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Gaps between values of positive definite quadratic forms

by

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1. Jarník and Walfisz [6] proved the following: Let $n \geqslant 5$ and let $\lambda_1, \ldots, \lambda_n$ be positive real numbers whose ratios are not all rational. Let $N(x)$ denote the number of solutions in integers of the inequality

$$\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2 \leqslant x.$$

Then

$$N(x) = Cx^{n/2} + o(x^{\frac{n}{2}-1}).$$

They showed also that "o" cannot be replaced by any specific function. The constant C is simply the volume of the ellipsoid $\lambda_1 x_1^2 + \ldots + \lambda_n x_n^2 \leqslant 1$ and hence $C = C_0(\lambda_1 \ldots \lambda_n)^{-\frac{1}{2}}$, where C_0 is a simple function of n alone.

It is natural to conjecture that a similar result holds true for a general positive definite form $\sum\limits_{j,k} \lambda_{jk} x_j x_k$, provided, once again, that not all ratios of the coefficients are rational.

A particular deduction from the theorem of Jarník and Walfisz is that for any fixed $\varepsilon > 0$,

$$N(x+\varepsilon) - N(x) \sim C \cdot \varepsilon \cdot \left(\frac{n}{2}\right) x^{\frac{n}{2}-1} \quad \text{as} \quad x \to \infty.$$

In particular, the gaps between successive values of the quadratic form $\sum \lambda_i x_i^2$ at integer points must tend to 0. One expects a similar result for general positive definite quadratic forms in 5 variables. In the present state of knowledge, one would be satisfied to prove this for a general form in n variables under the assumption that n is very large. This specific question was put to one of us some years ago by Professor T. Estermann.

* Work on this paper was begun while I was visiting Cambridge in the spring of 1969 shortly before Professor Davenport's death. Only now have I had the opportunity to complete it. While working on this paper, I have been partially supported by the National Science Foundation and by the Institute of Science and Technology, University of Michigan. I also wish to thank the Fellows of Trinity College for their generous hospitality. D.J.L.

In this paper we partially resolve this question regarding gaps. We prove:

THEOREM. *There exists an integer n_0 (absolute) with the following property:*

Let $Q(\boldsymbol{x}) = Q(x_1, \ldots, x_n)$ be a positive definite quadratic form with real coefficients and suppose that $n \geqslant n_0$. Then, if $x_1^*, \ldots, x_n^*$ are integers with $\max_i |x_i^*|$ sufficiently large, there exist integers $x_1, \ldots, x_n$, not all zero, such that

$$(1) \qquad |Q(\boldsymbol{x}+\boldsymbol{x}^*) - Q(\boldsymbol{x}^*)| < 1.$$

Clearly the inequality (1) can be replaced by $|Q(\boldsymbol{x}+\boldsymbol{x}^*) - Q(\boldsymbol{x}^*)| < \varepsilon$, where ε is any small positive real number, by applying the theorem to the form $\varepsilon^{-1}Q(\boldsymbol{x})$.

Our result is imperfect in two ways: (a) it may be that $Q(\boldsymbol{x}+\boldsymbol{x}^*) = Q(\boldsymbol{x}^*)$, and indeed this may well happen if Q represents an integral form in 4 variables; (b) even if this situation does not occur, no deduction can be made about the gaps since the result does not prevent the values occurring in clumps with decidedly large gaps between the clumps.

Our proof uses the Hardy–Littlewood circle method and the proof of a key proposition is modelled after a paper of Birch and Davenport [1].

An estimate for the size of n_0 could be obtained from our work. However, since such an estimate would be exceedingly large compared to the anticipated value of 5, we have been content to demonstrate the existence of n_0. Accordingly, we have often used very crude inequalities when sharper ones could have been obtained.

In the case where the quadratic form is indefinite, reasonably satisfactory results are known. In a long series of very complicated papers, Birch, Davenport and Ridout, separately and jointly (see [5] for full list of references) proved that an indefinite quadratic form in 21 or more variables takes on arbitrarily small values at integral points. It then follows from work of Oppenheim [7] that the values of any real indefinite quadratic form in 21 or more variables are either discrete (when the form is proportional to a form with integer coefficients) or are everywhere dense.

2. We can write

$$Q(\boldsymbol{x}) = \sum_{i,j} \lambda_{ij} x_i x_j = \boldsymbol{x} \Lambda \boldsymbol{x}^t,$$

where $\Lambda = (\lambda_{ij})$ is a positive definite symmetric matrix and $\boldsymbol{x} = (x_1, \ldots, x_n)$. Then

$$Q(\boldsymbol{x}+\boldsymbol{x}^*) - Q(\boldsymbol{x}^*) = \boldsymbol{x}\Lambda\boldsymbol{x}^t + 2\boldsymbol{x}^*\Lambda\boldsymbol{x}^t = Q(\boldsymbol{x}) + 2\sum_{j=1}^{n} A_j x_j,$$

where $A_j = \sum_{i=1}^{n} \lambda_{ij} x_i^*$. On relabelling we may suppose $|A_1| = \max |A_j|$, and by replacing $\boldsymbol{x}^*$ by $-\boldsymbol{x}^*$ if necessary, we may suppose $A_1 < 0$. We set

$$P = -2A_1, \quad B_2 = -2A_2, \ldots, \quad B_n = -2A_n.$$

Since Λ is nonsingular and the $A_j = \sum \lambda_{ij} x_j^*$, it follows that P and $\max |x_j^*|$ are of comparable size. Hence, in the proof of the theorem we may assume P to be fixed, but arbitrarily large. In the course of our argument, the notation $X \ll Y$ shall mean $X < cY$ where c is a constant independent of P.

Since Λ is positive definite,

$$\lambda_{ii} > 0, \quad i = 1, \ldots, n,$$

and hence we can write

$$\lambda_{ii} \gg 1, \quad i = 1, \ldots, n.$$

If we set

$$F(\boldsymbol{x}) = Q(\boldsymbol{x}+\boldsymbol{x}^*) - Q(\boldsymbol{x}^*),$$

then

$$(2) \qquad F(\boldsymbol{x}) = \sum \lambda_{ij} x_i x_j - Px_1 - B_2x_2 - \ldots - B_n x_n,$$

where $\Lambda = (\lambda_{ij})$ is a positive definite symmetric matrix and

$$(3) \qquad 1 \gg \lambda_{ii} \gg 1, \quad i = 1, \ldots, n,$$

$$(4) \qquad P \geqslant |B_j|, \quad j = 2, \ldots, n.$$

We seek to show the existence of integral $\boldsymbol{x} \neq \boldsymbol{0}$ such that

$$|F(\boldsymbol{x})| < 1.$$

LEMMA 1. *Suppose $n \geqslant 4m^2 \geqslant 16$ and that $L_1, \ldots, L_m$ are m linear forms in $x_1, \ldots, x_n$, say*

