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Received on 24. 1. 1970

ACTA ARITHMETICA
XVIII (1971)

On the least integers represented by the genera of binary quadratic forms

by

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In memory of our teacher and friend
Harold Davenport

1. Introduction. It is well known that every class of primitive binary quadratic forms with discriminant D represents a positive integer $\leqslant c\sqrt{|D|}$, where c is an absolute constant. In general the estimate cannot be improved either for $D<0$ or for $D\geqslant 0$; in the first case this is shown by the sequence $nx^2+(n+1)y^2$ $(n=1,2,\ldots)$ and in the second it follows as a consequence of the so-called Markov chain. The main aim of this paper is to show that for genera of forms in question, however, a better estimate obtains:

Theorem 1. *Every genus of primitive binary quadratic forms with discriminant D represents a positive integer $\leqslant c(\varepsilon)|D|^{3/8+\varepsilon}$ for any $\varepsilon>0$, where $c(\varepsilon)$ denotes a number depending only on ε.*

It seems likely that in fact every genus represents a positive integer $\leqslant c(\varepsilon)|D|^{\varepsilon}$ for any $\varepsilon>0$ but we are unable to establish this conjecture with our present arguments. Also our proof does not allow the number $c(\varepsilon)$ mentioned in the theorem to be effectively computed when $\varepsilon<1/8$; an effective estimate, on the other hand, would enable one to determine all the "numeri idonei" of Euler. This results at once from the following:

Theorem 2. *All negative discriminants D with one class of forms in every genus satisfy $|D|\leqslant C(\varepsilon)$, where $C(\varepsilon)$ is effectively computable in terms of $c(\varepsilon)$ for any $\varepsilon<1/8$.*

The proof of Theorem 1 involves an argument similar to that used by Linnik and Vinogradov in their paper [8] on the least prime quadratic residue; thus we shall appeal to the well-known result of Burgess [3] on character sums and also to Siegel's fundamental theorem [12] on L-functions. The proof of Theorem 2 is based on the work of [2]. As demonstrated there, for negative fundamental discriminants D with class number 1 or 2, where, in the latter case, D is assumed to be even, the integers referred to in Theorem 1 can be given explicitly and so, by virtue of Theorem 2, all such D can be completely determined. For odd discriminants D with class number 2 (whence $D \equiv 5 \pmod 8$ for $D < -15$) we do not know an explicit construction for the relevant integer represented by the non-principal genus; if, however, this integer could be determined then a bound for $|D|$ would follow at once from an old lemma of Heilbronn [5] without appeal to Theorem 2 (1). A result related to Theorem 2 has recently been published by Anferteva and Čudakov [1]. Their theorem allows one to give a bound in terms of $c(\varepsilon)$ for all negative discriminants with one class in every genus and a prescribed class number.

2. Lemmata. For the proof of Theorem 1 we shall need two lemmata.

LEMMA 1. *We have*

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{y^s}{s(s+1)(s+2)(s+3)}\,ds = \begin{cases} 0 & \textit{if} \quad 0 < y \leqslant 1, \\ \dfrac{1}{6}\left(1-\dfrac{1}{y}\right)^3 & \textit{if} \quad y \geqslant 1. \end{cases}$$

Proof. This is a special case given by $k = 3$, $c = 2$ of Theorem B on page 31 of Ingham [6].

LEMMA 2. *For any non-principal character χ with modulus k and conductor f and any $\varepsilon > 0$ we have*

(i) $$\Big|\sum_{d\leqslant x} \chi(d)d^{-s}\Big| \ll k^\varepsilon f^{3/16}|s|\log x,$$

(ii) $$\Big|\sum_{d=1}^{\infty} \chi(d)d^{-s}\Big| \ll k^\varepsilon f^{3/16}|s|,$$

(iii) $$\Big|\sum_{d>x} \chi(d)d^{-1}\Big| \ll k^\varepsilon f^{3/16}x^{-1/2},$$

where $s = \frac{1}{2}+it$ and the constants implied by $\ll$ depend only on ε.

