{9.23}$$

and (9.13) is

$$\ll Y^{4-4\sigma}l^{A}+Y_2^{12-24\sigma}D^3l^{A}. \tag{9.24}$$

The inequality (9.9) requires $\sigma>1/2$. The choice (9.14) is

$$Y_2 = D^{1/(6\sigma-2)}l^{A}, \tag{9.25}$$

and (9.16) is satisfied for $\sigma>17/21$ $(=0.809\ldots)$. If

$$\sigma \leqslant 1/3+1/(2+4\delta) = 5/6-O(\delta), \tag{9.26}$$

(9.18) can be satisfied, and

$$\sum_{q\leqslant Q}\sum_{\chi \bmod q}^{*} N(\sigma, T, \chi) \ll (Q^2T)^{3(1-\sigma)/(3\sigma-1)}l^A \tag{9.27}$$

uniformly in $17/21+\varepsilon < \sigma < 5/6-\varepsilon$, where A depends on ε.

For $\sigma \geqslant 5/6-\varepsilon$ we can satisfy (9.20) instead, and (9.21) gives the upper bound, a weak form of the density hypothesis.

For $3/4 < \sigma < 17/21$ we must choose Y_2 to satisfy (9.15):

$$Y_2 = D^{1/(46\sigma-36)}l^A, \tag{9.28}$$

giving

$$\sum_{q\leqslant Q}\sum_{\chi \bmod q}^{*} N(\sigma, T, \chi) \ll (Q^2T)^{(1-\sigma)/(8\sigma-6)}l^A, \tag{9.29}$$

which is still better than the Ingham–Montgomery exponent $3(1-\sigma)/(2-\sigma)$ and the Montgomery exponent $2(1-\sigma)/\sigma$ [6] for $\sigma > 4/5$.

10. Zeros of $\zeta(s)$. For $\zeta(s)$ Theorem 2 is available. With $k_1 = 1$, $k_2 = 2$, $k_3 = 2$

$$\begin{aligned}(\alpha_{03}, \beta_{03}, \gamma_{03}) &= (1/2, 1/2, 0),\\ (\alpha_{13}, \beta_{13}, \gamma_{13}) &= (1/8, 0, 1/4),\\ (\alpha_{23}, \beta_{23}, \gamma_{23}) &= (1/32, 1/4, 1/16),\\ (\alpha_{33}, \beta_{33}, \gamma_{33}) &= (1/64, 1/8, 9/64),\\ (0, B_3, \Gamma_3) &= (0, 3/16, 13/128).\end{aligned} \tag{10.1}$$

The result of Theorem 2 is contained in Theorem 1 if T satisfies (1.24), that is, if

$$N > T^{\frac{13-\varepsilon}{8(16\sigma-11)}}, \tag{10.2}$$

where we have used

$$GV^{-2} \ll N^{1-2\sigma+\varepsilon}, \tag{10.3}$$

the exponents ε being not necessarily the same; but each can be made arbitrarily small if T is large enough. Theorem 2 gives

$$\begin{aligned}R \ll N^{2-2\sigma+\varepsilon} &+ N^{(114-154\sigma)/13+\varepsilon}T^{1+\varepsilon} + N^{(24\sigma-36)/13+\varepsilon}T^{1+\varepsilon} +\\ &+ N^{(224-288\sigma)/13+\varepsilon}T^{1+\varepsilon} + N^{(192\sigma-184)/13+\varepsilon}T^{1+\varepsilon}.\end{aligned} \tag{10.4}$$

The term $i = 0$ exceeds the term $i = 1$ for $\sigma < 75/89$ $(= 0.842\ldots)$, the term $i = 2$ for $\sigma > 55/67$ $(= 0.820\ldots)$ and the term $i = 3$ for $\sigma < 149/173$ $(= 0.860\ldots)$. In the range

$$55/67 \leqslant \sigma \leqslant 75/89 \tag{10.5}$$

we need only consider the first two terms in (10.4), and have in place of (9.10)

$$\ll Y_{a-1}^{(2-2\sigma)a+\varepsilon} + Y_a^{(114-154\sigma)a/13+\varepsilon}T^{1+\varepsilon}. \tag{10.6}$$

Corresponding to (9.11) we put

$$Y_a^{26(a+1)(1-\sigma)-(114-154\sigma)a} = T^{13}. \tag{10.7}$$

As before, the critical value of N in (10.2) is

$$N = Y_2^2 = T^{26/(230\sigma-150)}, \tag{10.8}$$

and (10.2) holds for $\sigma > 1-\varepsilon$ only. Hence

$$N(\sigma, T) \ll Y^{4-4\sigma+\varepsilon} + T^{39(1-\sigma)/(115\sigma-75)+\varepsilon}. \tag{10.9}$$