$$L_i = \sum_{j=1}^{n} \gamma_{ij} x_j, \quad i = 1, \ldots, m.$$

Then, for each positive real number $P \geqslant 4$, there exist integers $x_1, \ldots, x_n$, not all 0, such that

$$|x_j| < 2P^{5m/n}, \quad j = 1, \ldots, n,$$

and

$$|L_i| < 2\sum_{j=1}^{n} |\gamma_{ij}| P^{(m/n)-4}, \quad j = 1, \ldots, m.$$

Proof. Let $x = (x_1, \ldots, x_n)$ run through all integer points in the n-dimensional cube: $|x_i| \leqslant \frac{1}{2}P^{5m/n}+1$. The number of such points exceeds P^{5m}. For each such point we obtain a point $L = (L_1(x), \ldots, L_m(x))$ in the m-dimensional box: $|L_i| \leqslant \sum_j |\gamma_{ij}|(\frac{1}{2}P^{5m/n}+1)$. Divide this box into equal subboxes by dividing the range of each coordinate into $[P^{4+(4m/n)}]+1$ equal parts. For each i, the length of the side of the subboxes is less than

$$\sum |\gamma_{ij}|\,(P^{5m/n}+2)P^{-4-(4m/n)} \leqslant 2\sum_j |\gamma_{ij}| P^{(m/n)-4}.$$

Since the number of subboxes is less than P^{5m}, some subbox will contain at least two distinct points $x^{(1)}, x^{(2)}$, and the point $x = x^{(1)} - x^{(2)}$ satisfies the conclusion of the lemma.

Lemma 2. *Let $F(x)$ be as in (2) and suppose $n \geqslant 176$. Then there exists an integral linear transformation T expressing $x_1, \ldots, x_n$ in terms of $y_1, \ldots, y_6$ such that rank $T = 6$ and*

$$(5)\qquad H(y) = H(y_1, \ldots, y_6) = F(Ty) = \sum_1^6 \mu_i y_i^2 + \sum_{1\leqslant i<j\leqslant 6} \varepsilon_{ij} y_i y_j - Py_1 - \sum_2^6 \varepsilon_k y_k,$$

where

$$(6)\qquad |\varepsilon_{ij}| \ll P^{-3}, \quad 1 \leqslant i < j \leqslant 6,$$

$$(7)\qquad |\varepsilon_k| \ll P^{-2}, \quad 2 \leqslant k \leqslant 6,$$

and

$$(8)\qquad \mu_1 = \lambda_{11}, \quad 1 \ll \mu_i \ll P^{10i/n}, \quad 2 \leqslant i \leqslant 6,$$

whence

$$(9)\qquad 1 \ll \prod_1^6 \mu_i \ll P^{200/n}.$$

Proof. Since Q is positive definite, the values of Q at non-zero integral points z must be positive and bounded away from zero. In fact $Q(z) = O(\max|z_i|^2)$. For otherwise there would exist an infinity of rational points $z^{(\nu)}$ lying on the boundary of the cube: $|z_i| \leqslant 1$ for which $Q(z^{(\nu)}) \to 0$. But the points $z^{(\nu)}$ have an accumulation point $z^{(0)}$ also on the boundary of this cube and $Q(z^{(0)}) = 0$; contrary to Q being positive definite.

If A is an integral matrix, then the coefficient of y_i^2 in $Q(Ay)$ is the value of Q at $a^{(i)}$, where $a^{(i)}$ is the ith column vector of A. Hence the coefficient of each y_i^2 in $Q(Ay)$ is $\gg 1$.

We now proceed to determine the integral matrix specified in the Lemma. We do so using Lemma 1 and a method introduced by Birch and Davenport [2]. Let P be a large real number (> 4). We shall choose in succession 6 integer points $a^{(1)}, \ldots, a^{(6)}$ in n-dimensional space which shall be the 6 column vectors of T. We take $a^{(1)} = (1, 0, \ldots, 0)$. To determine $a^{(2)}$, we apply Lemma 1 (with $m = 2$) to the linear forms

$$L_0 = Px_1 + B_2x_2 + \ldots + B_nx_n,$$

$$L_1 = \sum_{j=1}^n \gamma_{1j}x_j = a^{(1)}\Lambda x^t,$$

so that

$$(\gamma_{11}, \ldots, \gamma_{1n}) = a^{(1)}\Lambda.$$

By Lemma 1, there exists a non-zero integral point $a^{(2)}$ such that

$$|a_j^{(2)}| \ll P^{10/n}, \quad j = 1, \ldots, n,$$

$$|L_0(a^{(2)})| < 2\big(P + \sum |B_j|\big)P^{-4+(2/n)} \ll P^{-5/2},$$

$$|L_1(a^{(2)})| < 2\big(\sum |\gamma_{1j}|\big)P^{-4+(2/n)} < 2\big(\sum |\lambda_{1j}|\big)P^{-4+(2/n)} \ll P^{-7/2}.$$

For the choice of $a^{(3)}$ we apply Lemma 1 with L_0, L_1 as above and $L_2 = a^{(2)}\Lambda x^t$. We obtain a non-zero integral point $a^{(3)}$ such that

$$|a_j^{(3)}| \ll P^{15/n}, \quad j = 1, \ldots, n,$$

$$|L_0(a^{(3)})| = |Pa_1^{(3)} + B_2a_2^{(3)} + \ldots + B_na_n^{(3)}| < 2nP^{-3+(3/n)} \ll P^{-5/2},$$

$$|L_1(a^{(3)})| = |a^{(1)}\Lambda a^{(3)}| < 2n\lambda P^{-4+(3/n)} \ll P^{-7/2},$$

$$|L_2(a^{(3)})| = |a^{(2)}\Lambda a^{(3)}| < 2n\lambda P^{10/n}P^{-4+(3/n)} \ll P^{-7/2},$$

where $\lambda = \max|\lambda_{ij}|$.

We continue inductively, applying Lemma 1, to find $a^{(4)}, a^{(5)}, a^{(6)}$ such that

$$|a_j^{(\nu)}| \ll P^{5\nu/n}, \quad \nu = 1, \ldots, 6;\ j = 1, \ldots, n,$$

$$(10)\qquad |Pa_1^{(\nu)} + B_2a_2^{(\nu)} + \ldots + B_na_n^{(\nu)}| \ll P^{-5/2},$$

$$(11)\qquad |a^{(\nu)}\Lambda a^{(\mu)t}| \ll P^{-7/2}, \quad 1 \leqslant \nu < \mu \leqslant 6.$$

Then we also have

$$(12)\qquad 1 \ll |a^{(\nu)}\Lambda a^{(\nu)t}| \ll P^{10\nu/n}, \quad \nu = 2, \ldots, 6.$$

Let T be the n by 6 integral matrix whose νth column is $a^{(\nu)}$. Then

$$H(y) = F(Ty)$$

has the form (5) where $\mu_1 = \lambda_{11}$, $\mu_i = a^{(i)}\Lambda a^{(i)t}$, $\varepsilon_{ij} = 2a^{(i)}\Lambda a^{(j)t}$ and

$$\varepsilon_r = Pa_1^{(r)} + \ldots + B_na_n^{(r)}.$$

The relations (11), (10), (12) imply the relations (6), (7), and (8).

