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Interpolation series for continuous functions on π-adic completions of $\mathrm{GF}(q, x)$*

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1. Introduction. In 1944 Dieudonné [7] proved an analogue of the Weierstrass Approximation Theorem for continuous functions of a p-adic variable. In 1958 Mahler [8] sharpened this result by exhibiting a series expansion for continuous functions defined on the p-adic integers. He showed that every such function f is the uniform limit of an interpolation series

$$f(t) = \sum_{n=0}^{\infty} A_n \binom{t}{n} \tag{1.1}$$

where the coefficients A_n are uniquely determined by

$$A_n = \Delta^n f(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k). \tag{1.2}$$

In the present paper we choose an irreducible element π from the polynomial ring $\mathrm{GF}[q, x]$ over the finite field $\mathrm{GF}(q)$ and use it to equip the function field $\mathrm{GF}(q, x)$ with a π-adic absolute value. We denote by F_π the completion of $\mathrm{GF}(q, x)$ for this absolute value and by I_π the valuation ring of F_π. The aforementioned theorem of Dieudonné may easily be seen to generalize to the case of a locally compact non-archimedean field. Hence, every continuous function $f\colon K \to F_\pi$, where K is a compact subset of F_π, is the uniform limit of some sequence of polynomials over F_π. Our aim in this paper is to prove some Mahler type theorems for such functions.

We mention that Amice [1] has already constructed a certain type of series approximation for continuous functions defined on locally compact non-archimedean fields. In the process, Amice characterized those sequences ("suites très bien réparties") in the domain of a continuous function with respect to which a Newton type interpolation procedure will yield

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a uniformly convergent series approximation for that function. In particular, the nonnegative rational integers, ordered in the usual way, constitute such a sequence in the p-adic integers, and so Mahler's result appears as a special case of Amice's Interpolation Theorem [1].

In what follows, we exhibit a "suite très bien répartie" in I_π, denoted $\{m_i\}$, consisting of a special sequential ordering of $GF[q, x]$. Specializing Amice, we prove (Theorem 4.4) that for every continuous function $f\colon I_\pi \to F_\pi$ there exists a unique sequence $\{A_i\}$ in F_π such that

$$f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \tag{1.3}$$

where $Q_i(t)$ is the ith Newton interpolation polynomial for the interpolation sequence $\{m_i\}$, and (1.3) converges uniformly on I_π. We add that $\{A_i\}$ is always a null sequence, i.e., $\lim_{i\to\infty} A_i = 0$. Moreover, the above result may be extended to continuous functions $f\colon K \to F_\pi$, where K is any compact subset of F_π, by employing a Urysohn type theorem for totally disconnected spaces due to Dieudonné [7].

We may regard the foregoing approach to constructing function field analogues of Mahler's result as deriving from the observation that the polynomials $\binom{t}{n}$ are the Newton interpolation polynomials for the nonnegative rational integers. From this standpoint, the crucial problem, completely solved by Amice, is that of identifying those sequences in I_π for which the associated Newton polynomials yield interpolation series for continuous functions.

If, instead, one regards the sequence $\left\{\binom{t}{n}\right\}$ merely as an ordered basis of the Q_p-vector space $Q_p[t]$, then one is led to ask which ordered bases of the F_π-vector space $F_\pi[t]$ yield interpolation series for continuous functions on I_π. In this connection, it is of interest to recall that the sequence $\left\{\binom{t}{n}\right\}$ has the further property of being an ordered basis of the Z-module of polynomials over Q which map Z into Z (and also of the Z_p-module of polynomials over Q_p that map Z_p into Z_p, where Z_p is the valuation ring of Q_p).

The function field analogue of the latter property is that of being an ordered basis of the I_π-module of polynomials over F_π that map I_π into I_π. Let $\{H_i(t)\}$ be such a basis. We prove (Theorem 4.5) that for every continuous function $f\colon I_\pi \to I_\pi$ there exists a unique null sequence $\{B_i\}$ in I_π such that

$$f(t) = \sum_{i=0}^{\infty} B_i H_i(t), \tag{1.4}$$

where (1.4) converges uniformly on I_π.

The above theorem may be applied to a sequence of polynomials $\{G_i(t)/g_i\}$ introduced in 1948 by Carlitz [4]. This leads to the following characterization (Theorem 5.1) of continuous linear operators on the $GF(q)$-vector space I_π: Let $f\colon I_\pi \to I_\pi$ be continuous. If the (unique) interpolation series for f constructed from the Carlitz polynomials is given by

$$f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i}, \tag{1.5}$$

then f is a linear operator on the $GF(q)$-vector space I_π if and only if $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$.

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2. Preliminaries. Let $GF(q)$ be a finite field of cardinality q. Denote by $GF[q, x]$ the ring of polynomials in an indeterminate x over $GF(q)$, and by $GF(q, x)$ the quotient field of $GF[q, x]$. Let $\pi \in GF[q, x]$ be an irreducible polynomial of degree d. Then every nonzero $a \in GF(q, x)$ may be written, in essentially unique fashion,

$$a = \pi^n \frac{m_1}{m_2}, \tag{2.1}$$

where n is integral, and m_1 and m_2 are polynomials prime to each other and to π.