For the part

$$55/67 \leqslant \sigma \leqslant 189/230 \ (= 0.821\ldots) \tag{10.10}$$

of the range (10.5) we put

$$X = T^5, \tag{10.11}$$

$$Y = T^{39/(460\sigma-300)+\varepsilon}, \tag{10.12}$$

chosen so that (8.15) holds. Then (8.16) and (10.9) give

$$N(\sigma, T) \ll T^{39(1-\sigma)/(115\sigma-75)+\varepsilon}, \tag{10.13}$$

in the range (10.10). For

$$189/230 \leqslant \sigma \leqslant 75/89 \tag{10.14}$$

we replace (10.12) by

$$Y = T^{1/2+\varepsilon} \tag{10.15}$$

and have

$$N(\sigma, T) \ll T^{2-2\sigma+\varepsilon} \tag{10.16}$$

which was proved in [4] only for $\sigma \geqslant 5/6$. In fact for

$$75/89 \geqslant \sigma \geqslant 61/74 \ (= 0.8243\ldots) \tag{10.17}$$

we can use (8.11) and chose

$$Y = T^{12/(74\sigma-37)+\varepsilon} \ (< T^{1/2+\varepsilon}) \tag{10.18}$$

and obtain

$$N(\sigma, T) \ll T^{\frac{48(1-\sigma)}{37(2\sigma-1)}+\varepsilon}. \tag{10.19}$$

The estimate (10.19) also holds in a short interval to the right of 75/89.

Since (10.16) was established in [4] for $\sigma \geqslant 5/6$, we have (10.16), a weak form of the density hypothesis, for $\sigma \geqslant 189/230$.

Added in proof. 1. To prove (6.6) $M^{-c}B(R, M, D)$ was assumed to be a decreasing function of M for $c > 1$. This may be false for $M < D$. The difficulty may be resolved by redefining $B(R, M, D)$ or by moving the contour to the right, increasing c to $1/\varepsilon$ and permitting an extra factor M^{ε} in the upper bounds. The upper bounds substituted for $B(R, M, D)$ give decreasing functions, and the zero-density theorems are unaffected.

2. Jutila [*On a density theorem of H. L. Montgomery for L-functions*, Acta Acad. Sci. Fenn. Series AI 520] has new density theorems for L-functions, including the density hypothesis for $\sigma > 5/6$. Jutila's ideas will be discussed in the second part of this paper.

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Zur Methode von Stepanov

von

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Herrn Professor C. L. Siegel zum 75. Geburtstag gewidmet

1. Einleitung. S. A. Stepanov hat mit seiner neuen Methode einen wichtigen Fall des Satzes von Hasse–Weil elementar bewiesen ([2], [3], [4]). Es sei F_q der endliche Körper mit q Elementen, und es sei $f(X)$ ein Polynom vom Grad m mit Koeffizienten in F_q. Ist nun n eine positive, zu m relativ prime Zahl, und ist $q > 4m^2n(n-1)^2$, dann hat Stepanov gezeigt, daß die Anzahl A der Lösungen der Gleichung

$$y^n = f(x)$$

in Elementen x, y aus F_q die Ungleichung

$$|A-q| < (3mn)^{3/2} q^{1/2}$$

befriedigt. Weiter hat er ein ähnliches Resultat für Gleichungen $y^p - y = f(x)$ hergeleitet [5], wobei p die Charakteristik von F_q ist(1).

In der vorliegenden Arbeit wollen wir allgemein Polynome $f(X, Y)$ zulassen, die absolut irreduzibel sind, die also in keinem algebraischen Erweiterungskörper reduzibel sind. Wir werden die Methode von Stepanov weiter führen und das folgende Ergebnis elementar beweisen.

Satz. *Das Polynom $f(X, Y)$ mit Koeffizienten in F_q sei absolut irreduzibel, und es habe Grad $m > 0$ in X and Grad $n > 0$ in Y. Ist nun*

$$q > 9(m+1)^2(n+1)^2 \quad (2), \tag{1}$$

dann gilt für die Anzahl A der Lösungen der Gleichung

$$f(x, y) = 0 \tag{2}$$

in Elementen x, y aus F_q die Ungleichung

$$|A-q| < 2\,\mathrm{Min}(m^2n, n^2m)\, q^{1/2}. \tag{3}$$

(1) In einem Brief vom 22.6.1972 hat mir Herr Stepanov mitgeteilt, daß er nunmehr Gleichungen $y^n + g_1(x)y^{n-1} + \ldots + g_n(x) = 0$ behandeln kann, für die der Grad m von $g_n(X)$ zu n relativ prim ist, und für die weiter der Grad von $g_i(X)$ kleiner ist als $(i/n)m$ $(1 \leqslant i \leqslant n-1)$.*

(2) Mit zusätzlichem Aufwand könnte diese Bedingung gemildert werden.

* Bemerkung bei der Korrektur. Dieses Ergebnis erschien in Izv. Akad. Nauk SSSR, Ser. Mat 36 (1972), S. 683–711.