Now

$$TAT^t = \begin{pmatrix} \mu_1 & \tfrac{1}{2}\varepsilon_{12} & \dots & \tfrac{1}{2}\varepsilon_{16} \\ \tfrac{1}{2}\varepsilon_{12} & \mu_2 & \dots & \tfrac{1}{2}\varepsilon_{26} \\ \dots & \dots & \dots & \dots \\ \tfrac{1}{2}\varepsilon_{16} & & \dots & \mu_6 \end{pmatrix}$$

and since $\mu_i \geqslant 1$ and $|\varepsilon_{ij}| \ll P^{-7/2}$ it follows that TAT^t, and hence T, has rank 6. This completes the proof of the lemma.

Since T carries non-zero integral points into non-zero integral points, it follows that if

$$|F(\boldsymbol{x})| = |Q(\boldsymbol{x}+\boldsymbol{x}^*) - Q(\boldsymbol{x}^*)| \geqslant 1$$

for all non-zero integral $\boldsymbol{x}$, then

$$|H(\boldsymbol{y})| = |F(T\boldsymbol{y})| \geqslant 1$$

for all non-zero integral $\boldsymbol{y}$.

PROPOSITION 1. *Suppose $H(\boldsymbol{y})$ satisfies the conditions specified in Lemma 2 and suppose $|H(\boldsymbol{y})| \geqslant 1$ for all non-zero integral $\boldsymbol{y}$. Then, if P is sufficiently large, there is no solution of*

$$|\mu_1 y_1^2 + \dots + \mu_6 y_6^2 - Py_1| < \tfrac{1}{2} \tag{13}$$

in integers $y_1, \dots, y_6$, with $y_1 \neq 0$.

Proof. If there were such a solution $\boldsymbol{y}$ of (13), we should have $y_1 > 0$, and

$$\mu_1 y_1^2 < Py_1 + \tfrac{1}{2},$$

whence

$$0 < y_1 \ll P;$$

and

$$\mu_2 y_2^2 + \dots + \mu_6 y_6^2 \ll Py_1 \ll P^2,$$

whence

$$y_k \ll P, \quad k = 1, \dots, 6.$$

But then

$$|\varepsilon_{ij} y_i y_j| \ll P^{-1} \quad \text{and} \quad |\varepsilon_k y_k| \ll P^{-1},$$

and if P is sufficiently large, we have $|H(\boldsymbol{y})| < 1$; contrary to our hypothesis.

3. It follows from (9), on taking n sufficiently large, independently of P, that we can assume

$$P \geqslant (\mu_0 \dots \mu_6)^C, \tag{14}$$

where C is a suitable absolute constant to be determined later.

In view of Proposition 1, it suffices, to prove our theorem, to show that the assumption

$$|\mu_1 y_1^2 + \dots + \mu_6 y_6^2 - Py_1| \geqslant \tfrac{1}{2}$$

for all integers $y_1 \neq 0, y_2, \dots, y_6$, where $\mu_i \geqslant 1$ and P satisfies (14), leads to a contradiction. We shall show that this assumption leads to a real α such that $\alpha\mu_1, \dots, \alpha\mu_6$ are each very well approximable by rational numbers (Proposition 2), and then, in Section 11, we show that this leads to a contradiction.

PROPOSITION 2. *Suppose*

$$1 \ll \mu_1 \ll 1, \quad 1 \ll \mu_j, \quad j = 2, \dots, 6, \tag{15}$$

and

$$P \geqslant (\mu_1 \dots \mu_6)^{1/4\delta} \tag{16}$$

for a suitable constant $\delta < 1/10$. Suppose further that

$$|\mu_1 y_1^2 + \dots + \mu_6 y_6^2 - Py_1| \geqslant \tfrac{1}{2} \tag{17}$$

for all integers $y_1 \neq 0, y_2, \dots, y_6$. Then there exists a real number α with

$$P^{-9\delta} \ll \alpha < P^{\delta} \tag{18}$$

such that, for $j = 1, \dots, 6$,

$$\alpha\mu_j = \frac{a_j}{q_j} + \beta_j, \quad (a_j, q_j) = 1, \quad a_j \neq 0, \tag{19}$$

$$q_j \ll P^{9\delta} \tag{20}$$

and

$$|\beta_j| \ll P^{-2+8\delta}. \tag{21}$$

The proof of Proposition 2 uses the Hardy–Littlewood circle method and is modelled on a paper by Birch and Davenport [1]. The proof consists of a sequence of lemmas and we shall refer to that paper for proofs of a number of the lemmas. Throughout this sequence of lemmas, the hypotheses of Proposition 2 are assumed even if they are not explicitly stated.

4. Let

$$M = \max \mu_i, \quad m = \min \mu_i \quad \text{and} \quad \Pi = \mu_1 \dots \mu_6. \tag{22}$$

We define exponential sums

$$S_j(\alpha) = \sum_{P/7\mu_1^{\frac{1}{2}}\mu_j^{\frac{1}{2}} < y_j < P/2\mu_1^{\frac{1}{2}}\mu_j^{\frac{1}{2}}} e[\alpha\mu_j y_j^2], \quad j = 1, \dots, 6, \tag{23}$$

and

$$T(\alpha) = \sum_{9P/14\mu_1 < y_1 < P/\mu_1} e[\alpha(\mu_1 y_1^2 - Py_1)] = e[-\alpha P^2/4\mu_1] S_1(\alpha), \tag{24}$$

where $e[\theta] = e^{2\pi i\theta}$.

LEMMA 3. *For each positive integer k, there exists a real function $K(\alpha)$ of the positive real variable α, satisfying*

$$|K(\alpha)| < C(k)\min(1, \alpha^{-1-k}), \tag{25}$$

[$C(k)$ a constant depending on k] with the following property. Let

$$\psi(\theta) = \mathscr{R}\int_0^\infty e[\theta\alpha]K(\alpha)\,d\alpha; \tag{26}$$

then

$$0 \leqslant \psi(\theta) \leqslant 1 \quad \textit{for all real } \theta, \tag{27}$$

$$\psi(\theta) = 0 \quad \textit{for } |\theta| \geqslant \tfrac{1}{2}, \tag{28}$$

$$\psi(\theta) = 1 \quad \textit{for } |\theta| \leqslant \tfrac{1}{6}. \tag{29}$$

For a proof, see [3], Lemma 1.