Define a function $v_\pi\colon GF(q, x) - \{0\} \to Z$ by

$$v_\pi(a) = n, \tag{2.2}$$

where a is written as in (2.1). It follows that

$$v_\pi(a\beta) = v_\pi(a) + v_\pi(\beta) \quad (a\beta \neq 0) \tag{2.3}$$

and

$$v_\pi(a+\beta) \geqslant \min\{v_\pi(a), v_\pi(\beta)\} \quad (a, \beta, a+\beta \neq 0). \tag{2.4}$$

Fixing a real number b such that $0 < b < 1$, define the π-adic absolute value $|\ |_\pi$ on $GF(q, x)$ as follows:

$$|0|_\pi = 0, \tag{2.5}$$

$$|a|_\pi = b^{v_\pi(a)} \quad (a \neq 0). \tag{2.6}$$

By familiar methods $GF(q, x)$ may be embedded as a dense subfield in an essentially unique complete field, denoted F_π. With respect to the extended absolute value, F_π is a discrete non-archimedean field. Equipped with the metric d_π, defined by

$$d_\pi(a, \beta) = |a-\beta|_\pi, \tag{2.7}$$

F_π is a metric field. In particular, polynomial functions over F_π are continuous.

Denote by I_π the valuation ring of F_π, i.e.,

$$I_\pi = \{a \in F_\pi\colon\ |a|_\pi \leqslant 1\}.$$

Then the valuation ideal

$$(\pi) = \{a \in I_\pi\colon\ |a|_\pi < 1\}$$

is maximal and the residue class field $I_\pi/(\pi)$ is isomorphic to $\mathrm{GF}(q^d)$, where $d = \deg\pi$.

Let Γ be a complete set of representatives of $I_\pi/(\pi)$ in I_π. Then every nonzero $a \in F_\pi$ may be uniquely represented as a π-series,

$$a = \pi^n \sum_{i=0}^{\infty} a_i \pi^i, \tag{2.8}$$

where $a_i \in \Gamma$, $\pi \nmid a_0$ in I_π, and $|a| = b^n$ [6]. In particular, Γ may be taken to be the set of polynomials in $\mathrm{GF}[q, x]$ having degree less than d.

For $a \in F_\pi$ and k any integer, let

$$B_k(a) = \{\beta \in F_\pi\colon\ |\beta - a|_\pi \leqslant b^k\} = \{\beta \in F_\pi\colon\ |\beta - a|_\pi < b^{k-1}\}. \tag{2.9}$$

Then the collection $\{B_k(a)\colon k \geqslant 0\}$ is a fundamental system of open-closed neighborhoods of a; hence F_π is totally disconnected.

Again, let Γ be a complete set of representatives of $I_\pi/(\pi)$ in I_π. Given $\varepsilon > 0$, let k be a positive integer such that $b^k < \varepsilon$. Let

$$\Delta = \{a \in I_\pi\colon\ a = a_0 + a_1\pi + \ldots + a_{k-1}\pi^{k-1}\} \tag{2.10}$$

where $a_i \in \Gamma$. Then Δ has q^{kd} elements and the collection

$$\{B_k(a)\colon\ a \in \Delta\} \tag{2.11}$$

is a pairwise disjoint open cover of I_π, all of the members of which have radius less than ε. It follows that I_π (and, therefore, every closed and bounded subset of F_π) is compact. (In fact, the Heine–Borel Theorem holds in all locally compact non-archimedean fields, a result due to Schöbe [9].)

In the special case $\pi = x$, the complete field F_x may be identified with the field of formal power series over $\mathrm{GF}(q)$, for by (2.8) every nonzero $a \in F_x$ may be written

$$a = \sum_{i=-\infty}^{\infty} a_i x^i, \tag{2.12}$$

where $a_i \in \mathrm{GF}(q)$, all but a finite number of the a_i vanish for $i < 0$, and $|a|_x = b^n$, for n the smallest integer such that $a_n \neq 0$.

There would, in fact, be no loss of generality in restricting the investigation we have in mind to the case of x-adic absolute values; for it is known that every locally compact Hausdorff field having nonzero characteristic is topologically isomorphic to a field of formal power series in one indeterminate over some finite field ([10], pp. 12–22). In the case of the fields F_π we may specialize this result as follows.

THEOREM 2.1. *Let F_π be the completion of* $\mathrm{GF}(q, x)$ *for the absolute value* $|\ |_\pi$, *where π is an irreducible polynomial of degree d. Then F_π is topologically isomorphic to a field of formal power series in one indeterminate over the finite field* $\mathrm{GF}(q^d)$.

Proof. In view of representations (2.8) and (2.12), it suffices to show that Γ, a complete set of representatives of $I_\pi/(\pi)$ in I_π, may be chosen in such a way that Γ is a *subfield* of I_π.

