Since submitting this paper, the authors [3] have proved that, for $d > 5$, $N_d(5) = 10d-27$ or $10d-29$ according as d is even or odd. It is also shown that the normal DS sequence of length $N_{2d+1}(5)$ is unique but that there are exactly two normal DS lengths $N_{2d+1}(4)$ and $N_{2d}(5)$.

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A combinatorial problem connected with differential equations II

by

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1. Let us call a sequence *admissible* if it has no immediate repetition and contains no subsequence of the form a, b, a, b, a with $a \neq b$. Let $N(n)$ be the greatest length (that is, greatest number of terms) of an admissible sequence formed from n distinct elements.

The problem of estimating $N(n)$ has been investigated in [1] and it has been proved there that

$$5n-C < N(n) = O(n\log n).$$

($N(n)$ was denoted by $N_4(n)$.) The aim of this paper is to improve the above result in both directions. We prove

THEOREM 1. *We have*

$$N(n) = O\left(\frac{n\log n}{\log\log n}\right).$$

THEOREM 2. *We have*

$$\lim \frac{N(n)}{n} \geqslant 8.$$

THEOREM 3. *For positive integers l, m the following inequality holds*

$$N(lm+1) \geqslant 6lm-m-5l+2.$$

Theorem 3, found in collaboration with J. H. Conway, gives in general a weaker bound for $N(n)$ than that which can be obtained from the proof of Theorem 2. It is included as useful for small values of n. In particular, it implies

COROLLARY. *We have $N(n) \geqslant 5n-8$ and the equality sign is excluded for odd $n \geqslant 13$ and even $n \geqslant 18$.*

It is interesting to note that $N(n) = 5n-8$ for $n = 4, \ldots, 10$ (cf. [2]).

2. Let $M(n)$ be the maximum length of a sequence formed from the integers $1, 2, \ldots, n$ with the following property: for some r $(0 \leqslant r \leqslant n)$ there exists an admissible sequence of which the given sequence is a section, and the integers $1, 2, \ldots, r$ occur before this section and the integers $r+1, \ldots, n$ occur after it.

LEMMA 1. $M(n) < 5n$.

Proof. We can write the given sequence as

$$\mathscr{A}_1, \mathscr{B}_1, \mathscr{A}_2, \mathscr{B}_2, \ldots, \mathscr{A}_s, \mathscr{B}_s,$$

where the elements of each $\mathscr{A}_i$ are from $1, \ldots, r$ and the elements of each $\mathscr{B}_i$ are from $r+1, \ldots, n$, and $\mathscr{A}_1$ or $\mathscr{B}_s$ may be empty but the others are not.

If we remove the $\mathscr{B}$'s and eliminate any immediate repetitions we get a sequence formed from $1, \ldots, r$ of length $\geqslant \sum L(\mathscr{A}_i) - s$, where $L(\mathscr{A}_i)$ is the length of $\mathscr{A}_i$. Since this sequence is admissible when preceded by a sequence containing $1, \ldots, r$, it contains no a, a', a, a'. Hence by Theorem 1 of [1]

$$\sum_{i=1}^{s} L(\mathscr{A}_i) \leqslant s + (2r-1).$$

Similarly

$$\sum_{i=1}^{s} L(\mathscr{B}_i) \leqslant s + (2n-2r-1),$$

whence

$$M(n) \leqslant 2s + 2n - 2.$$

It remains to estimate s. We select one element α from each $\mathscr{A}_i$ and one element β from each $\mathscr{B}_i$. The elements α selected from consecutive $\mathscr{A}_i$ may be equal and we can enumerate all the selected elements as

$$(*) \qquad \alpha_1, \beta_1^{(1)}, \alpha_1, \beta_1^{(2)}, \ldots, \alpha_1, \beta_1^{(\nu_1)}, \alpha_2, \beta_2^{(1)}, \alpha_2, \ldots, \alpha_h, \beta_h^{(1)}, \ldots, \alpha_h, \beta_h^{(\nu_h)},$$