COROLLARY. *We have*

$$\mathscr{R}\int_0^\infty T(\alpha)S_2(\alpha)\dots S_6(\alpha)K(\alpha)\,d\alpha = 0. \tag{30}$$

Proof. By (23), (24), and (26), the left hand side of (30) is

$$\sum_{y_1}\dots\sum_{y_6}\psi(\mu_1y_1^2+\dots+\mu_6y_6^2-Py_1),$$

where the summations are over the intervals specified in (23) and (24). By (28) and (17), the sum is 0.

Define

$$I(\alpha) = \int_{P/7\mu_1^{\frac{1}{2}}}^{P/2\mu_1^{\frac{1}{2}}} e[\alpha\xi^2]\,d\xi. \tag{31}$$

LEMMA 4. *For*

$$0 \leqslant \alpha \leqslant (8P\mu_1^{-\frac{1}{2}}\mu_j^{\frac{1}{2}})^{-1} \tag{32}$$

we have

$$S_j(\alpha) = \mu_j^{-\frac{1}{2}}I(\alpha)+O(1), \quad j = 2, \dots, 6, \tag{33}$$

$$T(\alpha) = e[-\alpha P^2/4\mu_1]\mu_1^{-\frac{1}{2}}I(\alpha)+O(1). \tag{34}$$

Proof. These results are special cases of van der Corput's lemma (see [4], p. 65, Lemma 16). The proofs of (33) and (34) follow that given for Lemma 2 of [1].

LEMMA 5. *For $\alpha > 0$ we have*

$$|I(\alpha)| \ll \min(P, \alpha^{-1}P^{-1}). \tag{35}$$

For a proof see [1], Lemma 3.

LEMMA 6. *We have*

$$\mathscr{R}\int_0^{1/(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)} T(\alpha)S_2(\alpha)\dots S_6(\alpha)K(\alpha)\,d\alpha = N+R, \tag{36}$$

where

$$N \geqslant 5.3\times 10^{-8}\mu_1^{-2}\Pi^{-\frac{1}{2}}P^4, \qquad |R| \ll (M/\Pi)^{\frac{1}{2}}P^3. \tag{37}$$

Proof. The interval of integration in (36) lies in the interval (32), and hence (33) and (34) hold. Furthermore, by (35),

$$\mu_j^{-\frac{1}{2}}|I(\alpha)| \ll \mu_j^{-\frac{1}{2}}\min(P, \alpha^{-1}P^{-1})$$

and the right-hand side is $\gg 1$ for all α in the interval of integration. Hence (33) and (34) give

$$|T(\alpha)S_2(\alpha)\dots S_6(\alpha)-e[-\alpha P^2/4\mu_1]\Pi^{-\frac{1}{2}}I^6(\alpha)|$$
$$\ll \Pi^{-\frac{1}{2}}\Big(\sum\mu_j^{\frac{1}{2}}\Big)\min(P^5, \alpha^{-5}P^{-5}) \ll (M/\Pi)^{\frac{1}{2}}\min(P^5, \alpha^{-5}P^{-5}).$$

Since $|K(\alpha)| \ll 1$, it follows that

$$\Pi^{\frac{1}{2}}\mathscr{R}\int_0^{1/(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)} T(\alpha)S_2(\alpha)\dots S_6(\alpha)K(\alpha)\,d\alpha$$
$$= \mathscr{R}\int_0^{1/(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)} e[-\alpha P^2/4\mu_1]I^6(\alpha)K(\alpha)\,d\alpha+O\Big(M^{\frac{1}{2}}\int_0^\infty\min(P^5, \alpha^{-5}P^{-5})\,d\alpha\Big).$$

The integral in the error term is $O(P^3)$ and hence this term can be absorbed in R. Thus it suffices to consider

$$\Pi^{-\frac{1}{2}}\mathscr{R}\int_0^{1/(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)} e[-\alpha P^2/4\mu_1]I^6(\alpha)K(\alpha)\,d\alpha. \tag{38}$$

The error introduced in extending the range of integration of (38) to ∞ is

$$\ll \Pi^{-\frac{1}{2}}\int_{1/(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)}^\infty P^{-6}\alpha^{-6}\,d\alpha$$
$$\ll \Pi^{-\frac{1}{2}}P^{-6}(8\mu_1^{-\frac{1}{2}}M^{\frac{1}{2}}P)^5$$
$$\ll (M/\Pi)^{\frac{1}{2}}P^3,$$

since $1 \ll \mu_1 \ll 1$ and $\delta < 1/10$, whence $M^2 \ll P$.

It remains to find a lower bound for

$$N = \Pi^{-1/2}\mathscr{R}\int_0^\infty e[-\alpha P^2/4\mu_1]I^6(\alpha)K(\alpha)\,d\alpha. \tag{39}$$

By (31) and (26), we have

$$\Pi^{1/2}N = \int_{P/7\mu_1^{1/2}}^{P/2\mu_1^{1/2}} \cdots \int_{P/7\mu_6^{1/2}}^{P/2\mu_6^{1/2}} \psi\left(\xi_1^2+\ldots+\xi_6^2-\frac{P^2}{4\mu_1}\right)d\xi_1\ldots d\xi_6$$
$$= 2^{-6}\int_{P^2/49\mu_1}^{P^2/4\mu_1}\cdots\int_{P^2/49\mu_6}^{P^2/4\mu_6}\psi\left(\eta_1+\ldots+\eta_6-\frac{P^2}{4\mu_1}\right)\frac{d\eta_1\ldots d\eta_6}{(\eta_1\ldots\eta_6)^{1/2}}.$$

The parallelepiped $\mathscr{P}$:

$$P^2/49\mu_1<\eta_j<9P^2/200\mu_1,\quad j=2,\ldots,6,$$
$$\left|\eta_1+\ldots+\eta_6-\frac{P^2}{4\mu_1}\right|<\frac{1}{6}$$

lies in the region of integration. Hence, by (29),

$$\Pi^{1/2}N\geqslant 2^{-6}\int\underset{\mathscr{P}}{\cdots}\int\frac{d\eta_1\ldots d\eta_6}{(\eta_1\ldots\eta_6)^{1/2}}\geqslant 2^{-6}\cdot 3^{-1}(P^2/49\mu_1)^{-1/2}\left\{\int_{P^2/49\mu_1}^{9P^2/200\mu_1}\eta^{-1/2}d\eta\right\}^5$$
$$\geqslant 2^{-6}\cdot 3^{-1}\cdot 7\cdot(.068)^5\mu_1^{-2}P^4\geqslant 5.3\times 10^{-8}\mu_1^{-2}P^4.$$

This completes the proof of Lemma 6.

