with $j \neq k$, $\mathscr{S}$ contains no subsequence $a, b, a$. Similarly if $a \in C_j$, $b \in C_k$ or $a \in C_j$, $b = x_k$ with $k \neq j, j+1$. In the remaining cases the maximal subsequence of $\mathscr{S}$ formed from $a$ and $b$ is

$$b, \ldots, b, a, a, b, \ldots, b, a, a, a \quad \text{if} \quad a \in C_j,\ b = x_j,$$
$$a, a, a, b, \ldots, b, a, a, b, \ldots, b \quad \text{if} \quad a \in C_j,\ b = x_{j+1}$$

with the first or the last $a$ missing in both cases if $j = 1$ or $l$, respectively,

$$a, a, b, b, b, a, a, a, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j)},$$
$$a, a, a, b, b, a, a, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i < h,$$
$$a, a, a, a, b, a, b, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i = h,$$
$$a, a, a, a, a, b, b, b, b, b \quad \text{if} \quad a = c_h^{(j)},\ b = c_i^{(j+1)},\ i > h$$

with the first or the last term missing in all four cases if $j = 1$ or $l$, respectively.

None of the above sequences contains either $a, b, a, b, a$ or $b, a, b, a, b$, which completes the proof.

Proof of Corollary. Taking $l = 1$, $m = n-1$ we get

$$N(n) \geqslant 5n-8. \tag{$*$}$$

Taking $l = 2$ we get

$$N(2m+1) \geqslant 11m-8.$$

For $m \geqslant 6$ we have $11m-8 > 5(2m+1)-8$, hence the equality sign is excluded in ($*$) for all odd $n \geqslant 13$. For even $n \geqslant 18$ we use the inequality $N(n) \geqslant N(n-1)+3$ and obtain

$$N(n) \geqslant N(n-1)+3 \geqslant 11(\tfrac{1}{2}n-1)-5 > 5n-8.$$

Note added in proof. Mr. Z. Kolba has recently shown that $N(2m) \geqslant 11m-13$, thus $N(n) > 5n-8$ for all $n \geqslant 12$.

### References


[1] H. Davenport and A. Schinzel, *A combinatorial problem connected with differential equations*, Amer. J. Math. 87 (1965), pp. 684–694.

[2] D. P. Roselle and R. G. Stanton, *Some properties of Davenport–Schinzel sequences*, Acta Arith., this volume, pp. 355–362.


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# Density inequalities for a restricted sum of sets of lattice points

by

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**§ 1. Introduction.** Let $Q$ be the set of all 2-dimensional lattice points $(x, y)$ such that $x$ and $y$ are nonnegative integers and either $x$ or $y$ is positive. Addition and subtraction of elements of $Q$ will be done componentwise.

Let a set $B$ of positive integers be a basis of order $k$ for the positive integers. Then clearly any subset of $Q$ containing all points $(b, 0)$ and $(0, b)$ with $b \in B$ is a basis for $Q$, and is of order no more than $2k$, if addition of subsets of $Q$ is done as in [2]. For example, Schnirelmann has shown ([3], p. 680) that the set consisting of 1 and all positive primes $4t+3$ is a basis for the positive integers. Therefore the set $P$ consisting of $1, i$, and the Gaussian primes $p+qi$ where $(p, q) \in Q$ is a basis for the set of all Gaussian integers $a+bi$ where $(a, b) \in Q$.

However, it might be of interest to know whether these Gaussian integers can be written as sums of elements of $P$ in some less trivial way than as sums of elements on the axes. More specifically, we might ask which subsets $A$ of $Q$ have the property that each point $(x, y)$ of $Q$ can be written as a sum of no more than $k$ elements of $A$, and in such a way that no two of its summands are on different axes. This question leads us to make the following definition of sums of sets in $Q$. These restricted sum sets are not only smaller than the sum sets used in [1] and [2], but this addition of sets is not, in general, associative. In particular, we cannot assume $kA+A = (k+1)A$.

