Since $M(s, 2r)$ denotes the number of $s\times s$ upper triangular matrices such that rank $(B+B^T) = 2r$, it follows from Theorem 2.5 that

$$(4.7)\qquad N_s(R, 0)q^{\frac{s(s+1)}{2}} = q^{s(n-2\varrho)}\sum_{r=0}^{[s/2]} M(s, 2r)\,(q^{2(s-r)})^{\varrho}.$$

From Lemma 2.1, it follows that

$$(4.8)\qquad N_s(R, 0)q^{\frac{s(s+1)}{2}} = q^{s(n-2\varrho)}\sum_{r=0}^{[s/2]} q^{s}L_0(s, 2r)\,(q^{2(s-r)})^{\varrho}.$$

This completes the proof of the following theorem.

THEOREM 4.1. *Let A be an $n\times n$ alternate matrix of rank 2ϱ over* GF(q). *The number of $s\times n$ matrices X over* GF(q) *such that $XAX^T = 0$ is*

$$N_s(A, 0) = \frac{q^{s(n+1)}}{q^{\frac{s(s+1)}{2}}}\sum_{r=0}^{[s/2]} L_0(s, 2r)q^{-2\varrho r}$$

where $L_0(s, 2r)$ is given by (2.8).

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Slowly growing sequences and discrepancy modulo one

by

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§ 1. Introduction. Let $y_1, y_2, \ldots, y_k \ldots$ be numbers in the interval

$$[0, 1) = \{x\colon 0\leqslant x<1\}.$$

We say that $y_1, y_2, \ldots$ is a *uniformly distributed sequence* if for any $[a, b)$ $(0\leqslant a<b\leqslant 1)$, the number k' of $y_1, \ldots, y_k$ falling in $[a, b)$ satisfies

$$(1.1)\qquad k' = (b-a)k+o(k)\quad \text{as } k\to\infty.$$

One can prove [3] that if (1.1) is true for all a and b $(0\leqslant a<b\leqslant 1)$, it holds uniformly in a and b: that is, the *discrepancy* $D(k)$ of the sequence $(y_k)_{k=1}^{\infty}$, defined by

$$(1.2)\qquad D(k) = \sup_{0\leqslant a<b\leqslant 1}\left|\frac{k'}{k}-(b-a)\right|,$$

satisfies $\lim\limits_{k\to\infty} D(k) = 0$.

The behaviour of $D(k)$ is closely related to that of the exponential sums

$$(1.3)\qquad s(k, h) = \left|\sum_{j=1}^{k} e^{2\pi i y_j h}\right|\qquad (k\geqslant 1,\ h\geqslant 1).$$

It can be shown that

$$(1.4)\qquad \lim_{k\to\infty} D(k) = 0\quad \text{iff}\ \lim_{k\to\infty}\frac{s(k, h)}{h} = 0\quad \text{for all } h\geqslant 1$$

and, more precisely,

$$(1.5)\qquad \frac{1}{2\pi}\sup_{h\geqslant 1}\frac{s(k, h)}{h}\leqslant kD(k)\leqslant 150\left(\frac{k}{m+1}+\sum_{h=1}^{m}\frac{s(k, h)}{h}\right)$$

for all integers $m\geqslant 1$ ([7], Theorem III and [1], p. 14).

Now suppose that

$$(1.6)\qquad \lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\ldots$$

is an infinite sequence of real numbers. If

(1.7) $$\lambda_{n+k}-\lambda_n \geqslant c > 0 \quad \text{whenever } k \geqslant \frac{Cn}{\log^a(n+1)},\ n \geqslant 1$$

where $c > 0$, $C > 0$ and $a > 1$ are constants, then the sequence of fractional parts

(1.8) $$\{\lambda_1 x\}, \{\lambda_2 x\}, \ldots, \{\lambda_n x\}, \ldots \quad (x \text{ real})$$

is uniformly distributed in $[0, 1)$, except for a set of real x having Lebesgue measure zero. We call this set $E = E(\lambda_1, \lambda_2, \ldots)$.

Hardy and Littlewood [9] were the first to prove a theorem of this "almost everywhere" type (with $\lambda_n = b^n$, b an integer $\geqslant 2$). The above result was discovered by Weyl [16] in 1916. Since then a good many papers have appeared which give more information about the sequences (1.8). Thus Cassels [2] and Erdös and Koksma [5] discovered independently that if (1.7) is strengthened to

(1.9) $$\lambda_{n+1}-\lambda_n \geqslant c > 0 \quad (n = 1, 2, \ldots)$$

or

(1.10) $\lambda_1, \lambda_2, \ldots$ are distinct integers, not necessarily increasing,

then the discrepancy $D(k; x)$ of the sequence (1.8) satisfies

(1.11) $$D(k; x) = o\left(k^{-1/2}\left(\log(k+1)\right)^{5/2+\varepsilon}\right) \quad \text{as } k \to \infty$$

for almost all x, for every $\varepsilon > 0$. If $(\lambda_n)_{n=1}^{\infty}$ is lacunary, that is

(1.12) $$\lambda_1 > 0, \quad \lambda_{n+1}/\lambda_n \geqslant q > 1 \quad \text{for } n = 1, 2, \ldots,$$

then the term $\left(\log(k+1)\right)^{5/2+\varepsilon}$ can be improved [4], though not beyond $\left(\log\log(k+2)\right)^{1/2}$ [10]. However, it is known that (1.12) also implies

(1.13) $$\dim E = 1$$

where 'dim' denotes Hausdorff dimension. (See [6]).