5. LEMMA 7. *We have*

$$\int_0^{\mu_j^{-1}}|S_j(\alpha)|^4d\alpha\ll\mu_j^{-2}P^2\log P,\quad j=1,\ldots,6. \tag{40}$$

For a proof see [1], Lemma 5. As an immediate consequence, we also have

$$\int_0^{\mu_1^{-1}}|T(\alpha)|^4d\alpha\ll P^2\log P. \tag{41}$$

LEMMA 8. *We have*

$$\int_{1/(8\mu_1^{-1/2}M^{1/2}P)}^{1/(8\mu_1^{-1/2}m^{1/2}P)}|T(\alpha)S_2(\alpha)\ldots S_6(\alpha)|\,d\alpha\ll(M/\Pi)^{1/2}P^3\log P. \tag{42}$$

Proof. Since $|T(\alpha)|=|S_1(\alpha)|$, we can replace the $T(\alpha)$ in (42) by $S_1(\alpha)$.

Let

$$m=\lambda_1\leqslant\ldots\leqslant\lambda_6=M,$$

where the λ_i are the μ_j in some order. Let

$$S_j^*(\alpha)=\sum_{P/7\mu_1^{1/2}\lambda_j^{1/2}<x<P/2\mu_1^{1/2}\lambda_j^{1/2}}e[\alpha\lambda_jx^2].$$

Our lemma reduces to proving

$$\int_{1/(8\mu_1^{-1/2}M^{1/2}P)}^{1/(8\mu_1^{-1/2}m^{1/2}P)}|S_1^*(\alpha)\ldots S_6^*(\alpha)|\,d\alpha\ll(M/\Pi)^{1/2}P^3\log P. \tag{43}$$

We note that by Lemmas 7 and 4,

$$\int_0^{1/\lambda_j}|S_j^*(\alpha)|^4d\alpha\ll\lambda_j^{-2}P^2\log P, \tag{44}$$

$$S_j^*(\alpha)=\lambda_j^{-1/2}I(\alpha)+O(1)\quad\text{on}\quad 0\leqslant\alpha\leqslant(8\mu_1^{-1/2}\lambda_j^{1/2}P)^{-1}; \tag{45}$$

and in general

$$|S_j^*(\alpha)|\ll\lambda^{-1/2}P. \tag{46}$$

We split the interval of integration in (43) into 5 subintervals

$$I_k\colon (8\mu_1^{-1/2}\lambda_{k+1}^{1/2}P)^{-1}\leqslant\alpha\leqslant(8\mu_1^{-1/2}\lambda_k^{1/2}P)^{-1},\quad k=1,\ldots,5. \tag{47}$$

If $j\leqslant k$, then

$$I_k\subset[0,(8\mu_1^{-1/2}\lambda_j^{1/2}P)^{-1}]$$

and we can use (45) and Lemma 5 to estimate $|S_j^*(\alpha)|$. Since the left-hand endpoint of I_k exceeds P^{-2}, for α in I_k we have

$$|S_j^*(\alpha)|\ll\lambda_j^{-1/2}P^{-1}\alpha^{-1},\quad 1\leqslant j\leqslant k. \tag{48}$$

For $j>k$, we use (46). Hence in I_k, we have

$$|S_1^*(\alpha)\ldots S_6^*(\alpha)|\ll\Pi^{-1/2}P^{-k}\alpha^{-k}P^{6-k}=\Pi^{-1/2}P^{6-2k}\alpha^{-k}.$$

Thus, provided $k\geqslant 2$,

$$\int_{I_k}|S_1^*(\alpha)\ldots S_6^*(\alpha)|\,d\alpha\ll\Pi^{-1/2}\lambda_k^{\frac12(k-1)}P^{5-k}\ll(M/\Pi)^{1/2}P^3, \tag{49}$$

since $P>\Pi^2>M^2$.

There remains the case $k=1$. On I_1, $|S_1^*(\alpha)|$ still satisfies (48), and so

$$|S_1^*(\alpha)|\ll\lambda_2^{1/2}\quad\text{on}\quad I_1,$$

since $1\ll\lambda_1\leqslant\mu_1\ll 1$. We use (46) to estimate $|S_6^*(\alpha)|$. For $j=2,\ldots,5$, and P large, we have

$$\lambda_j^{-1}>(8\mu_1^{-1/2}\lambda_1^{1/2}P)^{-1},$$

and hence $I_1 \subset [0, \lambda_j^{-1}]$. Hence, by (44), we have

$$\int_{I_1} |S_j^*(\alpha)|^4 d\alpha \ll \lambda_j^{-2} P^2 \log P, \qquad j = 2, \ldots, 5.$$

Combining these estimates with Hölder's inequality, we obtain

$$\int_{I_1} |S_1^*(\alpha) \ldots S_6^*(\alpha)| d\alpha \ll (M/\Pi)^{\frac{1}{2}} P^3 \log P.$$

This last inequality, together with (49), gives (43) and completes the proof of Lemma 8.

6. Lemma 9. *For any fixed positive $\delta < 1/10$, there exists a constant $C_0(\delta)$ such that if P is sufficiently large*

$$\int_{1/(8\mu_1^{-\frac{1}{2}} m^{\frac{1}{2}} P)}^{P^\delta} |S_1(\alpha) S_2(\alpha) \ldots S_6(\alpha)| d\alpha > C_0(\delta) \mu_1^{-2} \Pi^{-\frac{1}{2}} P^4. \tag{50}$$

Proof. Let

$$R' = \mathscr{R} \int_{P^\delta}^{\infty} T(\alpha) S_2(\alpha) \ldots S_6(\alpha) K(\alpha) d\alpha.$$

By (25) and the trivial estimates for $|T(\alpha)|$ and $|S_j(\alpha)|$, we see that

$$|R'| \leqslant \int_{P^\delta}^{\infty} \Pi^{-\frac{1}{2}} P^6 C(k) \min(1, \alpha^{-1-k}) d\alpha = kC(k) \Pi^{-\frac{1}{2}} P^{6-k\delta}.$$

If we choose $k = [6/\delta]+1$ and set $C_1(\delta) = C(k)$, $C_2(\delta) = kC(k)$, we obtain

$$|R'| \leqslant C_2(\delta) \Pi^{-\frac{1}{2}}. \tag{51}$$

Using the Corollary to Lemma 3, Lemmas 6 and 8, and the estimate (51), we obtain

$$\Big| N + \mathscr{R} \int_{1/(8\mu_1^{-\frac{1}{2}} m^{\frac{1}{2}} P)}^{P^\delta} T(\alpha) S_2(\alpha) \ldots S_6(\alpha) K(\alpha) d\alpha \Big| \tag{52}$$
$$\ll (M/\Pi)^{\frac{1}{2}} P^3 \log P + C_2(\delta) \Pi^{-\frac{1}{2}} \ll P^3 \log P + C_2(\delta) \Pi^{-\frac{1}{2}}.$$