where possibly α_1 or $\beta_h^{(\nu_h)}$ may be missing. We have

$$s = \sum_{i=1}^{h} \nu_i.$$

If $\beta_i^{(j)} = \beta_k^{(l)} = \beta$, $\nu_k > 1$ and either $i < k$ or $i = k, j \leqslant l < \nu_k$ then the sequence $(*)$ contains the subsequence $\beta, \alpha_k, \beta, \alpha_k$, which is impossible. Therefore, the elements $\beta_i^{(j)}$ with $j < \nu_i$ are distinct and

$$\sum_{i=1}^{h} (\nu_i - 1) \leqslant n - r.$$

Also the sequence $\alpha_1, \alpha_2, \ldots, \alpha_h$ forms part of an admissible sequence when preceded by $1, \ldots, r$, whence

$$h \leqslant 2r-1.$$

Finally, we have

$$s \leqslant (n-r) + h \leqslant n + r - 1,$$

whence

$$M(n) \leqslant 2(n+r-1) + 2n - 2.$$

By symmetry this implies

$$M(n) \leqslant 5n - 4.$$

This proves Lemma 1.

We now consider any admissible sequence $\mathscr{S}$ of length $N(n)$ formed from $1, \ldots, n$, and construct a partition of $\mathscr{S}$ depending on an arbitrary integer m with $1 \leqslant m < n$. First we take the minimal left hand section $\mathscr{U}$ of $\mathscr{S}$ with m distinct terms, then the minimal right hand section $\mathscr{V}$ of $\mathscr{S}$ including all the elements of $\mathscr{S}$ not appearing in $\mathscr{U}$. We now write

$$\mathscr{S} = (\mathscr{U}, \mathscr{C}, \mathscr{V}).$$

Let for a given set A, $\complement A$ be its complement, $|A|$ its cardinality and for a given sequence $\mathscr{A}$, $\mathscr{A}^*$ be the set of its elements. We put

$$\begin{aligned}
m' &= |\mathscr{V}^*|, & & \\
m_1 &= |\complement\mathscr{C}^* \cap \complement\mathscr{V}^*|, & m_1' &= |\complement\mathscr{C}^* \cap \complement\mathscr{U}^*|, \\
m_2 &= |\mathscr{U}^* \cap \mathscr{C}^* \cap \complement\mathscr{V}^*|, & m_2' &= |\mathscr{V}^* \cap \mathscr{C}^* \cap \complement\mathscr{U}^*|, \\
m_3 &= |\mathscr{U}^* \cap \mathscr{V}^* \cap \complement\mathscr{C}^*|, & & \\
m_4 &= |\mathscr{U}^* \cap \mathscr{V}^* \cap \mathscr{C}^*|. & &
\end{aligned}$$

Then

$$\begin{aligned}
m &= m_1 + m_2 + m_3 + m_4, \\
m' &= m_1' + m_2' + m_3 + m_4, \\
n &= m_1 + m_1' + m_2 + m_2' + m_3 + m_4.
\end{aligned}$$

We note that of the $m_2 + m_2' + m_4$ distinct elements of $\mathscr{C}$, m_2 occur also to the left in $\mathscr{U}$, and m_2' occur also to the right in $\mathscr{V}$, and m_4 occur in both $\mathscr{U}$ and $\mathscr{V}$.

LEMMA 2. $N(n) < N(m_1) + N(m_1') + 13n$.

Proof. By Lemma 1 we have

$$L(\mathscr{C}) < 5(m_2 + m_2' + m_4) \leqslant 5n.$$

Now enumerate the terms of $\mathscr{V}$, picking out explicitly those that have occured already in $\mathscr{U}$ or $\mathscr{C}$, the number of such terms (distinct) being $m_2'+m_3+m_4 = m'-m_1'$. Write

$$\mathscr{V} = (a_1, \mathscr{F}_1^{(1)}, a_1, \ldots, a_1, \mathscr{F}_1^{(\nu_1)}, a_2, \ldots, a_h, \mathscr{F}_h^{(\nu_h)}),$$