Suppose we make a growth assumption on λ_n which prevents lacunarity, e.g.

(1.14) $$\lambda_n \leqslant Cn^p \quad (n = 1, 2, \ldots)$$

where $C > 0$, $p \geqslant 1$ are constants. Salem [14] and Erdös and Taylor [6] discovered independently that (1.10) and (1.14) together imply

(1.15) $$\dim E \leqslant 1 - 1/p.$$

This estimate is stated in [6] to be sharp.

In [1] I studied the larger sets

(1.16) $$E_q = \{x \in (-\infty, \infty):\ D(k, x) \text{ is not } o(k^{-q})\}.$$

We have of course $E \subset E_q \subset E_r$ for $0 < q < r$, and, if we assume (1.9) or (1.10), we have by (1.11):

(1.17) $$E_q \text{ is null} \quad (0 < q < \tfrac{1}{2}).$$

In [1] I gave the estimate

(1.18) $$\dim E_q \leqslant 1 - \frac{1-4q}{p+\frac{1}{2}} \quad (0 < q < \tfrac{1}{4}),$$

assuming (1.10) and (1.14).

This was rather an uneconomical estimate, and the method of [1] may easily be sharpened to give

(1.19) $$\dim E_q \leqslant 1 - \frac{1-4q}{p+q} \quad (0 < q < \tfrac{1}{4}).$$

(My estimates were for capacity dimension, but this equals Hausdorff dimension for Borel sets [15], such as E_q). In particular (1.19) gives the theorem of Salem and Erdös–Taylor (since $E \subset E_q$ for every $q > 0$).

The method of [1] was essentially a refinement of Salem's argument. By adapting the method of Erdös and Taylor, I have now improved (1.19) to

(1.20) $$\dim E_q \leqslant 1 - \frac{1-3q}{p+2q} \quad (0 < q < \tfrac{1}{3})$$

under either assumption (1.10) or (1.11); since $p+5q > 1$ this is a definite improvement on (1.19). But I cannot show $\dim E_q < 1$ for $\frac{1}{3} \leqslant q < \frac{1}{2}$ under these conditions. The proof of (1.20) will be given in § 5.

This method can be successfully applied to rather slowly growing sequences, e.g.

(1.21) $$cn^{-a} \leqslant \lambda_{n+1}-\lambda_n, \quad \lambda_n = O(n^p) \quad (0 < a < 1,\ p \geqslant 1-a)$$

or

(1.22) $$\frac{c\log^a(n+1)}{n} \leqslant \lambda_{n+1}-\lambda_n, \quad \lambda_n = O\left(\log^p(n+1)\right) \quad (a > 1,\ p \geqslant 1+a).$$

(For such sequences, the capacity dimension approach fails). I shall show in § 5 that (1.21) yields:

(1.23) $$\dim E_q \leqslant 1 - \frac{1-3q-a}{p+2q} \quad \left(0 < q < \frac{1-a}{3}\right)$$

and thus

(1.24) $$\dim E \leqslant 1 - \frac{1-a}{p}.$$

In particular, if λ_n roughly resembles n^{1-a} $(0<a<1)$ in the sense that

$$c(\delta)n^{-a-\delta}\leqslant \lambda_{n+1}-\lambda_n\leqslant C(\delta)n^{-a+\delta}\qquad (n=1,2,\ldots) \tag{1.25}$$

for every $\delta>0$, we obtain from (1.24) that

$$\dim E=0. \tag{1.26}$$

If we assume (1.22), then we obtain the similar inequality

$$\dim E\leqslant 1-\frac{a-1}{p}. \tag{1.27}$$

I shall outline the proof of (1.27), together with other deductions from (1.22), in § 5. Not much space will be devoted to 'logarithmic growth rates' as the method is essentially the same here as for the case (1.21).

I have no idea whether an estimate such as (1.20) or (1.24) is sharp. At present I cannot construct any sequence $(\lambda_n)_{n=1}^{\infty}$ for which I can show $E_q \supsetneqq E$ for some $q<\frac{1}{2}$.

§ 2. Notations. Throughout the rest of this paper $(\lambda_n)_{n=1}^{\infty}$ is a sequence of real numbers such that

$$\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_n\leqslant\lambda_{n+1}\leqslant\ldots, \tag{2.1}$$

$$\lambda_{n+k}-\lambda_n\geqslant c>0\qquad\text{whenever } k\geqslant f(n),\ n\geqslant 1. \tag{2.2}$$

Here $c>0$ is a constant and $\big(f(n)\big)_{n=1}^{\infty}$ is a positive sequence that is monotonic nondecreasing for $k\geqslant k_0=k_0(f)$. Hence

$$\sup_{r\leqslant k} f(r)\leqslant Cf(k)\qquad (k=1,2\ldots),\qquad C=C(f) \tag{2.3}$$

where e.g. $C=\big(\max_{k\leqslant k_0} f(k)\big)/\big(\min_{k\leqslant k_0} f(k)\big)$.