It follows from (25) and (51) that

$$C_1(\delta) \int_{1/(8\mu_1^{-\frac{1}{2}} m^{\frac{1}{2}} P)}^{P^\delta} |T(\alpha) S_2(\alpha) \ldots S_6(\alpha)| d\alpha$$
$$\geqslant \Big| \mathscr{R} \int_{1/(8\mu_1^{-\frac{1}{2}} m^{\frac{1}{2}} P)}^{P^\delta} T(\alpha) S_2(\alpha) \ldots S_6(\alpha) K(\alpha) d\alpha \Big| \geqslant N - C_3 P^3 \log P - C_3 C_2(\delta) \Pi^{-\frac{1}{2}},$$

where C_3 is the implied constant in (52). This inequality exceeds

$$5 \cdot 10^{-8} \mu_1^{-2} \Pi^{-\frac{1}{2}} P^4$$

provided $P > C_4 \Pi^{\frac{1}{2}} \log P$ and $P^4 > C_5 C_2(\delta)$, for suitable constants C_4 and C_5. Since $P^{\frac{1}{2}} \geqslant \Pi^{1/8\delta} \geqslant \Pi^{\frac{1}{2}}$, this is clearly the case when P is taken sufficiently large. On setting $C_0(\delta) = 5 \cdot 10^{-8}/C_1(\delta)$ and substituting $|S_1(\alpha)|$ for $|T(\alpha)|$ we obtain (50).

From now onwards we shall be concerned solely with values of α in the interval

$$\mathscr{I}\colon (8\mu_1^{-\frac{1}{2}} m^{\frac{1}{2}} P)^{-1} < \alpha < P^\delta. \tag{53}$$

7. For any integers a, q, with $q > 0$ and $(a, q) = 1$, define

$$S_{a,q} = \sum_{x=1}^{q} e[ax^2/q]. \tag{54}$$

Lemma 10. *We have*

$$|S_{a,q}| \ll q^{\frac{1}{2}}. \tag{55}$$

For a proof see [9], Chapter 2, Lemma 6.

Lemma 11. *Suppose that $A > 1$ and that α is a real number satisfying*

$$\alpha = \frac{a}{q} + \beta, \tag{56}$$

where

$$(a, q) = 1, \quad 0 < q \ll A, \quad 8|\beta| < q^{-1} A^{-1}. \tag{57}$$

Then

$$\sum_{\frac{1}{7}A < x < \frac{1}{2}A} e[\alpha x^2] = q^{-1} S_{a,q} \int_{\frac{1}{7}A}^{\frac{1}{2}A} e(\beta \xi^2) d\xi + O(q^{\frac{1}{2}} \log q). \tag{58}$$

Proof. If one makes allowance for the difference in the range of summation and integration here and in Lemma 9 of [1], then the same proof suffices.

Corollary. *Suppose that*

$$\alpha \mu_j = \frac{a_j}{q_j} + \beta_j, \quad j = 1, \ldots, 6, \tag{59}$$

where

$$(a_j, q_j) = 1, \quad 0 < q_j \ll P\mu_1^{-\frac{1}{2}} \mu_j^{-\frac{1}{2}}, \quad 8|\beta_j| < q_j^{-1} P^{-1} \mu_1^{\frac{1}{2}} \mu_j^{\frac{1}{2}}. \tag{60}$$

Then

$$|S_j(\alpha)| \ll q_j^{-\frac{1}{2}} (\log P) \min(P\mu_j^{-\frac{1}{2}}, P^{-1} \mu_j^{\frac{1}{2}} |\beta_j|^{-1}). \tag{61}$$

Proof. The hypotheses of Lemma 11 are satisfied when $A = P\mu_1^{-\frac{1}{2}}\mu_j^{-\frac{1}{2}}$, $a = a_j$, $q = q_j$ and α is replaced by $\alpha\mu_j$. The sum on the left in (58) becomes $S_j(\alpha)$. Hence

$$S_j(\alpha) = q_j^{-1} S_{a_j,q_j} \int\limits_{P/7\mu_1^{\frac{1}{2}}\mu_j^{\frac{1}{2}}}^{P/2\mu_1^{\frac{1}{2}}\mu_j^{\frac{1}{2}}} e[\beta\xi^2]\,d\xi + O(q_j^{\frac{1}{2}}\log q_j).$$

We can estimate the integral as in the proof of Lemma 5 and bound S_{a_j,q_j} by (55), to obtain

$$|S_j(\alpha)| \ll q_j^{-\frac{1}{2}}\min(P\mu_j^{-\frac{1}{2}}, P^{-1}\mu_j^{\frac{1}{2}}|\beta_j|^{-1}) + q_j^{\frac{1}{2}}\log q_j.$$

Now $q_j \ll P\mu_j^{-\frac{1}{2}}$ so that $q_j^{\frac{1}{2}} \ll q_j^{-\frac{1}{2}}P\mu_j^{-\frac{1}{2}}$, and $|\beta_j| \ll q_j^{-1}P^{-1}\mu_j^{\frac{1}{2}}$ so that $q_j^{\frac{1}{2}} \ll q_j^{-\frac{1}{2}}P^{-1}\mu_j^{\frac{1}{2}}|\beta_j|^{-1}$. Hence

$$|S_j(\alpha)| \ll q_j^{-\frac{1}{2}}\cdot(\log q_j)\cdot\min(P\mu_j^{-\frac{1}{2}}, P^{-1}\mu_j^{\frac{1}{2}}|\beta_j|^{-1}),$$

and this gives (61), since $q_j \ll P$.

8. For any α in the interval $\mathscr{I}$, defined in (53), and for each $j = 1, \ldots, 6$, there exist integers a_j, q_j such that

$$(a_j, q_j) = 1, \quad 0 < q_j \leqslant 8P\mu_1^{-\frac{1}{2}}\mu_j^{-\frac{1}{2}}, \tag{62}$$

and

$$\mu_j\alpha = \frac{a_j}{q_j} + \beta_j, \tag{63}$$

where

$$|\beta_j| < q_j^{-1}(8P\mu_1^{-\frac{1}{2}}\mu_j^{-\frac{1}{2}})^{-1}. \tag{64}$$

Thus, for α in $\mathscr{I}$, (59) and (60) are satisfied, and consequently (61) is valid.