where $a_1, \ldots, a_h$ are terms just mentioned (not necessarily distinct) and the $\mathscr{F}_i^{(j)}$ are formed from the m' distinct terms of $\mathscr{V}$ which do not occur in $\mathscr{U}$ or $\mathscr{C}$ (a_1 may be missing and $\mathscr{F}_i^{(\nu_i)}$ may be empty). By the arguments used in the proof of Lemma 1

$$h \leqslant 2(m'-m_1')-1,$$

$$\sum_{i=1}^{h} (\nu_i - 1) \leqslant m_1'.$$

If we remove from $\mathscr{V}$ the a's and eliminate any immediate repetitions we get an admissible sequence formed from m_1' distinct integers of length $\sum\sum L(\mathscr{F}_i^{(j)})-r$, where r is the number of immediate repetitions. However, r does not exceed h since (cf. the proof of Lemma 1)

$$\mathscr{F}_i^{(j)*} \cap \mathscr{F}_k^{(l)*} = \emptyset,$$

if $\nu_k > 1$ and either $i < k$ or $i = k$, $j \leqslant l < \nu_k$. Hence

$$\sum_{i=1}^{h} \sum_{j=1}^{\nu_i} L(\mathscr{F}_i^{(j)}) \leqslant N(m_1') + h$$

and

$$L(\mathscr{V}) \leqslant N(m_1') + h + \sum_{i=1}^{h} \nu_i \leqslant N(m_1') + m_1' + 2h$$

$$\leqslant N(m_1') + m_1' + 4(m'-m_1') \leqslant N(m_1') + 4n.$$

Similarly

$$L(\mathscr{U}) \leqslant N(m_1) + 4n$$

and on addition we obtain the result.

LEMMA 3. $N(n) \leqslant N(m) + N(n-m) + (n-m) + 4(m-m_1)$.

Proof. We set

$$\mathscr{S} = (\mathscr{U}, a_1, \mathscr{S}_1^{(1)}, a_1, \ldots, a_k, \mathscr{S}_k^{(r_k)}),$$

where the a_i are terms that have occured in $\mathscr{U}$, the $\mathscr{S}_i^{(j)}$ do not contain such terms, a_1 may be missing and $\mathscr{S}_i^{(r_i)}$ may be empty. Since the number of distinct terms available for the a_i is $m-m_1$, we have

$$k \leqslant 2(m-m_1)-1.$$

As in the proof of Lemma 2

$$\sum_{i=1}^{k} (r_i - 1) \leqslant n-m,$$

$$\sum_{i=1}^{k} \sum_{j=1}^{\nu_i} L(\mathscr{S}_i^{(j)}) \leqslant N(n-m) + k.$$

Hence

$$N(n) \leqslant N(m) + \sum_{i=1}^{k} r_i + \sum_{i=1}^{k} \sum_{j=1}^{\nu_i} L(\mathscr{S}_i^{(j)})$$

$$\leqslant N(m) + (n-m) + N(n-m) + 2k$$

$$< N(m) + N(n-m) + (n-m) + 4(m-m_1).$$

LEMMA 4. *If* $1 < h < n$ *then*

$$N(n) < \max\big(N(n-h)+13n,\; N(n-h)+N(h)+5h\big).$$

Proof. By Lemma 2,

$$N(n) < N(m_1) + N(m_1') + 13n < N(m_1+m_1') + 13n.$$

By Lemma 3, with $m = n-h$,

$$N(n) < N(n-h) + N(h) + h + 4(m-m_1).$$

Put $m_1 + m_1' = n-t$. Then

$$m - m_1 = m_2 + m_3 + m_4 \leqslant n - m_1 - m_1' = t.$$

Hence

$$N(n) < N(n-t) + 13n$$

and

$$N(n) < N(n-h) + N(h) + h + 4t.$$

If $t \geqslant h$ the former inequality gives the result, and if $t < h$ the latter inequality gives the result.