We shall use x to denote a real number in a fixed but arbitrary interval $A\leqslant x\leqslant B$. We write

$$s(k,h)=s(k,h;x)=\Big|\sum_{j=1}^{k} e^{2\pi i\lambda_j hx}\Big|\qquad (k\geqslant 1,\ h\geqslant 1\ \text{integers}), \tag{2.4}$$

$$s_1(k,h)=s_1(k,h;x)=\Big|\sum_{j=1}^{k}\cos 2\pi\lambda_j hx\Big|\qquad (k,h\ \text{as above}), \tag{2.5}$$

$$s_2(k,h)=s_2(k,h;x)=\Big|\sum_{j=1}^{k}\sin 2\pi\lambda_j hx\Big|\qquad (k,h\ \text{as above}). \tag{2.6}$$

We write $D(k)=D(k;x)$ for the discrepancy of

$$\{\lambda_1 x\},\ldots,\{\lambda_k x\}.$$

Thus (1.5) remains true in this notation.

We shall also consider a positive monotonic nondecreasing sequence $\big(Q(k)\big)_{k=1}^{\infty}$ and the functions

$$g(k)=g(k;x)=\frac{Q(k)}{k}\sum_{h=1}^{Q(k)}\frac{s(k,h;x)}{h}\qquad (k\geqslant 1,\ h\geqslant 1). \tag{2.7}$$

(Throughout, $\sum_{n=u}^{v} y_n$ means $\sum\{y_n:\ u\leqslant n\leqslant v\}$ if u and v are nonnegative reals. The empty sum is 0.) We observe that by (1.5), with $m=Q(k)$

$$D(k;x)\leqslant 150\left(\frac{1}{Q(k)}+\frac{g(k;x)}{Q(k)}\right) \tag{2.8}$$

and so

$$D(k;x)=O\big(Q^{-1}(k)\big)\quad\text{ if } g(k;x)=O(1)\text{ as } k\to\infty. \tag{2.9}$$

We shall write

$$G(k,d)=\{x\in[A,B]:\ |g(k,x)|\geqslant d\}\qquad (k\geqslant 1,\ d>0) \tag{2.10}$$

$g_1(k)$, $g_2(k)$, $G_1(k,d)$, $G_2(k,d)$ are defined similarly using s_1 and s_2 instead of s. We note that, if $k\geqslant 1$, $d>0$,

$$\big(G_1(k,2d)\cup G_2(k,2d)\big)\subset G(k,2d)\subset\big(G_1(k,d)\cup G_2(k,d)\big). \tag{2.11}$$

If F is a function in $L^2(A,B)$, we shall write

$$M_1(F)=\int_A^B|F(x)|\,dx,\qquad M_2(F)=\Big(\int_A^B|F(x)|^2\,dx\Big)^{1/2} \tag{2.12}$$

We note that $M_1(F)\leqslant M_2(F)(B-A)^{1/2}$ by Cauchy's inequality.

We use $|W|$ to denote the Lebesgue measure of a measurable set W of real numbers. We note that

$$|G(k,d)|\leqslant d^{-2}M_2^2\big(g(k)\big)\qquad (k\geqslant 1,\ d>0). \tag{2.13}$$

If W is any set of real numbers we define for $0<a<1$

$$H^a(W)=\liminf_{r\to 0+}\Big\{\sum_{k=1}^{\infty}|I_k|^a:\ I_1,I_2\ldots\text{ are intervals of length}\leqslant r\text{ covering } W\Big\}.$$

The *Hausdorff dimension* of W is

$$\dim W=\sup\{a:\ H^a(W)>0\}.$$

We shall use $A_1, A_2,\ldots$ for constants which may depend on c, A, B, the sequences f or Q, but not on x or on integer variables such as $h, j, k, m, \boldsymbol{n}$, unless otherwise stated.

Finally, we agree that $\min(1/0,\ a)$ means a if a is a real number.

§ 3. Integrals of exponential sums.

LEMMA 3.1. *With the hypotheses and notations of § 2,*

$$(3.1)\qquad \sum_{1\leqslant m<j\leqslant k}\min\left(\frac{1}{\lambda_j-\lambda_m},\,A_1\right)\leqslant A_2kf(k)\left(1+\frac{\log(k+1)}{c}\right)\qquad (k\geqslant 1)$$

where A_2 is independent of c if A_1 is.

$$(3.2)\qquad M_2^2\big(s(k,h)\big)\leqslant A_3kf(k)\left(1+\frac{\log(k+1)}{h}\right)\qquad (k\geqslant 1,\ h\geqslant 1)$$

where A_3 depends on c.