Let $a \in I_\pi$. Since $I_\pi/(\pi)$ is isomorphic to $\mathrm{GF}(q^d)$, it follows that $\pi \mid a^{q^d} - a$, and hence that

$$\pi^{q^{(n-1)d}} \mid a^{q^{nd}} - a^{q^{(n-1)d}},$$

for all natural numbers n. Therefore, the series

$$a + (a^{q^d} - a) + (a^{q^{2d}} - a^{q^d}) + \ldots \tag{2.13}$$

converges, i.e., $\lim_{n\to\infty} a^{q^{nd}}$ exists for all $a \in I_\pi$.

Define a function $w\colon I_\pi \to I_\pi$ by

$$w(a) = \lim_{n\to\infty} a^{q^{nd}}. \tag{2.14}$$

Then w is an endomorphism of the ring I_π with kernel (π), and so $w(I_\pi)$ is a subfield of I_π isomorphic to $\mathrm{GF}(q^d)$. By (2.13) and (2.14), it follows that $w(a) \equiv a \pmod{\pi}$; hence we may take $\Gamma = w(I_\pi)$, as desired.

To conclude this section, we recall that, in addition to the π-adic absolute values, $\mathrm{GF}(q, x)$ admits only one other non-trivial absolute value, $|\ |_\infty$, defined by

$$\left|\frac{m_1}{m_2}\right|_\infty = b^{\deg m_2 - \deg m_1}, \tag{2.15}$$

for m_1, m_2 nonzero elements of $\mathrm{GF}[q, x]$ and $0 < b < 1$ ([6], pp. 45–47). The completion of $\mathrm{GF}(q, x)$ for $|\ |_\infty$, denoted by F_∞, may be seen to consists of the set of all descending formal power series over $\mathrm{GF}(q)$,

$$a = \sum_{i=-\infty}^{\infty} a_i x^{-i}, \tag{2.16}$$

where $a_i \in \mathrm{GF}(q)$, all but a finite number of these coefficients vanish for $i < 0$, and $|a|_\infty = b^n$, n the smallest integer such that $a_n \neq 0$.

In what follows, we shall appeal to the obvious topological isomorphism between F_x and F_∞ to omit an explicit treatment of the problem of approximating continuous functions in F_∞. There appears, however, to be no particular advantage in a similar appeal to Theorem 2.1, and so we shall state our results for the fields F_π.

3. A special ordering of $\mathrm{GF}[q, x]$. Let $\pi \in \mathrm{GF}[q, x]$ be an irreducible polynomial of degree d. We define a sequential ordering of $\mathrm{GF}[q, x]$ which has the property of being, in the terminology of Amice [1], "très bien répartie" in I_π. Let $(a_0, a_1, \ldots, a_{q^d-1})$ be a fixed ordering of the polynomials in $\mathrm{GF}[q, x]$ of degree $< d$ such that $a_0 = 0$, $a_1 = 1$, and $\deg a_i \leqslant \deg a_j$ for $1 \leqslant i \leqslant j$. The special sequence $\{m_n\}$, running through $\mathrm{GF}[q, x]$, is defined as follows. If

(3.1) $$n = k_0 + k_1 q^d + \ldots + k_s q^{sd} \quad (0 \leqslant k_i < q^d),$$

set

(3.2) $$m_n = a_{k_0} + a_{k_1}\pi + \ldots + a_{k_s}\pi^s.$$

THEOREM 3.1. *For any integers $s \geqslant 0$ and $k \geqslant 1$, the set*

(3.3) $$\{m_{i+sq^{kd}}:\ 0 \leqslant i < q^{kd}\}$$

is a complete residue system $(\mathrm{mod}\,\pi^k)$.

Proof. As there is no "overlap" in the q^d-adic expansions (3.1) of i and sq^{kd}, it follows that

(3.4) $$m_{i+sq^{kd}} = m_i + m_{sq^{kd}}.$$

The set $\{m_i:\ 0 \leqslant i < q^{kd}\}$ is a complete residue system $(\mathrm{mod}\,\pi^k)$, and this property is preserved under shifting by the additive constant $m_{sq^{kd}}$.

Let

(3.5) $$S_n = \{m_0, m_1, \ldots, m_{n-1}\} \quad (n \geqslant 1),$$

and let

(3.6) $$\varrho(a; k, n) = \mathrm{card}\big(B_k(a) \cap S_n\big),$$

with $a \in I_\pi$, $n, k \geqslant 1$, and $B_k(a)$ as in (2.9). Then the following theorem is a straightforward consequence of Theorem 3.1.