Furthermore, it should be noted that when α is in $\mathscr{I}$, none of the a_j are 0. For if $a_j = 0$, then

$$|\mu_j\alpha| = |\beta_j| < \frac{\mu_1^{\frac{1}{2}}\mu_j^{\frac{1}{2}}}{8P},$$

contrary to (53).

Let $\mathscr{J}$ denote the subset of the interval $\mathscr{I}$ consisting of those α for which

$$|S_j(\alpha)| > P^{1-3\delta}\Pi^{-\frac{1}{4}}\mu_j^{-\frac{1}{2}}, \quad j = 1, \ldots, 6. \tag{65}$$

Lemma 12. *We have*

$$\int\limits_{\mathscr{I}-\mathscr{J}} |S_1(\alpha)\ldots S_6(\alpha)|\,d\alpha \ll P^{4-\delta}\Pi^{-\frac{1}{2}}. \tag{66}$$

Proof. In $\mathscr{I}-\mathscr{J}$, one of the inequalities (65) is false, say for $j = 6$. Thus

$$|S_6(\alpha)| \leqslant P^{1-3\delta}\Pi^{-\frac{1}{4}}\mu_6^{-\frac{1}{2}}. \tag{67}$$

By Lemma 7 and the periodicity of $S_j(\alpha)$, with period μ_j^{-1}, we have

$$\int\limits_0^{P^\delta} |S_j(\alpha)|^4\,d\alpha \ll \mu_j^{-1}P^{2+\delta}\log P.$$

It follows from this and Hölder's inequality, that

$$\int\limits_0^{P^\delta} |S_2(\alpha)\ldots S_5(\alpha)|\,d\alpha \ll (\mu_2\ldots\mu_5)^{-\frac{1}{4}}P^{2+\delta}\log P \ll \Pi^{-\frac{1}{4}}\mu_1^{\frac{1}{4}}\mu_6^{\frac{1}{4}}P^{2+\delta}\log P.$$

From this, the inequality (67) and the trivial estimate for $|S_1(\alpha)| \ll P/\mu_1$, it follows that

$$\int\limits_{\mathscr{I}-\mathscr{J}} |S_1(\alpha)\ldots S_6(\alpha)|\,d\alpha \ll \Pi^{-\frac{1}{2}}P^{4-2\delta}\log P,$$

whence (66) is valid.

Lemma 13. *The set $\mathscr{J}$ has positive measure and hence is a non-empty set.*

Proof. It follows from Lemmas 9 and 12 that

$$\int\limits_{\mathscr{J}} |S_1(\alpha)\ldots S_6(\alpha)|\,d\alpha > 0,$$

whence the measure of $\mathscr{J}$ must be positive.

9. Proof of Proposition 2. By Lemma 13, the set $\mathscr{J}$ is non-empty. Let α be an element of $\mathscr{J}$. Then there exist integers a_j, q_j, with $a_j \neq 0$, satisfying (62), (63) and (64), hence (61) is valid. But (65) is also valid for α in $\mathscr{J}$ and hence we have

$$P^{1-3\delta}\Pi^{-\frac{1}{4}}\mu_1^{-\frac{1}{2}} \ll q_j^{-\frac{1}{2}}P(\log P)\mu_1^{-\frac{1}{2}},$$

whence

$$q_j \ll P^{6\delta}(\log P)^2\Pi^{\frac{1}{2}} \ll P^{9\delta}, \tag{68}$$

since

$$\Pi < P^{4\delta}.$$

It also follows from (61) and (65) that, for $j = 1, \ldots, 6$, we have

$$P^{1-3\delta}\Pi^{-\frac{1}{4}}\mu_j^{-\frac{1}{2}} \ll q_j^{-\frac{1}{2}}(\log P)\mu_j^{\frac{1}{2}}P^{-1}|\beta_j|^{-1},$$

whence

$$|\beta_j| \ll P^{-2+3\delta}(\log P)\Pi^{\frac{1}{4}}\mu_j^{\frac{3}{4}} \leqslant P^{-2+3\delta}(\log P)\Pi$$

and therefore

$$|\beta_j| \ll P^{-2+8\delta}. \tag{69}$$

Finally, we note that since the $a_j \neq 0$ and since $\delta < 1/10$, by (68) and (69) we have

(70) $$a\mu_1 = \frac{a_1}{q_1} + \beta_1 \geqslant \frac{1}{q_1} - |\beta_1| \geqslant P^{-9\delta} - P^{-2+8\delta} \gg P^{-9\delta}.$$

The inequalities (68), (69) and (70) imply (20), (21) and (18). This completes the proof of the proposition.

10. LEMMA 14. *Given positive integers $b_1, \ldots, b_5$, there exists a positive integer $B \leqslant C(b_1 \ldots b_5)^{3/2}$, where C is an absolute constant, such that for all positive integers t, the equation*

(71) $$b_1x_1^2 + \ldots + b_5x_5^2 = Bt$$

is soluble in rational integers.

Proof. We first show that there exists a positive integer $N \leqslant 4b_1 \ldots b_5$ such that for each positive integer t and each prime p, the equation

(72) $$b_1x_1^2 + \ldots + b_5x_5^2 = Nt$$

is soluble in the ring of p-adic integers.

For each i and each prime p, let $\lambda_{p,i}$ be the integer such that $p^{\lambda_{p,i}} \| 4b_i$. Set

$$\lambda_p = \max_i \lambda_{p,i}.$$

For almost all p, we clearly have $\lambda_p = 0$. Set

$$N = \prod_p p^{\lambda_p}.$$

Clearly

$$N \leqslant 4b_1 \ldots b_5 \quad \text{and} \quad 4b_i | N.$$

As is well known, for every prime p, the equation $b_1^2x_1^2 + \ldots + b_5^2x_5^2 = 0$ has a p-adic integral solution $\alpha = (\alpha_1, \ldots, \alpha_5)$ with at least one coordinate, say $\alpha_{I(p)}$, a p-adic unit. Set

$$x_{I(p)} = \alpha_{I(p)}(1+u), \quad x_j = \alpha_j(1-u), \quad j \neq I(p).$$

Then for each positive integer t, there is a p-adic integer u such that

$$b_1x_1^2 + \ldots + b_5x_5^2 = 2u\Big(b_{I(p)}\alpha_{I(p)}^2 - \sum_{j \neq I(p)} b_j\alpha_j^2\Big) = 4b_{I(p)}\alpha_{I(p)}^2u = Nt.$$

Hence N has the desired property.