Proof. To obtain (3.1) it suffices to prove that given m, $1\leqslant m<k$,

$$(3.3)\qquad \sum_{m<j\leqslant k}\min\left(\frac{1}{\lambda_j-\lambda_m},\,A_1\right)\leqslant A_2f(k)\left(1+\frac{\log(k+1)}{c}\right).$$

We may assume $\lambda_1\geqslant 0$ (add a constant to each λ_j if need be). To see (3.3) let p_n be the number of terms λ_j $(m<j\leqslant k)$ falling in the interval $[nc,(n+1)c)$ $(n=0,1,2,\ldots)$. If λ_r is the greatest and λ_s the least of these,

$$\lambda_r-\lambda_s<c,\qquad \text{so } r-s<f(s)\leqslant\max_{r\leqslant k}f(r)\leqslant Cf(k);$$

and

$$p_n=r-s+1\leqslant Cf(k).$$

Thus

$$(3.4)\qquad \sum_{m<j\leqslant k}\min\left(\frac{1}{\lambda_j-\lambda_m},\,A_1\right)\leqslant A_1p_0+\sum_{n=1}^{M}\frac{p_n}{nc}\leqslant CA_1f(k)+\sum_{n=1}^{M}\frac{p_n}{nc},$$

where M is the largest integer with $p_n\neq 0$. Clearly there are at most k integers with $p_n\neq 0$, so

$$(3.5)\qquad \sum_{n=1}^{M}\frac{p_n}{nc}\leqslant\frac{Cf(k)}{c}\sum_{j=1}^{k}\frac{1}{j}\leqslant\frac{A_4f(k)\log(k+1)}{c}.$$

(3.3) follows from (3.4) and (3.5); clearly A_2 depends only on A_1 and $C(f)$.

To see (3.2), we write

$$(3.6)\qquad M_2^2\big(s(k,h)\big)=\sum_{m,j=1}^{k}\int_A^B e^{2\pi i(\lambda_j-\lambda_m)hx}\,dx$$

$$\leqslant k(B-A)+2\sum_{1\leqslant m<j\leqslant k}\min\left(\frac{1}{\lambda_jh-\lambda_mh},\,B-A\right)$$

since

$$\int_A^B e^{2\pi ipx}\,dx=\begin{cases}B-A & \text{if } p=0,\\[4pt] \dfrac{e^{2\pi ipB}-e^{2\pi ipA}}{2\pi ip} & \text{if } p\neq 0.\end{cases}$$

We now apply (3.1) to the sequence $(\lambda_jh)_{j=1}^{\infty}$, replacing c by ch, to get (3.2).

Note. The enumeration of constants $A_1, A_2, \ldots$ begins anew for each lemma.

LEMMA 3.2. *With the notations and hypotheses of § 2,*

$$(3.7)\qquad M_2^2\big(g(k)\big)\leqslant\frac{A_1Q^2(k)f(k)}{k}\Big(\log^2\big(Q(k)+1\big)+\log(k+1)\Big)\qquad (k\geqslant 1.$$

Proof.

$$M_2^2\big(g(k)\big)$$

$$=\frac{Q^2(k)}{k^2}\sum_{h=1}^{Q}\sum_{r=1}^{Q}\frac{M_1\big(s(k,h)s(k,r)\big)}{hr}\leqslant\frac{Q^2(k)}{k^2}\sum_{h=1}^{Q}\sum_{r=1}^{Q}\frac{M_2\big(s(k,h)\big)M_2\big(s(k,r)\big)}{hr}$$

(by Cauchy's inequality)

$$\leqslant\frac{A_2Q^2(k)f(k)}{k}\sum_{h=1}^{Q}\sum_{r=1}^{Q}\frac{1}{hr}\left(1+\frac{\log^{1/2}(k+1)}{h^{1/2}}\right)\left(1+\frac{\log^{1/2}(k+1)}{r^{1/2}}\right)$$

(using (3.2) and $(u+v)^{1/2}\leqslant u^{1/2}+v^{1/2}$ if $u,v\geqslant 0$)

$$\leqslant\frac{A_2Q^2(k)f(k)}{k}\sum_{h=1}^{Q}\sum_{r=1}^{Q}\left(\frac{1}{hr}+\frac{\log^{1/2}(k+1)}{hr^{3/2}}+\frac{\log^{1/2}(k+1)}{rh^{3/2}}+\frac{\log(k+1)}{r^2h^2}\right)$$

$$\leqslant\frac{A_3Q^2(k)f(k)}{k}\Big(\log^2\big(Q(k)+1\big)+\log\big(Q(k)+1\big)\log^{1/2}(k+1)+\log(k+1)\Big)$$

(using convergence of $\sum_{r=1}^{\infty}\frac{1}{r^{3/2}}$, $\sum_{r=1}^{\infty}\frac{1}{r^2}$).