THEOREM 3.2. *For every $a \in I_\pi$, and for all positive integers n and k,*

(3.7) $$\left[\frac{n}{q^{kd}}\right] \leqslant \varrho(a; k, n) \leqslant \left[\frac{n-1}{q^{kd}}\right] + 1.$$

Furthermore,

(3.8) $$\varrho(m_n; k, n) = \left[\frac{n}{q^{kd}}\right].$$

We now introduce an ordered basis of the F_π-vector space $F_\pi[t]$, consisting of the Newton interpolation polynomials for the interpolation sequence $\{m_n\}$, defined by (3.2). Set

(3.9) $$P_0(t) = 1, \quad P_n(t) = (t-m_0)(t-m_1)\ldots(t-m_{n-1}) \quad (n \geqslant 1),$$

and

(3.10) $$Q_0(t) = 1, \quad Q_n(t) = P_n(t)/P_n(m_n) \quad (n \geqslant 1).$$

Since $\deg Q_n(t) = n$, $\{Q_n(t)\}$ is an ordered basis of the F_π-vector space $F_\pi[t]$. Hence, every polynomial $g(t) \in F_\pi[t]$ of degree $\leqslant n$ may be written uniquely as

(3.11) $$g(t) = \sum_{i=0}^{n} A_i Q_i(t).$$

To derive a formula for the coefficients A_i, let $g_r(t)$ be the unique polynomial of degree $\leqslant r$ for which $g_r(m_j) = g(m_j)$ for $0 \leqslant j \leqslant r$. Then

(3.12) $$g_r(t) = \sum_{j=0}^{r} A_j Q_j(t) = \sum_{j=0}^{r} \frac{P_{r+1}(t)\, g(m_j)}{(t-m_j)P'_{r+1}(m_j)},$$

where the second equality above is the result of Lagrange interpolation. It follows from (3.12) that

(3.13) $$g_i(t) - g_{i-1}(t) = A_i Q_i(t) = \Big(P_i(m_i) \sum_{j=0}^{i} \frac{g(m_j)}{P'_{i+1}(m_j)}\Big) Q_i(t).$$

Hence

(3.14) $$A_i = P_i(m_i) \sum_{j=0}^{i} \frac{g(m_j)}{P'_{i+1}(m_j)}.$$

The following two theorems imply that the sequence $\{Q_n(t)\}$ is, in fact, an ordered basis of the I_π-module of polynomials over F_π that map I_π into itself. In the remainder of the paper the subscript π will be omitted from the symbols v_π and $|\ |_\pi$.

THEOREM 3.3. *For all $t \in I_\pi$, $|Q_n(t)| \leqslant 1$.*

Proof (Amice [1]). In virtue of (2.6) it suffices to show that

(3.15) $$v\big(P_n(t)\big) \geqslant v\big(P_n(m_n)\big).$$

By (3.7),

$$(3.16)\qquad v(P_n(t)) = \sum_{i=0}^{n-1} v(t-m_i) = \sum_{k=1}^{\infty} k(\varrho(t;k,n)-\varrho(t;k+1,n))$$
$$= \sum_{k=1}^{\infty} \varrho(t;k,n) \geqslant \sum_{k=1}^{\infty}\left[\frac{n}{q^{kd}}\right].$$

But, by (3.8),

$$(3.17)\qquad v(P_n(m_n)) = \sum_{k=1}^{\infty} \varrho(m_n;k,n) = \sum_{k=1}^{\infty}\left[\frac{n}{q^{kd}}\right],$$

from which the desired result follows.

THEOREM 3.4. *Let $g(t)\in F_\pi[t]$, and write*

$$(3.18)\qquad g(t) = \sum_{i=0}^{n} A_i Q_i(t).$$

Then g maps I_π into itself if and only if $A_i\in I_\pi$, for $0\leqslant i\leqslant n$.

Proof. Sufficiency. By Theorem 3.3, $|Q_i(t)|\leqslant 1$ if $|t|\leqslant 1$, so if $|A_i|\leqslant 1$, $|g(t)|\leqslant 1$, since $|\ |$ is non-archimedean.

Necessity. By (3.14), it suffices to show that, for all $j\leqslant i$,

$$(3.19)\qquad v(P_i(m_i)) \geqslant v(P'_{i+1}(m_j)).$$

By (3.17),

$$(3.20)\qquad v(P_i(m_i)) = \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right].$$

We show that

$$(3.21)\qquad v(P'_{i+1}(m_i)) \leqslant \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right].$$

Since

$$(3.22)\qquad P'_{i+1}(m_j) = (m_j-m_0)\dots(m_j-m_{j-1})(m_j-m_{j+1})\dots(m_j-m_i),$$

inequality (3.21) is obvious for $j=i$, so assume that $j<i$. Denote by $S(i,j)$ the set $S_{i+1}-\{m_j\}$. Then

$$(3.23)\qquad v(P'_{i+1}(m_j)) = \sum_{\substack{r=0\\ r\neq j}}^{i} v(m_j-m_r)$$
$$= \sum_{k=1}^{\infty} k\big(\mathrm{card}(B_k(m_j)\cap S(i,j)) - \mathrm{card}(B_{k+1}(m_j)\cap S(i,j))\big)$$
$$= \sum_{k=1}^{\infty} \mathrm{card}(B_k(m_j)\cap S(i,j)) \leqslant \sum_{k=1}^{\infty}\left[\frac{i}{q^{kd}}\right],$$

as desired.