Proof. Let χ be induced by the character χ_f with modulus f, and set

$$S(M,N) = \sum_{n=M+1}^{N} \chi(n).$$

The sum can be expressed alternatively in the form

$$\sum_{n=M+1}^{N} \chi_f(n) \sum_{d|(n,k)} \mu(d) = \sum_{d|k} \mu(d)\chi_f(d) \sum_{M/d<n\leqslant N/d} \chi_f(n).$$

(1) It will be seen from the proof of Theorem 1 that the integers referred to there can be chosen relatively prime to D.

Added in proof: The class number 2 problem has recently been resolved by some new results on linear forms in logarithms; see papers by Baker and Stark to appear in Annals of Math.

By Theorem 2 of Burgess [3] with $r = 2$ we have

$$\Big|\sum_{n=M+1}^{N} \chi_f(n)\Big| \ll N^{1/2}f^{3/16+\varepsilon}$$

and thus we see that

(1) $$|S(M,N)| \ll \sum_{d|k} \mu^2(d)N^{1/2}f^{3/16+\varepsilon} \ll k^\varepsilon f^{3/16}N^{1/2}.$$

Similarly from the Pólya–Vinogradov inequality we obtain

(2) $$|S(M,N)| \ll k^{(1/2)\varepsilon}f^{1/2}\log f \ll k^\varepsilon f^{1/2}.$$

Now the identity

$$\sum_{d=M+1}^{N} \chi(d)d^{-s} = s\int_M^N S(M,[u])u^{-1-s}du + S(M,N)N^{-s}$$

gives

$$\Big|\sum_{d=M+1}^{N} \chi(d)d^{-s}\Big| \leqslant |s|\int_M^N |S(M,[u])|u^{-1-\sigma}du + |S(M,N)|N^{-\sigma},$$

where $s = \sigma+it$. Applying this with $M = 0$, $N = x$, $\sigma = 1/2$ and using (1) we easily verify (i). Further, substituting $M = x$, $\sigma = 1$ and taking limits as $N \to \infty$ we obtain (iii). Finally, to prove (ii), we put $M = f$, $\sigma = 1/2$ and appeal to (2); again taking limits as $N \to \infty$ we get

$$\Big|\sum_{d>f} \chi(d)d^{-s}\Big| \ll |s|\int_f^\infty k^\varepsilon f^{1/2}u^{-3/2}\,du \ll |s|k^\varepsilon,$$

and (ii) now follows as a consequence of (i) on substituting $x = f$ in the latter.

3. Proof of Theorem 1. Let $D = e^2D_0$, where D_0 is a fundamental discriminant, and let χ_0 be the principal character $\bmod D$. We shall denote by U the set of all generic characters for D; the set U consists of all Legendre symbols (n/p), where p runs through the odd prime divisors of D together with at most two supplementary characters given by $(-1/n)$, $(2/n)$ or their product (2). By T we shall denote the subset of U given by the generic characters of D_0 (3) and, for any subset S of U, we put

$$\chi_S = \chi_0 \prod_{\chi\in S} \chi;$$

by the law of quadratic reciprocity we have then $\chi_T = (D/n)$.

(2) Cf. Dickson [4], p. 82 or Mathews [9], p. 135.

(3) The generic characters of D_0 are contained in U except possibly for the product of the supplementary characters mentioned above; in the exceptional case, however, both factors are included in U and we replace one of them by their product.

Now consider a genus determined by the values $\varepsilon_\chi = \pm 1$ to be taken by χ in U and thus satisfying

$$\prod_{\chi \in T} \varepsilon_\chi = 1. \tag{3}$$

We shall suppose that no positive integer $\leqslant x$ and relatively prime to D is represented by the genus and we shall obtain a contradiction if $x \gg |D|^{3/8+\varepsilon}$. The supposition implies that for each integer $n \leqslant x$ we have either $\chi(n) = -\varepsilon_\chi$ for some χ in U or, if n is not represented by any genus, $\chi_T(p) = -1$ for some prime p which occurs in the factorization of n with odd exponent. On writing

$$a_n = \sum_{d|n} \chi_T(d),$$

it is easily verified that the second possibility is eiquivalent to the condition $a_n = 0$ [4]. Hence we see that

$$\sum_{n \leqslant x} (1-n/x)^3 a_n \chi_0(n) \prod_{\chi \in U} \big(1+\varepsilon_\chi \chi(n)\big) = 0,$$