This completes the proof (since the product term

$$uv\leqslant u^2/2+v^2/2).$$

Lemma 3.2 and (2.8) may be combined to yield

$$M_2^2(kD(k)) \leqslant \frac{A_1k^2}{Q^2(k)}\left(1+2M_1(g(k))+M_2^2(g(k))\right) \leqslant \frac{A_2k^2}{Q^2(k)}\left(1+M_2^2(g(k))\right)$$

$$\leqslant \frac{A_3k^2}{Q^2(k)}\left\{1+\frac{Q^2(k)f(k)}{k}\left(\log^2(Q(k)+1)+\log(k+1)\right)\right\} \quad (k\geqslant 1).$$

If $D(m, k; x)$ denotes the discrepancy of the sequence

$$\{\lambda_{m+1}x\}, \{\lambda_{m+2}x\}, \ldots, \{\lambda_{m+k}x\}, \quad (m\geqslant 0, k\geqslant 1)$$

we have $\lambda_{m+n+k}\geqslant\lambda_{m+n}$ if $k\geqslant f(m+n)$ $(n\geqslant 1)$, so we replace $f(n)$ by $f(m+n)$ to estimate $M_2^2(kD(m, k))$. We notice that

$$\sup_{r\leqslant k} f(m+r)\leqslant C(f)f(m+k).$$

So we deduce the estimate

$$M_2^2(kD(m, k)) \leqslant \frac{A_3k^2}{Q^2(k)}\left\{1+\frac{Q^2(k)f(m+k)}{k}\left(\log^2(Q(k)+1)+\log(k+1)\right)\right\} \quad (k\geqslant 1, m\geqslant 1). \tag{3.8}$$

We shall refer to this again in § 5 and § 6.

§ 4. Lemmas on interpolation.

LEMMA 4.1. *If $1\leqslant k_1\leqslant k_2\leqslant\ldots$ is an increasing sequence, $k_m\to\infty$ and $k_{m+1}/k_m\to 1$ as $m\to\infty$, then*

$$\frac{s(k_m, h)}{k_m}\to 0 \text{ as } m\to\infty \text{ implies } \frac{s(k, h)}{k}\to 0 \text{ as } k\to\infty \quad (h\geqslant 1). \tag{4.1}$$

Proof. This is given in Weyl [16] but we repeat it here for convenience and to motivate Lemmas 4.2 and 4.3.

Let $k_m\leqslant k\leqslant k_{m+1}$. Then

$$\frac{s(k, h)}{k}\leqslant\frac{1}{k}\left\{s(k_m, h)+\left|\sum_{j=k_m+1}^{k} e^{2\pi i\lambda_j hx}\right|\right\}$$

$$\leqslant\frac{1}{k_m}s(k_m, h)+\frac{1}{k_m}(k-k_m)\leqslant\frac{s(k_m, h)}{k_m}+\frac{k_{m+1}-k_m}{k_m}.$$

Both summands in the last expression tend to 0 as $m\to\infty$, and so as $k\to\infty$.

LEMMA 4.2. *Suppose $0<q<1$ and $1\leqslant r<\frac{1}{q}$. Then, if $Q(k) = k^q$ $(k\geqslant 1)$,*

$$g(m^r) = O(1) \text{ as } m\to\infty \text{ implies } g(k) = O(1) \text{ as } k\to\infty. \tag{4.2}$$

Proof. Let $k_m = m^r$, and $k_m\leqslant k\leqslant k_{m+1}$. We have

$$g(k)-g(k_m) = k^{q-1}\sum_{h=1}^{k^q}\frac{s(k, h)}{h}-k_m^{q-1}\sum_{h=1}^{k_m^q}\frac{s(k_m, h)}{h}$$

$$= k^{q-1}\left\{\sum_{h=1}^{k^q}\frac{s(k, h)}{h}-\sum_{h=1}^{k_m^q}\frac{s(k_m, h)}{h}\right\}+(k^{q-1}-k_m^{q-1})\sum_{h=1}^{k_m^q}\frac{s(k_m, h)}{h}.$$

Thus

$$|g(k)-g(k_m)|$$

$$\leqslant k^{q-1}\sum_{h=1}^{k^q}\frac{|s(k, h)-s(k_m, h)|}{h}+k^{q-1}\sum_{h=k_m^q}^{k^q}\frac{s(k_m, h)}{h}+\frac{(k_m^{q-1}-k^{q-1})}{k_m^{q-1}}g(k_m)$$

$$\leqslant k^{q-1}\sum_{h=1}^{k^q}\frac{1}{h}(k-k_m)+\frac{k^{q-1}}{k_{m+1}^{q-1}}g(k_{m+1})+\left(1-\frac{k_{m+1}^{q-1}}{k_m^{q-1}}\right)g(k_m)$$

$$\leqslant A_1k^{q-1}\log(k+1)(k_{m+1}-k_m)+O(1)g(k_{m+1})+o(1)g(k_m).$$

To complete the proof we use the first Mean Value Theorem;

$$k_{m+1}-k_m = (m+1)^r-m^r = r(m+\theta)^{r-1}\leqslant r(m+1)^{r-1} \quad (0<\theta<1),$$

$$k^{q-1}\leqslant k_m^{q-1} = m^{r(q-1)}, \quad \log(k+1)\leqslant A_2r\log(m+1),$$

and, since $r(q-1)+r-1 = rq-1<0$,

$$m^{r(q-1)}(m+1)^{r-1}\log(m+1)\to 0 \quad \text{as } m\to\infty.$$

Hence

$$|g(k)-g(k_m)| = o(1)+O(1)g(k_{m+1})+o(1)g(k_m).$$

LEMMA 4.3. *Suppose $q>0$ and $0<r<\frac{1}{q+1}$. Then, if $Q(k) = \log^q(k+1)$ $(k\geqslant 1)$,*

$$g(e^{m^r}) = O(1) \text{ as } m\to\infty \text{ implies } g(k) = O(1) \text{ as } k\to\infty. \tag{4.3}$$