**§ 2. Definitions and notation.** For any $k$ subsets $A_1, \ldots, A_k$ of $Q$ let $A_1+\ldots+A_k$ be the set of all $a_1+\ldots+a_k$ in $Q$ such that (1) each $a_i \in A_i \cup \{(0, 0)\}$, and (2) if two of the summands have the forms $a_i = (a, 0)$ and $a_j = (0, b)$ then one of them is $(0, 0)$. If $A_1 = \ldots = A_k = A$ we write $kA$ instead of $A+\ldots+A$.

For any $p$ and $q$ in $Q$, $p < q$ if and only if $q-p \in Q$. Let $Lq = \{p \in Q\colon p \leqslant q\}$. We will also use the definitions and notation of [2], except that sums of sets are as defined above. For any subset $A$ of $Q$ the density of $A$, as defined in [2], will be denoted by $d(A)$.

**§ 3. Theorems and their corollaries.** In the following Theorem 1 and Corollaries 1, 2, 3, we let $A$ and $B$ denote subsets of $Q$ with densities $\alpha$ and $\beta$, respectively, and let $C = A+B$.

THEOREM 1. *If $g \epsilon Q-C$ and if $\alpha+\beta > 1$ then*

$$Q(\mathrm{L}g) \leqslant \frac{1}{\alpha+\beta-1}.$$

Proof. Suppose $g = (g_1, g_2) \epsilon Q-C$. If $a \epsilon A \cap \mathrm{L}g$ then $g-a \epsilon \mathrm{L}g-B$, or $a = (g_1, 0) \epsilon \mathrm{L}g$, or $a = (0, g_2) \epsilon \mathrm{L}g$. Also $g \notin B$, and $g-a \neq g$ for any $a \epsilon A \cap \mathrm{L}g$. Therefore

$$B(\mathrm{L}g) \leqslant Q(\mathrm{L}g)-A(\mathrm{L}g)-1+2,$$

or

$$A(\mathrm{L}g)+B(\mathrm{L}g) \leqslant Q(\mathrm{L}g)+1.$$

This implies

$$\alpha+\beta \leqslant \frac{A(\mathrm{L}g)+B(\mathrm{L}g)}{Q(\mathrm{L}g)} \leqslant 1+\frac{1}{Q(\mathrm{L}g)},$$

and, since $\alpha+\beta > 1$,

$$Q(\mathrm{L}g) \leqslant \frac{1}{\alpha+\beta-1}.$$

COROLLARY 1. *If $\alpha+\beta > 1$ then $Q-C$ is finite.*

Proof. If $g = (g_1, g_2) \epsilon Q-C$, then

$$Q(\mathrm{L}g) = (g_1+1)(g_2+1)-1 \leqslant \frac{1}{\alpha+\beta-1},$$

and $g_1+1$ and $g_2+1$ are positive integers.

COROLLARY 2. *If $\alpha+\beta > 4/3$ then $Q-C = \emptyset$.*

Proof. Suppose $g = (g_1, g_2) \epsilon Q-C$. Then

$$1 \leqslant Q(\mathrm{L}g) \leqslant \frac{1}{\alpha+\beta-1} < 3,$$

so $Q(\mathrm{L}g)$ is either 1 or 2. Thus $g \epsilon \{(1,0), (2,0), (0,1), (0,2)\}$. But $\alpha+\beta > 4/3$ and $\alpha \leqslant 1, \beta \leqslant 1$ imply that neither $\alpha$ nor $\beta$ is 0. Therefore $(1,0)$ and $(0,1)$ are in $A \cap B$, and $\{(1,0), (2,0), (0,1), (0,2)\} \subseteq C$.

As an alternative proof for Corollary 2, it might be noted that the 1-dimensional result obtained by Schnirelmann ([3], p. 654): "If $\alpha+\beta > 1$ then $d(C) = 1$" implies that $g_1 \neq 0$ and $g_2 \neq 0$. Hence $(1,1) \leqslant g$ and $Q(\mathrm{L}g) \geqslant 3$, which is a contradiction.