4. Interpolation theorems. We require a preliminary theorem, due to Amice [1], which specifies conditions under which certain finite subsets of $\{Q_i(t)\}$ are locally constant (mod π). As in the case of a previous theorem, we include, for completeness, a specialized version of the proof given by Amice.

THEOREM 4.1. *Let $\pi\in GF[q,x]$ be an irreducible polynomial of degree d, and let $|\pi| = b$. Then, for all $k\geqslant 1$ and for all i such that $0\leqslant i\leqslant q^{kd}-1$, if $t_1, t_2\in I_\pi$ and $|t_1-t_2|\leqslant b^k$, then*

$$|Q_i(t_1)-Q_i(t_2)|\leqslant b.$$

Proof. It suffices to show that for all i,j with $0\leqslant i,j\leqslant q^{kd}-1$, if $t\in B_k(m_j)$, then $|Q_i(t)-Q_i(m_j)|\leqslant b$. The cases (1) $j<i$ and (2) $j\geqslant i$ are treated separately.

(1) If $j<i$, then $|Q_i(t)-Q_i(m_j)| = |Q_i(t)|$, and so it suffices to show that, for $t\in B_k(m_j)$,

$$(4.1)\qquad v(t-m_0)+\dots+v(t-m_{i-1}) > v(m_i-m_0)+\dots+v(m_i-m_{i-1}),$$

or, as in (3.16), that

$$(4.2)\qquad \sum_{r=1}^{\infty}\varrho(t;r,i) > \sum_{r=1}^{\infty}\varrho(m_i;r,i).$$

By Theorem 3.2,

$$(4.3)\qquad \varrho(t;r,i)\geqslant \varrho(m_i;r,i) = \left[\frac{i}{q^{rd}}\right].$$

When $r=k$, however, inequality (4.3) is strict, since

$$\varrho(t;k,i) = 1 \quad\text{and}\quad \varrho(m_i;k,i) = 0.$$

(2) Let $i\leqslant j$. By hypothesis, $|t-m_j|\leqslant b^k$. For all r with $0\leqslant r\leqslant i-1 < j$, $m_r\not\equiv m_j \pmod{\pi^k}$, and so

$$(4.4)\qquad |t-m_j|\leqslant b\,|m_j-m_r|,$$

or

$$(4.5)\qquad |(t-m_r)-(m_j-m_r)|\leqslant b\,|m_j-m_r|,$$

or

$$(4.6)\qquad \left|\frac{t-m_r}{m_j-m_r}-1\right|\leqslant b.$$

Hence, for each r, there is an $\alpha_r\in I_\pi$ such that

$$(4.7)\qquad \frac{t-m_r}{m_j-m_r} = 1+\pi\alpha_r,$$

and so, there is a $\beta \in I_\pi$ such that

$$\prod_{r=0}^{i-1} \frac{t-m_r}{m_j-m_r} = \frac{Q_i(t)}{Q_i(m_j)} = 1+\pi\beta. \tag{4.8}$$

Therefore,

$$\left|\frac{Q_i(t)}{Q_i(m_j)} - 1\right| \leqslant b, \tag{4.9}$$

and, by Theorem 3.3,

$$|Q_i(t)-Q_i(m_j)| \leqslant b\,|Q_i(m_j)| \leqslant b. \tag{4.10}$$

The interpolation theorems announced in the Introduction are included in the following sequence of theorems.

THEOREM 4.2. *Let π, b, and d be as in Theorem 4.1. Let $f\colon I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geqslant 1$ and a continuous function $h\colon I_\pi \to I_\pi$ such that*

$$f(t) = \sum_{i=0}^{q^{kd}-1} f(m_i)\chi_i(t) + \pi h(t), \tag{4.11}$$

where χ_i is the characteristic function of the set $B_k(m_i)$.

Proof. Since I_π is compact, f is uniformly continuous. Hence, there is an integer $k \geqslant 1$ such that, for all i, $0 \leqslant i \leqslant q^{kd}-1$, if $t \in B_k(m_i)$, then $|f(t)-f(m_i)| \leqslant b$. Thus, there is a continuous function $h^i\colon B_k(m_i) \to I_\pi$ such that, for $t \in B_k(m_i)$,

$$f(t) = f(m_i) + \pi h^i(t). \tag{4.12}$$

Since the sets $B_k(m_i)$ are a pairwise disjoint open-closed cover of I_π, (4.11) may be gotten by setting $h(t) = h^i(t)$ for $t \in B_k(m_i)$.