LEMMA 5. *If* $F(n) = \dfrac{n\log n}{\log\log n}$ *and* $h < \frac{1}{2}n$ *then*

$$F(n) - F(n-h) > \tfrac{1}{2}h \frac{\log n}{\log\log n}$$

and

$$F(n) - F(n-h) - F(h) > \tfrac{1}{2}h \frac{\log\big((n-h)/h\big)}{\log\log n}$$

for all large n.

Proof. Put $F(n) = nL(n)$, where $L(n) = \dfrac{\log n}{\log\log n}$. We note that

$$L'(x) = \frac{1}{x\log\log x} - \frac{1}{x(\log\log x)^2},$$

and that this is a decreasing function and is greater than $\dfrac{1}{2x\log\log x}$.

The first result is easy:

$$\begin{aligned} F(n)-F(n-h) &= nL(n)-(n-h)L(n-h) \\ &= n\big(L(n)-L(n-h)\big)+hL(n-h) \\ &> hL(n-h) > hL(\tfrac{1}{2}n) > \tfrac{1}{2}h\frac{\log n}{\log\log n}. \end{aligned}$$

For the second result, using part of the preceding chain of inequalities, we have

$$\begin{aligned} F(n)-F(n-h)-F(h) &> hL(n-h)-hL(h) \\ &= h\int_h^{n-h} L'(t)\,dt > \tfrac{1}{2}h\int_h^{n-h}\frac{dt}{t\log\log t} > \tfrac{1}{2}\frac{h}{\log\log n}\int_h^{n-h}\frac{dt}{t}, \end{aligned}$$

whence the result.

Proof of Theorem 1. We suppose that $N(m) < AF(m)$ for $m < n$, where A is a suitable large constant, and prove that then this also holds for $n = m$. We take

$$h = \left[n\frac{\log\log n}{\log n}\right]$$

in Lemma 4. It suffices to prove that

$$AF(n-h)+13n < AF(n) \quad \text{and} \quad AF(n-h)+AF(h)+5h < AF(n).$$

By Lemma 5, the former holds if

$$\tfrac{1}{2}hA\frac{\log n}{\log\log n} > 13n$$

and this is so if A is a sufficiently large constant. Also the second inequality holds if

$$\tfrac{1}{2}hA\frac{\log\big((n-h)/h\big)}{\log\log n} > 5h.$$

Now

$$\log\frac{n-h}{h} > \log\frac{n}{2h} > \log\frac{\log n}{3\log\log n} > \tfrac{1}{2}\log\log n.$$

Hence the condition is again satisfied if A is a sufficiently large constant.

3. Proof of Theorem 2. Consider a sequence $\mathscr{A}$ formed from m^2 distinct terms, of length $4m^2$, typified by the following example

$$1,2,3;\ 3,2,1;\ 4,5,6;\ 6,5,4;\ 7,8,9;\ 9,8,7;$$
$$7,4,1;\ 1,4,7;\ 8,5,2;\ 2,5,8;\ 9,6,3;\ 3,6,9.$$

In general

$$\mathscr{A} = (\mathscr{B}_1, \mathscr{C}_1, \ldots, \mathscr{B}_m, \mathscr{C}_m, \mathscr{C}_{m+1}, \mathscr{B}_{m+1}, \ldots, \mathscr{C}_{2m}, \mathscr{B}_{2m}),$$

where

$$\mathscr{B}_k = \big((k-1)m+1, \ldots, km\big), \quad \mathscr{C}_k = \big(km, \ldots, (k-1)m+1\big) \quad (1 \leqslant k \leqslant m);$$
$$\mathscr{B}_k = (k-m, \ldots, k+m^2-2m),$$
$$\mathscr{C}_k = (k+m^2-2m, \ldots, k-m) \quad (m < k \leqslant 2m).$$

$\mathscr{A}$ contains no subsequence a, b, a, b, a. It contains some immediate repetitions, but they will disappear later.

The first appearances of all the integers are in the blocks $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_m$ and their last appearances are in the blocks $\mathscr{B}_{m+1}, \mathscr{B}_{m+2}, \ldots, \mathscr{B}_{2m}$. We shall expand each of these blocks.