COROLLARY 3. *If $\alpha+r\beta > 1$, $0 < r \leqslant 1/2$, then $Q-C = \emptyset$.*

Proof. Suppose $g = (g_1, g_2) \epsilon Q-C$. Then $g \epsilon Q-A$ and

$$1-r\beta < \alpha \leqslant \frac{A(\mathrm{L}g)}{Q(\mathrm{L}g)} \leqslant \frac{Q(\mathrm{L}g)-1}{Q(\mathrm{L}g)} = 1-\frac{1}{Q(\mathrm{L}g)}.$$



Therefore, $Q(\mathrm{L}g) > 1/r\beta$. Thus, from Theorem 1, $\dfrac{1}{\alpha+\beta-1} > \dfrac{1}{r\beta}$, or $\alpha+(1-r)\beta < 1 < \alpha+r\beta$. This implies $1-r < r$, or $r > 1/2$, which is a contradiction.

THEOREM 2. *Let $A_1, \ldots, A_k$ be subsets of $Q$ with densities $\alpha_1, \ldots, \alpha_k$, respectively. If $(1,1) \epsilon A_i$ for all $i = 1, \ldots, k-1$ then*

$$1-d(A_1+\ldots+A_k) \leqslant (1-\alpha_1)\ldots(1-\alpha_k).$$

Proof. If $k = 1$ the result is trivial. Assume $1-d(A_2+\ldots+A_k) \leqslant (1-\alpha_2)\ldots(1-\alpha_k)$ for some $k \geqslant 2$. Let $A = A_1$, let $B = A_2+\ldots+A_k$, and let $C = A_1+A_2+\ldots+A_k$. Let the densities of $A, B, C$ be $\alpha = \alpha_1$, $\beta$, and $\gamma$, respectively. We will first show

$$\gamma \geqslant \alpha+\beta-\alpha\beta. \tag{1}$$

The proof of (1) is similar to the proof of Theorem 2 of [2], with a modification necessitated by the fact that $C$ may be a proper subset of $A+B$. The entire proof will be presented here for the sake of completeness.

If $(1,0) \notin A$ or if $(0,1) \notin A$ then $\alpha = 0$, and (1) holds since $B \subseteq C$. Hence we suppose $(1,0), (0,1)$, and $(1,1)$ are all in $A$. Let $R$ be any fundamental set in $Q$. If $C(R) = Q(R)$ then $C(R)/Q(R) \geqslant \alpha+\beta-\alpha\beta$, since $(1-\alpha)(1-\beta) \geqslant 0$ implies $1 \geqslant \alpha+\beta-\alpha\beta$. It must be shown that $C(R)/Q(R) \geqslant \alpha+\beta-\alpha\beta$ for every fundamental set $R$ in $Q$; therefore we assume that $C(R) < Q(R)$ and, consequently, $A(R) < Q(R)$.

Let $H = R-A$, let $H_1$ be the set of all $(x, 0)$ in $H$, let $H_2$ be the set of all $(0, y)$ in $H$, and let $H_3$ be the remaining points of $H$. We will show that there exist lattice points $a^{(1)}, \ldots, a^{(s)}$ in $A$ and sets $L_1, \ldots, L_s$ with the following properties.

(i) For each $i = 1, \ldots, s$, $L_i \neq \emptyset$ and $L_i$ is a subset of one of the sets $H_1, H_2$, or $H_3$.

(ii) The sets $L_i' = \{x-a^{(i)}\colon x \epsilon L_i\}$ are fundamental sets.

(iii) $L_i \cap L_j = \emptyset$ for $i \neq j$.

(iv) $H = L_1 \cup \ldots \cup L_s$.