THEOREM 4.3. *Let $f\colon I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geqslant 1$, a continuous function $f_1\colon I_\pi \to I_\pi$, and a sequence $\{a_i\colon 0 \leqslant i \leqslant q^{kd}-1\}$ in I_π such that*

$$f(t) = \sum_{i=0}^{q^{kd}-1} a_i Q_i(t) + \pi f_1(t). \tag{4.13}$$

Proof. Using the uniform continuity of f, determine k as in Theorem 4.2. By Theorem 4.1, this k is also associated with the uniform continuity of the functions $Q_i(t)$, $0 \leqslant i \leqslant q^{kd}-1$. Applying Theorem 4.2 to these functions, we get

$$Q_i(t) = \sum_{j=0}^{q^{kd}-1} Q_i(m_j)\chi_j(t) + \pi h_i(t). \tag{4.14}$$

Since $Q_i(m_j) = 0$ when $j < i$, system (4.14) is triangular. Solving for the functions $\chi_i(t)$ in terms of the $Q_i(t)$ and the error functions $h_i(t)$, and substituting in (4.11), we get (4.13), where $f_1(t)$ is expressed in terms of the error functions $h_i(t)$.

THEOREM 4.4. *Let $f\colon I_\pi \to I_\pi$ be continuous. Then there is a unique sequence $\{A_i\}$ in F_π such that*

$$f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \tag{4.15}$$

where (4.15) converges uniformly on I_π. Moreover, for all i, $|A_i| \leqslant 1$ and $\lim_{i\to\infty} A_i = 0$.

Proof. By Theorem 4.3 there is an integer $k_0 \geqslant 1$, a sequence $\{a_i^0\colon 0 \leqslant i \leqslant q^{k_0 d}-1\}$, and a continuous function $f_1\colon I_\pi \to I_\pi$ such that

$$f(t) = \sum_{i=0}^{q^{k_0 d}-1} a_i^0 Q_i(t) + \pi f_1(t). \tag{4.16}$$

Similarly, we may write

$$f_1(t) = \sum_{i=0}^{q^{k_1 d}-1} a_i^1 Q_i(t) + \pi f_2(t). \tag{4.17}$$

Iterating and substituting in (4.16) at each stage, we get

$$f(t) = \sum_{i=0}^{M_{n-1}-1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t) + \pi f_n(t), \tag{4.18}$$

where

$$M_{n-1} = \max\{q^{k_0 d}, q^{k_1 d}, \ldots, q^{k_{n-1} d}\}. \tag{4.19}$$

Define the sequence $\{A_i\}$ by

$$A_i = \sum_{j=0}^{\infty} \pi^j a_i^j. \tag{4.20}$$

The series (4.20) converges to an element of I_π, for $|a_i^j| \leqslant 1$. Also

$$\lim_{i\to\infty} A_i = 0, \tag{4.21}$$

for if $i \geqslant M_{n-1}$, then $a_i^0 = a_i^1 = \ldots = a_i^{n-1} = 0$, and so $|A_i| \leqslant b^n$.

Let $k \geqslant M_{n-1}-1$. Then

$$\begin{aligned} &\left|\sum_{i=0}^{k} A_i Q_i(t) - \sum_{i=0}^{M_{n-1}-1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t)\right| \\ &\leqslant \max\left\{\left|\sum_{i=M_{n-1}}^{k} A_i Q_i(t)\right|, \left|\sum_{i=0}^{M_{n-1}-1} \big(A_i - (a_i^0 + \ldots + \pi^{n-1} a_i^{n-1})\big) Q_i(t)\right|\right\} \leqslant b^n, \end{aligned} \tag{4.22}$$

and by (4.18)

$$|f(t)-\sum_{i=0}^{k} A_iQ_i(t)| \leqslant b^n. \tag{4.23}$$

Hence (4.15) converges uniformly to f on I_π. The coefficients A_i are uniquely determined by f, since for each $n \geqslant 0$, the finite sum

$$\sum_{i=0}^{n} A_iQ_i(t) \tag{4.24}$$

is the unique polynomial of degree $\leqslant n$ which takes the same values as f on the set $\{m_0, \ldots, m_n\}$. Hence, by (3.14),

$$A_i = P_i(m_i)\sum_{j=0}^{i} \frac{f(m_j)}{P'_{i+1}(m_j)}. \tag{4.25}$$

In the slightly more general case of a continuous function $f\colon I_\pi \to F_\pi$, the boundedness of f implies the existence of an integer $k \geqslant 0$ such that $\pi^k f\colon I_\pi \to I_\pi$. Hence

$$\pi^k f(t) = \sum_{i=0}^{\infty}\left(P_i(m_i)\sum_{j=0}^{i}\frac{\pi^k f(m_j)}{P'_{i+1}(m_j)}\right)Q_i(t), \tag{4.26}$$

and so

$$f(t) = \sum_{i=0}^{\infty} A_iQ_i(t), \tag{4.27}$$

where A_i is defined by (4.25).