For this purpose we use for each block a new set U_k of l integers $u_1^{(k)}, \ldots, u_l^{(k)}$, where $l \geqslant m+1$. Thus there are $2ml$ new integers, and the total number of integers

$$n = m^2+2ml.$$

For each set U_k $(1 \leqslant k \leqslant m)$ we take an admissible sequence $\mathscr{S}_k$ of length $N(l)$ formed from the elements of U_k and arranged so that the last appearance of $u_i^{(k)}$ occurs before the last appearance of $u_j^{(k)}$ for $i < j$. We replace the last appearance of $u_j^{(k)}$ by $u_j^{(k)}$, $(k-1)m+j$, $u_j^{(k)}$, $(k-1)m+j$ for $j = 1, 2, \ldots, m$. Thus if $m = 3$ and $l = 4$ we can take

$$\mathscr{S}_1 = (u_1, u_2, u_1, u_3, u_1, u_3, u_2, u_4, u_2, u_4, u_3, u_4)$$

and this becomes

$$\mathscr{E}_1 = (u_1, u_2, u_1, u_3, u_1, 1, u_1, 1, u_3, u_2, u_4, u_2, 2, u_2,$$
$$2, u_4, u_3, 3, u_3, 3, u_4),$$

where the superscripts over u's are omitted. $\mathscr{E}_1$ replaces the block 1, 2, 3. Note that the last term is now not 3, so the immediate repetition of 3 in

$\mathscr{A}$ disappears and in general the same holds for the repetition of mk $(1 \leqslant k \leqslant m)$.

For each set U_k $(m < k \leqslant 2m)$ we take similarly an admissible sequence $\mathscr{S}_k$ of length $N(l)$ formed from the elements of U_k and arranged so that the first appearance of $u_i^{(k)}$ occurs before the first appearance of $u_j^{(k)}$ for $i < j$. We replace the first appearance of $u_j^{(k)}$ by $(j-2)m+(k-m), u_j^{(k)}, (j-2)m+(k-m), u_j^{(k)}$ for $j = 2, 3, \ldots, m+1$.

The number of terms of the expanded block $\mathscr{E}_k$ is

$$N(l)+3m.$$

Of these terms, m were already present in $\mathscr{B}_k$. So the length of the sequence

$$\mathscr{E}_1, \mathscr{C}_1, \ldots, \mathscr{E}_m, \mathscr{C}_m, \mathscr{C}_{m+1}, \mathscr{E}_{m+1}, \ldots, \mathscr{C}_{2m}, \mathscr{E}_{2m}$$

is $4m^2+2m\big(N(l)+2m\big)$. If the sequence $\mathscr{D}$ obtained from the above by cancelling the central term m^2-m+1 (7 in the example) is admissible, we get

$$N(m^2+2ml) \geqslant 8m^2+2mN(l)-1.$$

Since $N(n_1+n_2) \geqslant N(n_1)+N(n_2)$, $N(n)/n$ tends to a limit (finite or infinite). Choose $l = m+1$. If

$$\lambda = \lim_{n\to\infty}\frac{N(n)}{n} < \infty,$$

then $N(l) > (\lambda-\varepsilon)l$ and

$$\frac{N(m^2+2ml)}{m^2+2ml} \geqslant \frac{8m^2+2ml(\lambda-\varepsilon)-1}{m^2+2ml}$$

for $m > m_0(\varepsilon)$. Making $m \to \infty$ we get

$$\lambda \geqslant \frac{8+2\lambda}{3}, \quad \text{whence} \quad \lambda \geqslant 8.$$