Proof. Let $k_m = e^{m^r}$ and $k_m\leqslant k\leqslant k_{m+1}$. Exactly as in Lemma 4.2, we obtain

$$|g(k)-g(k_m)| \leqslant \frac{\log^q(k+1)}{k} \sum_{h=1}^{\log^q(k+1)} \frac{1}{h}(k-k_m)+O(1)g(k_{m+1})+o(1)g(k_m).$$

It is thus a question of showing that

$$\log^q(k+1)\log\log(k+2)\,\frac{(k_{m+1}-k_m)}{k} = o(1). \tag{4.4}$$

Now for $0<\theta<1$,

$$\frac{k_{m+1}-k_m}{k} = \frac{r(m+\theta)^{r-1}e^{(m+\theta)^r}}{k} \leqslant r(m+1)^{r-1}e^{(m+1)^r-m^r}$$
$$\leqslant r(m+1)^{r-1}e^{rm^{r-1}} \leqslant 2r(m+1)^{r-1} \quad \text{for large } m.$$

Also $\log^q(k+1)\log\log(k+1) = O(m^{rq+\varepsilon})$, any $\varepsilon>0$. Since $rq+\varepsilon+r-1<0$ for sufficiently small $\varepsilon>0$, (4.4) is proved.

§ 5. An estimate of the Hausdorff dimension of E_q. In this section we assume that $f(k) = A_1k^a$, $0<a<1$.

Thus $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and

$$\lambda_{n+k}-\lambda_n \geqslant c>0 \quad \text{when } k \geqslant A_1n^a \quad (n\geqslant 1). \tag{5.1}$$

The hypothesis is fulfilled in particular if

$$\lambda_{n+1}-\lambda_n \geqslant c_1n^{-a} \quad (n\geqslant 1), \text{ where } c_1>0, \tag{5.2}$$

since $n^{-a}+(n+1)^{-a}+\ldots+(n+k)^{-a} \geqslant 2^{-a}$ if $k\geqslant n^a$. We introduce (5.1) as it allows for more irregular growth than (5.2) and seems to lead to the same estimates. The same remark applies to the use of (2.2) in general.

Under the hypothesis (5.1), Theorem II of Cassels [2] (combined with Lemma 3.1), yields

$$D(k;x) = o\left(k^{-(1-a)/2}(\log(k+1))^{5/2+\varepsilon}\right) \tag{5.3}$$

for almost all x, for every $\varepsilon>0$. This can also be obtained from (3.8) by arguments similar to those of [5]. In fact, choosing $Q(k) = k^{(1-a)/2}$,

$$M_2^2\left(kD(m,k)\right) \leqslant A_1k(m+k)^a\log^2(k+1)$$

which implies (5.3) (cf. Theorem 2, [8]). Thus

$$|E_q| = 0 \quad \text{for } 0<q<\frac{1-a}{2}. \tag{5.4}$$

It is of interest to note that if $\lambda_n = n^{1-a}$ (a special case of (5.2)) then for all $x\neq 0$, $D(k;x) = O(k^{b-1})$ with $b = \max(a, 1-a)$ (see [13]). Disregarding the exceptional set, this is better than (5.3) for $a\geqslant \frac{1}{3}$ only.

To return to the problem of this section, (5.4) suggests an estimate of $\dim E_q$. To avoid lacunarity we assume $\lambda_n = O(n^p)$ for some $p>0$. We shall prove

THEOREM 5.1. *If $\lambda_1\leqslant\lambda_2\leqslant\ldots$ is a sequence satisfying*

$$\lambda_{n+k}-\lambda_n \geqslant c>0 \quad \textit{if } k\geqslant A_1n^a \;(n = 1, 2, \ldots)$$

where $0<a<1$, and

$$\lambda_n \leqslant A_2n^p \quad (n = 1, 2 \ldots) \quad \textit{where } p\geqslant 1-a,$$

then, in the notation of (1.16),

$$\dim E_q \leqslant 1-\left(\frac{1-3q-a}{p+2q}\right) \quad \left(0<q<\frac{1-a}{3}\right). \tag{5.5}$$

We need only show that the set

$$W_q = W_q(A, B) = \{x\in[A, B]\colon D(k;x) \text{ is not } O(k^{-q})\} \tag{5.6}$$

satisfies $\dim W_q \leqslant 1-\left(\frac{1-3q-a}{p+2q}\right)$. For then clearly

$$\dim E_q\cap[A, B] \leqslant \dim W_{q'} \leqslant 1-\frac{1-3q'-a}{p+2q'} \quad \text{for any } q'>q,$$

and this leads to (5.5).