In the case of a continuous function $f\colon B_k(0) \to F_\pi$, where $k < 0$, define $g\colon I_\pi \to F_\pi$ by $g(t) = f(\pi^k t)$. Then by (4.21) and (4.27), we have, for all $t \in I_\pi$,

$$f(\pi^k t) = g(t) = \sum_{i=0}^{\infty}\left(P_i(m_i)\sum_{j=0}^{i}\frac{f(\pi^k m_j)}{P'_{i+1}(m_j)}\right)Q_i(t). \tag{4.28}$$

Hence, for all $t \in B_k(0)$,

$$f(t) = f(\pi^k(\pi^{-k}t)) = \sum_{i=0}^{\infty}\left(P_i(m_i)\sum_{j=0}^{i}\frac{f(\pi^k m_j)}{P'_{i+1}(m_j)}\right)Q_i(\pi^{-k}t). \tag{4.29}$$

It follows that every continuous function $f\colon K \to F_\pi$, where K is a compact subset of F_π, has a series expansion of the form (4.29), for $K \subseteq B_k(0)$ for some $k \leqslant 0$ and, by a theorem of Dieudonné ([7], p. 82), any such f has a continuous extension to $B_k(0)$.

THEOREM 4.5. *Let $\{H_i(t)\}$ be an ordered basis of the I_π-module of polynomials over F_π that map I_π into itself. Let $f\colon I_\pi \to I_\pi$ be continuous. Then there exists a unique null sequence $\{B_i\}$ in I_π such that*

$$f(t) = \sum_{i=0}^{\infty} B_iH_i(t), \tag{4.30}$$

where (4.30) converges uniformly on I_π.

Proof. By Theorem 4.4,

$$f(t) = \sum_{j=0}^{\infty} A_jQ_j(t), \tag{4.31}$$

where $A_j \in I_\pi$ and $\lim_{j\to\infty} A_j = 0$. By Theorem 3.3, for all $j \geqslant 0$, $Q_j(t)$ may be written uniquely as

$$Q_j(t) = \sum_{i=0}^{n_j} D_i^j H_i(t), \tag{4.32}$$

where $D_i^j \in I_\pi$. Set

$$B_i = \sum_{j=0}^{\infty} A_jD_i^j. \tag{4.33}$$

Since $\lim_{j\to\infty} A_j = 0$ and $|D_i^j| \leqslant 1$, (4.33) converges to an element of I_π. Moreover, $\lim_{i\to\infty} B_i = 0$, for, given any integer $k \geqslant 0$, let r be such that $|A_j| \leqslant b^k$ if $j \geqslant r$. Let $i > \max\{n_0, \ldots, n_{r-1}\}$. Then $D_i^j = 0$ if $j < r$, and so $|B_i| \leqslant b^k$.

If $k \geqslant 0$, let r be such that $|A_j| \leqslant b^k$ for $j \geqslant r$ and

$$\left|\sum_{j=0}^{s} A_jQ_j(t)-f(t)\right| \leqslant b^k \tag{4.34}$$

for $s \geqslant r$. If $n \geqslant \max\{n_0, \ldots, n_r\}$, then

$$\left|\sum_{i=0}^{n} B_iH_i(t)-\sum_{j=0}^{r} A_jQ_j(t)\right| = \left|\sum_{u=r+1}^{\infty} A_u\sum_{v=0}^{n} D_v^u H_v(t)\right| \leqslant b^k. \tag{4.35}$$

Then (4.34) and (4.35) yield (4.30).

Moreover, $\{B_i\}$, as defined in (4.33), is the only null sequence in I_π for which (4.30) holds. For suppose that

$$f(t) = \sum_{i=0}^{\infty} C_iH_i(t), \tag{4.36}$$

where $\lim_{i\to\infty} C_i = 0$. For all $i \geqslant 0$, write

$$H_i(t) = \sum_{j=0}^{n_i} E_j^iQ_j(t), \tag{4.37}$$

where $E_j^i \in I_\pi$. A repetition of the preceding argument yields

$$f(t) = \sum_{j=0}^{\infty} Q_j(t) \sum_{i=0}^{\infty} C_i E_j^i. \tag{4.38}$$

By Theorem 4.4, however,

$$\sum_{i=0}^{\infty} C_i E_j^i = A_j \quad (j \geqslant 0), \tag{4.39}$$

where A_j is defined by (4.25). Since $\{C_i\}$ and $\{A_j\}$ are null sequences, the equations (4.39) may be written matrically,

$$MC = A, \tag{4.40}$$

where C and A are the infinite column vectors $[C_0, C_1, \ldots]^T$ and $[A_0, A_1, \ldots]^T$ and M is the column-finite matrix $[m_{rs}]$, where

$$m_{rs} = E_r^s \quad (r, s \geqslant 0), \tag{4.41}$$

and E_r^s is defined by (4.39).