In order to prove that $\mathscr{D}$ is admissible consider two distinct elements a and b. If $a \in U_k$, $b \in U_k$, $\mathscr{D}$ contains no subsequence a, b, a, b, a in view of the same property of $\mathscr{S}_k$. If $a \in U_k$, $b \in U_j$ with $k \neq j$, or $b \leqslant m^2$ and b does not occur in $\mathscr{C}_k$, $\mathscr{D}$ contains no subsequence a, b, a. If $a \in U_k$ and b occurs in $\mathscr{C}_k$, then the maximal subsequence of $\mathscr{D}$ formed from a and b is

$$\begin{array}{lll} a, \ldots, a, b, a, b, b, b, b, b & \text{if} & k \leqslant m, \\ b, b, b, b, b, a, b, a, \ldots, a & \text{if} & k > m, \\ \text{with one } b \text{ missing} & \text{if} & b = m^2-m+1. \end{array}$$

Finally, if $a < b \leqslant m^2$ the maximal subsequence of $\mathscr{D}$ formed from a and b is

$$a, a, a, b, b, b, a, a, a, b, b, b$$
$$\text{if} \quad [-a/m] > [-b/m] \text{ and } \{-a/m\} > \{-b/m\},$$
$$a, a, a, b, b, b, b, a, a, a, b, b$$
$$\text{if} \quad [-a/m] > [-b/m] \text{ and } \{-a/m\} = \{-b/m\},$$
$$a, a, a, b, b, b, b, b, b, a, a, a$$
$$\text{if} \quad [-a/m] > [-b/m] \text{ and } \{-a/m\} < \{-b/m\},$$
$$a, a, b, b, b, a, a, a, a, b, b, b \quad \text{if} \quad [-a/m] = [-b/m]$$

with one letter missing if $a = m^2-m+1$ or $b = m^2-m+1$. None of the above sequences contains either a, b, a, b, a or b, a, b, a, b, which completes the proof.

4. Proof of Theorem 3. We take l pairwise disjoint sets of $m-1$ integers $C_j = \{c_1^{(j)}, \ldots, c_{m-1}^{(j)}\}$, where

$$c_i^{(j)} = (j-1)(m-1)+i \quad (1 \leqslant i < m, 1 \leqslant j \leqslant l),$$

say, and $l+1$ other integers $x_k = l(m-1)+k$ $(1 \leqslant k \leqslant l+1)$. Set

$$\mathscr{A}_1 = (x_1, c_1^{(1)}, x_1, \ldots, c_{m-1}^{(1)}, x_1),$$

$$\mathscr{A}_k = (x_k, c_1^{(k-1)}, c_1^{(k)}, c_1^{(k-1)}, c_1^{(k)}, x_k, c_2^{(k-1)}, c_2^{(k)}, c_2^{(k-1)}, c_2^{(k)}, x_k, \ldots$$
$$\ldots, x_k, c_{m-1}^{(k-1)}, c_{m-1}^{(k)}, c_{m-1}^{(k-1)}, c_{m-1}^{(k)}, x_k) \quad (1 < k \leqslant l),$$

$$\mathscr{A}_{l+1} = (x_{l+1}, c_1^{(l)}, \ldots, x_{l+1}, c_{m-1}^{(l)}, x_{l+1}),$$

$$\mathscr{B}_j = (c_{m-1}^{(j)}, \ldots, c_1^{(j)}) \quad (1 \leqslant j \leqslant l)$$

and form the sequence $\mathscr{S}$

$$\mathscr{A}_1, \mathscr{B}_1, \mathscr{A}_2, \mathscr{B}_2, \ldots, \mathscr{A}_l, \mathscr{B}_l, \mathscr{A}_{l+1}.$$

The number of distinct terms in $\mathscr{S}$ is

$$n = l(m-1)+l+1 = lm+1$$

and the length of $\mathscr{S}$

$$N = \sum_{k=1}^{l+1} L(\mathscr{A}_k) + \sum_{j=1}^{l} L(\mathscr{B}_j)$$
$$= 2(2m-1)+(l-1)(5m-4)+l(m-1) = 6lm-m-5l+2.$$