LEMMA 5.1. *Let $Q(k) = k^q$, let $0<r<1/q$ and $0<a<1$, where $0<q<1$.*

Suppose $\left(I(k,j)\right)_{j=1}^{N_k}$ and $\left(J(k,j)\right)_{j=1}^{N_k}$ are finite sets of intervals covering $G_1(k,1)$ and $G_2(k,1)$ respectively. Let

$$u(k) = \sum_{j=1}^{N_k}|I(k,j)|^a, \quad v(k) = \sum_{j=1}^{N_k}|J(k,j)|^a. \tag{5.7}$$

If the series $\sum_{m=1}^{\infty}u([m^r])$ and $\sum_{m=1}^{\infty}v([m^r])$ converge, then

$$\dim W_q \leqslant a. \tag{5.8}$$

Proof. Let

$$X = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}G(m^r, 2).$$

Then $W_q\subset X$. For if $x\in W_q$, $D(k;x)$ is not $O(k^{-q})$, so $g(k;x)$ is not $O(1)$ by (2.8) and $g(m^r;x)$ is not $O(1)$ by Lemma 4.2, and finally $x\in G(m^r, 2)$ for infinitely many m.

Thus we need only show $\dim X \leqslant \alpha$. Now for $n \geqslant 1$,

$$X \subset \bigcup_{m=n}^{\infty} G(m^r, 2) \subset \bigcup_{m=n}^{\infty} G_1(m^r, 1) \cup G_2(m^r, 1),$$

by (2.11). We can thus cover X with intervals $I_1, I_2, \ldots$ such that

$$\sum_{k=1}^{\infty} |I_k|^\alpha \leqslant \sum_{m=n}^{\infty} \{u(m^r) + v(m^r)\}. \tag{5.9}$$

The right hand side of (5.9) tends to 0 as $n \to \infty$. This shows that $H_\alpha(X) = 0$ and $\dim X \leqslant \alpha$.

Proof of Theorem 5.1. We again take $Q(k) = k^q$. Let

$$1 > \alpha > 1 - \frac{1-3q-a}{p+2q} = \frac{5q+p+a-1}{2q+p}. \tag{5.10}$$

Thus $\alpha(2q+p) - (4q+p+a-1) > q$.

Choose r so that

$$\min\bigl(1,\ \alpha(2q+p) - (4q+p+a-1)\bigr) > \frac{1}{r} > q. \tag{5.11}$$

We now find a covering of $G_1(k, 1)$ of the type required in Lemma 5.1. First of all, by (2.13) and (3.7),

$$|G_1(k, d)| \leqslant d^{-2} A_1 k^{2q-1+a} \log^2(k+1) \qquad (d > 0,\ k \geqslant 1) \tag{5.12}$$

where A_1 is independent of d (and k!). Also, except at the finitely many 'corners' of $g_1(k; x)$,

$$\left|\frac{d}{dx} g_1(k; x)\right| \leqslant 2\pi k^{q-1} \sum_{h=1}^{k^q} \frac{1}{h} \sum_{j=1}^{k} h|\lambda_j| \tag{5.13}$$

$$\leqslant A_3 k^{2q-1} \sum_{j=1}^{k} j^p$$

(using $\lambda_n \leqslant A_2 n^p$)

$$\leqslant A_4 k^{2q+p} \qquad (k = 1, 2, \ldots).$$

By the fundamental theorem of calculus,

$$|g_1(k; y) - g_1(k; x)| \leqslant A_4 k^{2q+p} |y - x| \qquad (x, y \text{ real}).$$

Thus if $x \in G_1(k, 1)$ and $y \in [A, B]$, $|y - x| \leqslant \frac{1}{2} A_4^{-1} k^{-2q-p}$, then $y \in G_1(k, \frac{1}{2})$.

Now $G_1(k, 1)$ is a finite union of closed intervals J. Starting from the left endpoint of each J, cover it with closed intervals of length $\frac{1}{2} A_4^{-1} k^{-2q-p}$ as economically as possible. Thus no two intervals overlap in more than one point and all are contained in $G_1(k, \frac{1}{2})$. The total number N_k of intervals required to cover $G_1(k, 1)$ thus satisfies

$$N_k \leqslant \frac{|G_1(k, \frac{1}{2})|}{\frac{1}{2} A_4^{-1} k^{-2q-p}} + 1 \leqslant A_5 k^{4q+p-1+a} \log^2(k+1). \tag{5.14}$$

If $I(k, 1), \ldots, I(k, N_k)$ is the covering system, by Hölder's inequality,

$$u(k) = \sum_{j=1}^{N_k} |I(k, j)|^\alpha \leqslant \Bigl\{\sum_{j=1}^{N_k} 1\Bigr\}^{1-\alpha} \Bigl\{\sum_{j=1}^{N_k} |I(k, j)|^{\alpha \cdot \frac{1}{\alpha}}\Bigr\}^\alpha \tag{5.15}$$

$$\leqslant N_k^{1-\alpha} |G_1(k, \tfrac{1}{2})|^\alpha \leqslant A_6 k^{(4q+p-1+a) - \alpha(2q+p)} \log^2(k+1),$$

using (5.12) and (5.14). It is easy to see that $\sum_{m=1}^{\infty} u([m^r])$ converges, since

$$r(4q+p-1+a) - \alpha r(2q+p) < -1.$$

Thus $\dim W_q \leqslant \alpha$ by Lemma 5.1 and $\dim E_q \leqslant \alpha$. This completes the proof, since

$$1 > \alpha > 1 - \frac{1-3q-a}{p+2q}$$

is the only restriction on α.