that is

$$\sum_{S \subset U} \prod_{\chi \in S} \varepsilon_\chi E(x, S) = 0, \tag{4}$$

where

$$E(x, S) = \sum_{n \leqslant x} (1-n/x)^3 a_n \chi_S(n).$$

We have

$$a_n \chi_S(n) = \chi_S(n) \sum_{d|n} \chi_T(d) = \sum_{d|n} \chi_S(n/d) \chi_{T \dot{-} S}(d),$$

where $T \dot{-} S$ denotes the symmetric difference, that is the set of elements in precisely one of S and T, and thus it is clear that

$$E(x, S) = E(x, T \dot{-} S).$$

Furthermore we have

$$E(x, S) = \sum_{d \leqslant x} \chi_{T \dot{-} S}(d) \sum_{m \leqslant x/d} (1-md/x)^3 \chi_S(m)$$

and, by virtue of Lemma 1, the last sum can be expressed in the form

$$\sum_{m=1}^{\infty} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{6x^s}{s(s+1)(s+2)(s+3)} \frac{\chi_S(m)}{(md)^s}\, ds.$$

(4) Cf. Landau [7], Satz 204.

Hence we obtain

$$E(x, S) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(x, S, s)\, ds,$$

where

$$F(x, S, s) = \frac{6x^s}{s(s+1)(s+2)(s+3)} L(s, \chi_S) \sum_{d \leqslant x} \chi_{T \dot{-} S}(d) d^{-s}.$$

On moving the line of integration from $\sigma = 2$ to $\sigma = \frac{1}{2}$ we get

$$E(x, S) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} F(x, S, s)\, ds + R(S), \tag{5}$$

where $R(S) = 0$ if S is not $\emptyset$, the empty set, and

$$R(\emptyset) = \frac{x}{4} \cdot \frac{\varphi(|D|)}{|D|} \sum_{d \leqslant x} \frac{\chi_T(d)}{d}.$$

In view of Lemma 2 and a property of $\zeta(s)$ we have, on the line $\sigma = \frac{1}{2}$,

$$|L(s, \chi_S)| \ll |s|\, |D|^\varepsilon f_S^{3/16},$$

$$\Big|\sum_{d \leqslant x} \chi_{T \dot{-} S}(d) d^{-s}\Big| \ll |s|\, |D|^\varepsilon f_{T \dot{-} S}^{3/16} \log x \qquad (S \neq T),$$

where f_S denotes the conductor of χ_S. It follows that, if $S \neq T$, the above integral is [5]

$$\ll x^{1/2} \log x\, |D|^{2\varepsilon} (f_S f_{T \dot{-} S})^{3/16}.$$

Also from Lemma 2 we see that, if $T \neq \emptyset$,

$$\sum_{d \leqslant x} \frac{\chi_T(d)}{d} = L(1, \chi_T) + O(|D|^{3/16+\varepsilon} x^{-1/2}),$$

and obviously the left-hand side is at least 1 if $T = \emptyset$. Further we note that

$$|D|/\varphi(|D|) \ll |D|^\varepsilon.$$

But now, on observing that the number of elements in U is $\ll 2^{\nu(|D|)} \ll |D|^\varepsilon$, where $\nu(|D|)$ denotes the number of distinct prime divisors of $|D|$, and recalling also that $E(x, T) = E(x, \emptyset)$, we readily verify from (3), (4) and (5) that

$$L(1, \chi_T) \ll x^{-1/2} \log x\, |D|^{4\varepsilon} (f_{T \dot{-} S} f_S)^{3/16} + |D|^{3/16+\varepsilon} x^{-1/2}$$

(5) Here essential use is made of the four factors $(s+j)$; $s(s+1)$ as occurs in [8] does not suffice.

if $T \neq \emptyset$, and that the same holds if $T = \emptyset$ with $L(1, \chi_T)$ replaced by 1. Since finally $f_{T \dotplus S} f_S = f_T f^2$, where f denotes the conductor of $\chi_{S'}$ with S' defined as the set of elements in S but not in T, and f_T, f divide D_0, e respectively, we clearly have a contradiction to Siegel's estimate [12] for $L(1, \chi)$ if $x \gg |D|^{3/8+5\varepsilon}$. The contradiction proves the theorem.