It follows from a theorem of G. L. Watson [8] that if (72) is not soluble in rational integers, then

$$Nt < C_1 \cdot (b_1 \ldots b_4)(b_1 + \ldots + b_5)^{3/2},$$

where C_1 is an absolute constant independent of $b_1, \ldots, b_5, Nt$. Choose

$$t_0 = \left[\frac{C_1(b_1 \ldots b_4)(b_1 + \ldots + b_5)^{3/2}}{N}\right] + 1$$

and set

$$B = Nt_0.$$

Then for each positive integer t the equation

$$b_1x_1^2 + \ldots + b_5x_5^2 = Bt$$

is soluble in rational integers. Furthermore, without loss of generality, we can suppose

$$b_5 = \max_i b_i$$

and then we see that

$$B = Nt_0 \leqslant 2 \cdot 5^{3/2}C_1b_1 \ldots b_4b_5^{3/2}.$$

If we let $C = 2 \cdot 5^{3/2}C_1$, it would follow that

$$B \leqslant C(b_1 \ldots b_5)^{3/2}.$$

This completes the proof of the lemma.

11. Proof of the theorem. For a given δ, we choose

(73) $$n > \frac{50}{\delta}.$$

We saw in Sections 2 and 3 that if the theorem is false, there exist an infinity of P, tending to ∞, and $\mu_1, \ldots, \mu_6$ (depending on P and the quadratic form) such that

$$1 \ll \mu_1 \ll 1, \quad 1 \ll \mu_j \ll P^{10j/n}, \quad j = 2, \ldots, 6,$$

whence

$$\Pi = \mu_1 \ldots \mu_6 \leqslant P^{200/n} \leqslant P^{4\delta},$$

and such that

$$|\mu_1y_1^2 + \ldots + \mu_6y_6^2 - Py_1| \geqslant \tfrac{1}{2}$$

for all integers $y_1 \neq 0, y_2, \ldots, y_6$. But these, along with the assumption $\delta < 1/10$, are just the hypotheses of Proposition 2. Thus it follows from Proposition 2 that there exists a real number α such that

(74) $$|\alpha\mu_1y_1^2 + \ldots + \alpha\mu_6y_6^2 - \alpha Py_1| \gg P^{-9\delta},$$

for all integers $y_1 \neq 0, y_2, \ldots, y_6$. Hence, on putting $y_j = q_jz_j, j = 1, \ldots, 6$, we have

(75) $$|a_1q_1z_1^2 + \ldots + a_6q_6z_6^2 - \alpha Pq_1z_1 + \beta_1q_1^2z_1^2 + \ldots + \beta_6q_6^2z_6^2| \gg P^{-9\delta},$$

for all integers $z_1 \neq 0, z_2, \ldots, z_6$.

By Lemma 14, there exists a positive integer

$$B \ll (a_2 q_2 \ldots a_6 q_6)^{3/2},$$

such that for every positive integer t, we can solve

$$a_2 q_2 z_2^2 + \ldots + a_6 q_6 z_6^2 = Bt$$

in rational integers $z_2, \ldots, z_6$. We have

(76) $$B \ll (\mu_2 \ldots \mu_6 \cdot \alpha^5 \cdot q_2^2 \ldots q_6^2)^{3/2} \ll P^{150\delta}.$$

If

$$Bt < P^{2-27\delta},$$

then

$$\Big|\sum_2^6 \beta_j q_j^2 z_j^2\Big| \leqslant \sum_2^6 |\beta_j| q_j^2 z_j^2 \ll P^{-2+8\delta} P^{9\delta} P^{2-27\delta} \ll P^{-10\delta},$$

also

$$|\beta_1 q_1^2 z_1^2| \ll P^{-10\delta},$$

provided $0 < z_1 \leqslant P^{1-19\delta}$.

Hence, if we can find positive integers z_1, t such that

(77) $$0 < Bt < P^{2-27\delta},$$

(78) $$0 < z_1 < P^{1-19\delta},$$

and

(79) $$|a_1 q_1 z_1^2 + Bt - \alpha P q_1 z_1| \ll P^{-10\delta},$$

we will have contradicted (75), and thereby demonstrated the truth of the theorem.

Put $z_1 = Bu$, then the inequality (79) is equivalent to

$$|a_1 q_1 B u^2 + t - \alpha P q_1 u| \ll B^{-1} P^{-10\delta},$$

and it then follows from (76) that the inequality

(80) $$|t + a_1 q_1 B u^2 - \alpha P q_1 u| \ll P^{-160\delta}$$

implies (79).

There exist integers u and v such that

(81) $$0 < u \leqslant P^{160\delta}, \quad |\alpha P q_1 u - v| < P^{-160\delta}.$$

Put

(82) $$t = v - a_1 q_1 B u^2.$$

We now show that if

(83) $$348\delta \leqslant 1$$

then t and $z_1 = Bu$ satisfy (77), (78), and (80).

Clearly $\alpha q_1 P = a_1 P + \beta_1 q_1 P \gg P$ and $a_1 q_1 B u \ll P^{319\delta} \ll P^{1-\delta}$ provided (83) holds and P is sufficiently large. Hence $t > 0$. Also

$$Bt \leqslant Bv \ll B\alpha P q_1 u \ll P^{1+320\delta} \ll P^{2-27\delta}$$

and

$$z_1 = Bu \ll P^{310\delta} \ll P^{1-19\delta},$$

provided (83) holds. Hence, for our choice of u and t, (77) and (78) are satisfied if (83) holds.

Finally we note that by (82) and (81), we have

$$|t + a_1 q_1 B u^2 - \alpha P q_1 u| = |v - \alpha P q u| < P^{-160\delta}.$$

Hence (80), and therefore (79), hold for our choice of z_1 and t.

Thus it follows that if we choose δ to satisfy (83) and choose n to satisfy (73), then the assumption that the theorem is false for such n leads to a contradiction. Hence the theorem is true for sufficiently large n.

References

[1] B. J. Birch and H. Davenport, *On a theorem of Davenport and Heilbronn*, Acta Arith. 10 (1958), pp. 259–279.

[2] — — *Indefinite quadratic forms in many variables*, Mathematika 5 (1958), pp. 8–12.

[3] H. Davenport, *Indefinite quadratic forms in many variables*, Mathematika 3 (1956), pp. 81–101.

[4] — *Analytic Methods for Diophantine Equations and Diophantine Inequalities*, Ann Arbor 1962.

[5] — and D. Ridout, *Indefinite quadratic forms*, Proc. London Math. Soc. (3), 9 (1959), pp. 544–555.

[6] V. Jarník und A. Walfisz, *Über Gitterpunkte in mehrdimensionalen Ellipsoiden*, Math. Zeitschr. 32 (1930), pp. 152–160.

[7] A. Oppenheim, *Values of quadratic forms I, II*, Quart. J. Math. (2), 4 (1953), pp. 54–59, 60–66.

[8] G. L. Watson, *Quadratic diophantine equations*, Phil. Trans. Royal Soc. London, 253 A (1960), pp. 227–254.

[9] I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers* (trans. Roth and Davenport), London and New York 1954.

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