For every $h$ in $H_1$ let $A_h = \{a \epsilon A\colon (1,0) \leqslant a < h\}$, for every $h$ in $H_2$ let $A_h = \{a \epsilon A\colon (0,1) \leqslant a < h\}$, and for every $h$ in $H_3$ let $A_h = \{a \epsilon A\colon (1,1) \leqslant a < h\}$. The sets $A_h$ are finite and nonempty, hence each $A_h$ contains at least one lattice point which is maximal with respect to the ordering $\leqslant$. Let "the" largest element in each $A_h$ be the maximal element of $A_h$ having the largest first component, and let $a^{(1)}, \ldots, a^{(s)}$ be all the distinct lattice points that are largest elements in any $A_h$. Let

$$L_i = \{h \epsilon H\colon a^{(i)} \text{ is the largest element in } A_h\}, \quad i = 1, \ldots, s.$$

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the $L_i$. To show that (ii) is satisfied, we suppose that $x \in L_i$, $y \in Q$, and $a^{(i)} < y \leqslant x$. If $y \in A$ then $y \in A_x$, which contradicts the maximality of $a^{(i)}$. Thus $y \in H$, and if $x \in H_1$ $(H_2, H_3)$ then also $y \in H_1$ $(H_2, H_3)$. If $y \in L_k$ then $a^{(k)} < y \leqslant x$, $a^{(k)} \in A_x$, $a^{(k)} = a^{(i)}$, and $k = i$. Thus for any $x$ in $L_i$, $z \in Q$ and

$$z \leqslant x - a^{(i)} \in L_i' \Rightarrow a^{(i)} < a^{(i)} + z \leqslant x \Rightarrow a^{(i)} + z \in L_i \Rightarrow z \in L_i'.$$

If $b \in B \cap L_i'$ then $b$ must satisfy the conditions in the definition of a sum set for the sets $A_2, \ldots, A_k$: $b = a_2 + \ldots + a_k$, etc. Either $a^{(i)}$ has no 0 component, in which case $a^{(i)} + b \in C \cap L_i \subseteq C - A$, or one (and only one) component of $a^{(i)}$ is 0, in which case that same component is 0 in $b$, hence that same component is 0 in each of $a_2, \ldots, a_k$, hence $a^{(i)} + b \in C \cap L_i \subseteq C - A$. Therefore,

$$\begin{aligned} C(R) &\geqslant A(R) + B(L_1') + \ldots + B(L_s') \\ &\geqslant A(R) + \beta[Q(L_1') + \ldots + Q(L_s')] \\ &= A(R) + \beta[Q(L_1) + \ldots + Q(L_s)] \\ &= A(R) + \beta Q(H) \\ &= A(R) + \beta[Q(R) - A(R)] \\ &= (1-\beta)A(R) + \beta Q(R) \\ &\geqslant (1-\beta)\alpha Q(R) + \beta Q(R), \end{aligned}$$

and

$$C(R)/Q(R) \geqslant \alpha + \beta - \alpha\beta$$

for every fundamental set $R$ in $Q$. This completes the proof of (1).

Now $\gamma \geqslant \alpha + \beta - \alpha\beta$ implies

$$1 - \gamma \leqslant (1-\alpha)(1-\beta),$$

or

$$1 - d(A_1 + \ldots + A_k) \leqslant (1-\alpha_1)(1-\alpha_2) \ldots (1-\alpha_k).$$

COROLLARY 4. *If $A$ is any subset of $Q$ such that $(1,1) \in A$ and $\alpha = d(A) > 0$ then there exists a positive integer $m$ such that $mA + A = Q$.*

Proof. We have $0 < \alpha \leqslant 1$. Thus for any $\varepsilon > 0$ there exists a positive integer $m$ such that $(1-\alpha)^m < \varepsilon$. Theorem 2 then implies that $1 - d(mA) \leqslant (1-\alpha)^m < \varepsilon$, or $d(mA) > 1 - \varepsilon$. Choose $\varepsilon = \alpha/2$. Now $\alpha/2 + d(mA) > 1$, and $mA + A = Q$ from Corollary 3.