4. Proof of Theorem 2. Let k be a positive fundamental discriminant relatively prime to d and let $\chi(n) = (k/n)$. By $h(k)$ and a we denote respectively the class number and fundamental unit in the quadratic field $Q(\sqrt{k})$. Further we set $D = -d$, denote by $h(kd)$ the class number of quadratic forms with discriminant kD and suppose that

$$f = ax^2 + bxy + cy^2$$

runs through a complete set of inequivalent quadratic forms with discriminant D(6). We have (see [2])

$$(6) \qquad h(k)h(kd)\log a$$
$$= \tfrac{1}{12}\pi k\sqrt{d}\Big(\sum_f \chi(a)/a\Big)\prod_{p|k}(1-p^{-2}) + B_0 + \sum_f \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} B_r e^{\pi i r b/(ka)},$$

where

$$B_0 = -\log p \sum_f \chi(a)$$

if k is the power of a prime p, $B_0 = 0$ otherwise and, for $r \neq 0$(7)

$$|B_r| \leqslant k|r|e^{-\pi|r|\sqrt{d}/(ak)}.$$

By Theorem 1 every class represents a positive integer $< c(\varepsilon)d^{3/8+\varepsilon}$ and, after dividing by a common factor if necessary, we can assume that the representation is proper. Thus we can suppose that $a < c(\varepsilon)d^{3/8+\varepsilon}$ for every f. We have

$$(7) \qquad \Big|\sum_f \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} B_r e^{\pi i r b/(ka)}\Big| \leqslant h(d) \sum_{\substack{r=-\infty \\ r\neq 0}}^{\infty} k|r|\eta^{|r|} = 2h(d)\eta(1-\eta)^{-2},$$

where $h(d)$ denotes the number of f and $\eta = e^{-\pi\sqrt{d}/(ak)}$. Assuming that $k < d^{\delta}$, where $\delta = \frac{1}{2}(\frac{1}{8}-\varepsilon)$, it is easily seen that $\eta < e^{-c_1 d^{\delta}}$, where c_1, like $c_2, c_3, \ldots$, signifies a number effectively computable in terms of

(6) It would suffice to consider fundamental discriminants since by a result of Grube 1874, recently rediscovered by Grosswald (these Acta 8, pp. 295–306) any nonfundamental discriminant $D < -315$ with one class in every genus differs from a fundamental discriminant only by a factor 4.

(7) Though it is unimportant, a factor 2 which mistakenly entered in the work of [2] via the definition of A_r, has been omitted here.

$c(\varepsilon)$. If $d > c_2$ then $\eta < 1/2$ and, recalling that $h(d)$ is also the number of genera, it follows that the expression on the right of (7) is at most

$$8h(d)\eta \leqslant 2^{\nu(d)+4}\eta < e^{-c_3 d^{\delta}}.$$

Now there exist four distinct primes p, p', p'', p''' say, not dividing d and less than $d^{(1/16)\delta}$; for there are $\ll d^{(1/64)\delta}$ prime divisors of d and Čebyšev's estimates, for instance, show that there are $\gg d^{(1/32)\delta}$ primes below $d^{(1/16)\delta}$. Let $k_1 = pp'$, $k_2 = pp''$ or $p''p'''$ so that, for $k = k_1$ or k_2 we have $k \equiv 1 \bmod 4$, $(k, d) = 1$ and $B_0 = 0$. We introduce the suffix j ($= 1$ or 2) to distinguish the quantities defined above corresponding to k_1 or k_2. We write

$$b_1 = h(k_1)h(k_1 d)k_2 P_1 Q_2\Big(\prod_f a\Big)\sum_f \chi_2(a)/a,$$

where

$$P_1 = \prod_{p|k_1} p^2, \qquad Q_2 = \prod_{p|k_2}(p^2-1),$$

and we define b_2 similarly by interchanging the suffixes 1 and 2. Then clearly b_1, b_2 are rational integers and from (6) and (7) we obtain

$$|b_1\log a_1 - b_2\log a_2| < 2He^{-c_3 d^{\delta}},$$

where H is some number exceeding the maximum of $|b_1|$, $|b_2|$. Further, if $d > c_4$, we have