It remains to prove that $\mathscr{S}$ is admissible. Clearly it contains no immediate repetitions. Consider two elements $a < b$. If $a = x_j$, $b = x_k$

with $j \neq k$, $\mathscr{S}$ contains no subsequence a, b, a. Similarly if $a \in C_j$, $b \in C_k$ or $a \in C_j$, $b = x_k$ with $k \neq j, j+1$. In the remaining cases the maximal subsequence of $\mathscr{S}$ formed from a and b is

$$b, \ldots, b, a, a, b, \ldots, b, a, a, a \quad \text{if} \quad a \in C_j,\ b = x_j,$$
$$a, a, a, b, \ldots, b, a, a, b, \ldots, b \quad \text{if} \quad a \in C_j,\ b = x_{j+1}$$

with the first or the last a missing in both cases if $j = 1$ or l, respectively,

$$a, a, b, b, b, a, a, a, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j)},$$
$$a, a, a, b, b, a, a, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i < h,$$
$$a, a, a, a, b, a, b, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i = h,$$
$$a, a, a, a, a, b, b, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i > h$$

with the first or the last term missing in all four cases if $j = 1$ or l, respectively.

None of the above sequences contains either a, b, a, b, a or b, a, b, a, b, which completes the proof.

Proof of Corollary. Taking $l = 1$, $m = n-1$ we get

$$N(n) \geqslant 5n-8. \tag{$\ddagger$}$$

Taking $l = 2$ we get

$$N(2m+1) \geqslant 11m-8.$$

For $m \geqslant 6$ we have $11m-8 > 5(2m+1)-8$, hence the equality sign is excluded in ($\ddagger$) for all odd $n \geqslant 13$. For even $n \geqslant 18$ we use the inequality $N(n) \geqslant N(n-1)+3$ and obtain

$$N(n) \geqslant N(n-1)+3 \geqslant 11(\tfrac{1}{2}n-1)-5 > 5n-8.$$

Note added in proof. Mr. Z. Kolba has recently shown that $N(2m) \geqslant 11m-13$, thus $N(n) > 5n-8$ for all $n \geqslant 12$.

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Density inequalities for a restricted sum of sets of lattice points

by

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§ 1. Introduction. Let Q be the set of all 2-dimensional lattice points (x, y) such that x and y are nonnegative integers and either x or y is positive. Addition and subtraction of elements of Q will be done componentwise.

Let a set B of positive integers be a basis of order k for the positive integers. Then clearly any subset of Q containing all points $(b, 0)$ and $(0, b)$ with $b \in B$ is a basis for Q, and is of order no more than $2k$, if addition of subsets of Q is done as in [2]. For example, Schnirelmann has shown ([3], p. 680) that the set consisting of 1 and all positive primes $4t+3$ is a basis for the positive integers. Therefore the set P consisting of $1, i$, and the Gaussian primes $p+qi$ where $(p, q) \in Q$ is a basis for the set of all Gaussian integers $a+bi$ where $(a, b) \in Q$.

However, it might be of interest to know whether these Gaussian integers can be written as sums of elements of P in some less trivial way than as sums of elements on the axes. More specifically, we might ask which subsets A of Q have the property that each point (x, y) of Q can be written as a sum of no more than k elements of A, and in such a way that no two of its summands are on different axes. This question leads us to make the following definition of sums of sets in Q. These restricted sum sets are not only smaller than the sum sets used in [1] and [2], but this addition of sets is not, in general, associative. In particular, we cannot assume $kA+A = (k+1)A$.

§ 2. Definitions and notation. For any k subsets $A_1, \ldots, A_k$ of Q let $A_1+\ldots+A_k$ be the set of all $a_1+\ldots+a_k$ in Q such that (1) each $a_i \in A_i \cup \{(0, 0)\}$, and (2) if two of the summands have the forms $a_i = (a, 0)$ and $a_j = (0, b)$ then one of them is $(0, 0)$. If $A_1 = \ldots = A_k = A$ we write kA instead of $A+\ldots+A$.

For any p and q in Q, $p < q$ if and only if $q-p \in Q$. Let $Lq = \{p \in Q\colon p \leqslant q\}$. We will also use the definitions and notation of [2], except that sums of sets are as defined above. For any subset A of Q the density of A, as defined in [2], will be denoted by $d(A)$.