It will be seen that Corollary 4 also follows directly from Theorem 3. The integer $m$ obtained in the proof given for Corollary 4 can be expected to be smaller in some cases than the $m$ obtained in the proof of Theorem 3.

THEOREM 3. *If $A$ is any subset of $Q$ such that $\alpha = d(A) > 0$ and $(1,1) \in A$ then there exists a positive integer $m$ such that $mA = Q$.*



Proof. As in the proof of Corollary 4, there exists an integer $h_1$ such that $d(h_1 A) > 1 - \alpha$.

Let $Q_1 = \{(x, 0) \in Q\}$, $Q_2 = \{(0, y) \in Q\}$. Since $\alpha > 0$, the one-dimensional densities of $A \cap Q_1$ and $A \cap Q_2$ are positive. Hence, from a corollary to the one-dimensional Schnirelmann result quoted previously, there exists an integer $h_2$ such that $h_2 A \supseteq Q_1 \cup Q_2$.

Let $h$ be the larger of $h_1$ and $h_2$, and let $B = hA$, $\beta = d(hA)$. We have $\alpha + \beta > \alpha + (1-\alpha) = 1$. Suppose $g = (g_1, g_2) \in Q - (h+1)A$. Then $g \notin hA$. For any $a \in A \cap \mathrm{L}g$, either $g - a \notin B$ or $a \in Q_1$ or $a \in Q_2$. Hence

$$\begin{aligned} B(\mathrm{L}g) &\leqslant Q(\mathrm{L}g) - A(\mathrm{L}g) - 1 + A(\mathrm{L}g \cap Q_1) + A(\mathrm{L}g \cap Q_2) \\ &\leqslant Q(\mathrm{L}g) - A(\mathrm{L}g) - 1 + g_1 + g_2. \end{aligned}$$

This yields

$$A(\mathrm{L}g) + B(\mathrm{L}g) \leqslant Q(\mathrm{L}g) + g_1 + g_2 - 1,$$

$$\alpha + \beta \leqslant \frac{A(\mathrm{L}g) + B(\mathrm{L}g)}{Q(\mathrm{L}g)} \leqslant 1 + \frac{g_1 + g_2 - 1}{Q(\mathrm{L}g)},$$

$$\frac{Q(\mathrm{L}g)}{g_1 + g_2 - 1} \leqslant \frac{1}{\alpha + \beta - 1},$$

and

$$\frac{(g_1+1)(g_2+1) - 1}{g_1 + g_2 - 1} \leqslant \frac{1}{\alpha + \beta - 1}.$$

If for every real number $K > 0$ there exists $(g_1, g_2) \in Q - (h+1)A$ such that $g_1 > K$ and $g_2 > K$ then $\dfrac{(g_1+1)(g_2+1)-1}{g_1+g_2-1}$ can be made arbitrarily large, hence is greater than $1/(\alpha+\beta-1)$ for some $(g_1, g_2)$ in $Q - (h+1)A$, which is a contradiction.

Thus there exists a positive real number $K_0$ such that $(g_1, g_2) \in Q - (h+1)A$ implies either $g_1 \leqslant K_0$ or $g_2 \leqslant K_0$. We may suppose that $K_0$ is an integer. It will be recalled that $Q_1 \cup Q_2 \subseteq hA$ and $(1,1) \in A$. Therefore $\{(p, q) \in Q\colon \text{ either } 0 \leqslant p \leqslant K_0 \text{ or } 0 \leqslant q \leqslant K_0\} \subseteq (h + K_0)A$, and we may let $m = h + K_0$.

### References


[1] L. P. Cheo, *On the density of sets of Gaussian integers*, Amer. Math. Monthly 58 (1951), pp. 618–620.

[2] B. Kvarda, *On densities of sets of lattice points*, Pacific J. Math. 13 (1963), pp. 611–615.

[3] L. Schnirelmann, *Über additive Eigenschaften von Zahlen*, Math. Ann. 107 (1933), pp. 649–690.


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