$$\prod_f a \leqslant (\sqrt{d})^{h(d)} \leqslant d^{d^{(1/16)\delta}}$$

and, for $\chi = \chi_1$ or χ_2,

$$\Big|\sum_f \chi(a)/a\Big| \leqslant \sum_f 1/a \leqslant h(d) < d.$$

Since also $h(k) < d^{\delta}$, $h(kd) < d^{1+\delta}$, $P_1 < d^{2\delta}$, $Q_2 < d^{2\delta}$ we see that a possible value for H is $e^{d^{\frac{1}{2}\delta}}$. This gives

$$|b_1\log a_1 - b_2\log a_2| < e^{-c_5 d^{\delta}}.$$

On the other hand, by Theorem 2 of [10], a version of well-known theorem of Gelfond in which the dependence on the logarithms is specified explicitly, we have(8)

$$|b_1\log a_1 - b_2\log a_2| \geqslant e^{-c_6(\log H)^7},$$

provided that $\log H > a_0 + 1 + a_0^{-1}$, where

$$a_0 = \log\max\{|eD_0|^{1/4}, 1, \overline{|\alpha_1|}, \overline{|\alpha_2|}, \overline{|\alpha_1\alpha_2|}\}$$

(8) To apply the result of [10], note that $|e^z - 1| \leqslant |z|e^{|z|}$, where $z = b_1\log a_1 - b_2\log a_2$, and that a_1, a_2 are multiplicatively independent.

and D_0 is the discriminant of the field $Q(\sqrt{k_1}, \sqrt{k_2})$. Furthermore, the classical estimate of fundamental units in real quadratic fields (cf. Schur [11]) gives

$$\overline{|a_j|} \leqslant k_j^{\sqrt{k_j}} \quad (j = 1, 2),$$

and it is readily verified that

$$|D_0|^{1/2} \leqslant 64 k_1 k_2 |k_1 - k_2|.$$

Thus we obtain $a_0 < c_7 d^{(1/16)\delta} \log d$, whence $\log H = d^{(1/8)\delta}$ satisfies the above condition.

It is clear that the inequalities are consistent only if $d < c_8$, and this completes the proof of Theorem 2.

We conclude by expressing our thanks to Professor Heilbronn for pointing out a mistake in an earlier draft of this paper.

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Received on 25. 1. 1970

Subgroups of the modular group generated by parabolic elements of constant amplitude

by

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Dedicated to the memory of Harold Davenport

1. Let $\Gamma(1) = \mathrm{SL}(2, \boldsymbol{Z})$ denote the modular group of unimodular 2×2 matrices with rational integral entries and put

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

For each positive integer n we write $\Delta(n)$ for the normal closure of U^n; it is the subgroup of $\Gamma(1)$ generated by the conjugate parabolic matrices $L^{-1}U^nL(L \in \Gamma(1))$. Further, $\Delta(n)$ is a subgroup of the principal congruence group

$$\Gamma(n) = \{T \in \Gamma(1)\colon\ T \equiv I (\mathrm{mod}\, n)\}.$$

The purpose of this paper is to provide a new proof of the well known

THEOREM. (i) *For* $1 \leqslant n \leqslant 5$, $\Delta(n) = \Gamma(n)$. (ii) *For* $n \geqslant 6$, *the index* $[\Gamma(n)\colon \Delta(n)] = \infty$.

If G is any subgroup of $\Gamma(1)$, we denote by $\hat{G}$ the corresponding inhomogeneous group. Thus $\hat{\Gamma}(1) = \mathrm{LF}(2, \boldsymbol{Z})$ and $\hat{G}$ is the image of G under the natural mapping from $\Gamma(1)$ to $\Gamma(1)/\Lambda$, where $\Lambda = \{I, -I\}$ is the centre of $\Gamma(1)$. A corresponding theorem holds for $\hat{\Delta}(n)$ and $\hat{\Gamma}(n)$.

As pointed out by Knopp [5], who gave an independent proof of part (ii) of the theorem, the results stated can be excavated by the persevering reader from the first volume of Klein and Fricke's monumental treatise ([4], pp. 354–360). An alternative proof of part (i) of the theorem has been given by Brenner ([1], pp. 215–217). The quickest and most elegant proof of the theorem is obtained by using the canonical presentation of a Fuchsian group; see Wohlfahrt [10]. For an application of properties of tesselation groups to prove part (ii), see Mennicke [6].

The proof of part (i) given below uses elementary properties of indefinite binary quadratic forms. It is, perhaps, worth pointing out that