Of course (1.20) follows for a sequence satisfying (1.9) or even a weaker hypothesis $\lambda_{n+1} - \lambda_n \geqslant c\bigl(\log(n+1)\bigr)^{-b}$, $b > 0$. If we assumed (1.10) instead we would arrive at the same estimate using $M_2^2\bigl(s(k, h)\bigr) = k(B-A)$ for $k \geqslant 1$ (after expansion of $[A, B]$ to have integer endpoints) instead of Lemma 3.1.

We now state the result analogous to Theorem 5.1 for sequences growing "like $\log^{a+1} n$".

THEOREM 5.2. *If $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ is a sequence satisfying*

$$\lambda_{n+k} - \lambda_n \geqslant c > 0 \quad \textit{if } k \geqslant \frac{A_1 n}{\log^a(n+1)} \qquad (n \geqslant 1) \tag{5.16}$$

and

$$\lambda_n \leqslant A_2 \log^p(n+1) \qquad (n \geqslant 1)\ (p \geqslant a+1,\ a > 1), \tag{5.17}$$

then

$$\dim E \leqslant 1 - \frac{a-1}{p}. \tag{5.18}$$

If we further assume that $a > 2$ and $\alpha > 1 - \dfrac{a-2}{p}$, then

$$D(k; x) = o\Bigl((\log k)^{-\left\{\frac{a-2-p(1-\alpha)}{5-2\alpha}\right\} + \varepsilon}\Bigr) \tag{5.19}$$

for every $\varepsilon > 0$, except for a set of dimension $\leqslant \alpha$.

Note. (i). If we have a sequence satisfying

$$c_1 \frac{\log^a(n+1)}{n} \leqslant \lambda_{n+1} - \lambda_n \leqslant c_2 \frac{\log^a(n+1)}{n} \quad (c_1, c_2 > 0)$$

for $n \geqslant 1$, then

$$c_3 \log^{a+1}(n+1) \leqslant \lambda_n \leqslant c_4 \log^{a+1}(n+1) \quad (n \geqslant 1)$$

since the derivative of $\log^{a+1} y$ is $(a+1)(\log^a y)/y$. This is the reason for assuming $p \geqslant a+1$.

(ii). There is a discontinuity between (5.19) and (5.20) in that $\dim E \leqslant 1 - \frac{a-2}{p}$ is all that follows from (5.19). It seems to be difficult to improve Weyl's estimate $D(k; x) = o(1)$ a.e. for $1 < a < 2$.

The proof of (5.18) is carried out in two stages: first we assume $\lambda_1, \lambda_2, \ldots$ are integers. If

$$S(k, h, d) = \left\{ x \in [A, B]: \frac{s(k, h)}{k} > d \right\} \quad (d > 0,\ k \geqslant 1,\ h \geqslant 1)$$

then

$$E \cap [A, B] \subset \bigcup_{h=1}^{\infty} \bigcup_{t=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} S\left(e^{m^r}, k, \frac{1}{t}\right) \quad (0 < r < 1).$$

Provided $ap + a - p > \frac{1}{r}$, we can show that

$$H^a\Big(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} S(e^{m^r}, h, d)\Big) = 0 \tag{5.20}$$

by a process exactly analogous to Theorem 5.1. It is important to use the estimate

$$M_2^2\big(s(k, h)\big) \leqslant \frac{A_3 k^2}{\log^a(k+1)} \quad (k \geqslant 1)$$

which is better than (3.2) because $\lambda_1, \lambda_2, \ldots$ are integers. (5.20) implies (5.18), which thus holds equally if $\lambda_1, \lambda_2, \ldots$ are fractions with the same denominator.

In the second stage we assume $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ are any reals satisfying (5.16) and (5.17), and use an approximation argument identical to that of Weyl in ([16], § 7) to deduce that (5.18) still holds.

The proof of (5.19) is even more closely allied to that of Theorem 5.1. We simply apply Lemma 4.3 instead of Lemma 4.2. Details are left for the interested reader to fill in. It is worth noticing that (5.19) leads to

$$D(k; x) = o\Big((\log k)^{-\frac{(a-2)}{3}+\varepsilon}\Big) \tag{5.21}$$

for almost all x. Actually one can prove this with the assumption (5.16), that is, without (5.17), simply by applying the Borel–Cantelli lemma. The

method of Cassels [2] gives an estimate similar to (5.21) with $(a-5)/2$ instead of $(a-2)/3$; while (3.8) and [8] yield $(a-4)/2$. The latter result is better than (5.21) for $a > 8$.

For further information about the simplest case, $\lambda_n = \log^{a+1}(n+1)$, see [12], p. 89, and [13].

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