Using (4.32) and (4.37) the matrix M may be seen to possess the two-sided inverse $Q = [q_{rs}]$, where

$$q_{rs} = D_r^s \quad (r, s \geqslant 0), \tag{4.42}$$

and D_r^s is defined by (4.32). Hence the relation (4.40) determines C uniquely, and

$$C_i = B_i = \sum_{j=0}^{\infty} A_j D_i^j. \tag{4.43}$$

We stress that Theorem 4.5 asserts the uniqueness of the coefficients B_i on the assumption that $\{B_i\}$ is null. The unqualified uniqueness of these coefficients (which we have been able to prove only in special cases) is equivalent to the assertion that a series

$$\sum_{i=0}^{\infty} C_i H_i(t) \tag{4.44}$$

converges uniformly on I_π only if $\{C_i\}$ is null.

5. Applications. Define the sequence of polynomials $\psi_r(t)$ over $\mathrm{GF}[q, x]$ by

$$\psi_r(t) = \prod_{\deg m < r} (t - m), \quad \psi_0(t) = t, \tag{5.1}$$

where the product in (5.1) extends over all polynomials $m \in \mathrm{GF}[q, x]$ (including 0) having degree $< r$. It follows [3] that

$$\psi_r(t) = \sum_{i=0}^{r} (-1)^{r-i} \begin{bmatrix} r \\ i \end{bmatrix} t^{q^i}, \tag{5.2}$$

where

$$\begin{bmatrix} r \\ i \end{bmatrix} = \frac{F_r}{F_i L_{r-i}^{q^i}}, \quad \begin{bmatrix} r \\ 0 \end{bmatrix} = \frac{F_r}{L_r}, \quad \begin{bmatrix} r \\ r \end{bmatrix} = 1, \tag{5.3}$$

and

$$\begin{aligned} F_r &= [r][r-1]^q \ldots [1]^{q^{r-1}}, & F_0 &= 1, \\ L_r &= [r][r-1] \ldots [1], & L_0 &= 1, \\ [r] &= x^{q^r} - x. \end{aligned} \tag{5.4}$$

Let K be any extension field of $\mathrm{GF}(q, x)$. By (5.2), the functions associated to the polynomials $\psi_r(t)$ are linear operators on the $\mathrm{GF}(q)$-vector space K. Furthermore, $\psi_r(x^r) = \psi_r(m) = F_r$, for m monic of degree r, so that F_r is the product of all monic polynomials in $\mathrm{GF}[q, x]$ of degree r. On the other hand, L_r may be seen to be the l.c.m. of all polynomials in $\mathrm{GF}[q, x]$ of degree r [2].

Following Carlitz [4], we define $g_k \in \mathrm{GF}[q, x]$, and polynomials $G_k(t)$, $G_k^*(t)$ over $\mathrm{GF}[q, x]$. Let k be a positive integer, and write

$$k = e_0 + e_1 q + \ldots + e_s q^s \quad (0 \leqslant e_i < q). \tag{5.5}$$

Define g_k by

$$g_k = F_1^{e_1} \ldots F_s^{e_s}, \quad g_0 = 1, \tag{5.6}$$

and $G_k(t)$ and $G_k^*(t)$ by

$$G_k(t) = \psi_0^{e_0}(t) \ldots \psi_s^{e_s}(t), \quad G_0(t) = 1 \tag{5.7}$$

and

$$G_k^*(t) = \prod_{i=0}^{s} G_{e_i q^i}^*(t), \tag{5.8}$$

where

$$G_{eq^i}^*(t) = \begin{cases} \psi_i^e(t) & \text{for } 0 \leqslant e < q-1, \\ \psi_i^e(t) - F_i^e & \text{for } e = q-1. \end{cases} \tag{5.9}$$

Let K be any extension field of $\mathrm{GF}(q, x)$. Since $\deg G_n(t) = \deg G_n^*(t) = n$, the sequences $\{G_n(t)/g_n\}$ and $\{G_n^*(t)/g_n\}$ are ordered bases of the K-vector space $K[t]$. Indeed, for any $f(t) \in K[t]$ of degree $\leqslant n$, we have [4] the unique representations

$$f(t) = \sum_{i=0}^{n} A_i \frac{G_i(t)}{g_i} \tag{5.10}$$

and

$$f(t) = \sum_{i=0}^{n} A_i^* \frac{G_i^*(t)}{g_i}, \tag{5.11}$$

where A_i is uniquely determined by choosing any r such that $i < q^r$, and setting

$$(5.12)\qquad A_i = (-1)^r \sum_{\deg m<r} \frac{G^*_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (m \in \mathrm{GF}[q, x]),$$

and A_i^* is uniquely determined by choosing any r such that $n < q^r$, and setting

$$(5.13)\qquad A_i^* = (-1)^r \sum_{\deg m<r} \frac{G_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (m \in \mathrm{GF}[q, x]).$$

Note the difference between the defining conditions for r in (5.12) and (5.13).

An important property of the polynomials $G_i(t)/g_i$ and $G_i^*(t)/g_i$ is the fact that for all $m \in \mathrm{GF}[q, x]$, $G_i(m)/g_i \in \mathrm{GF}[q, x]$ and $G_i^*(m)/g_i \in \mathrm{GF}[q, x]$ [4]. With (5.12) and (5.13), this implies that $\{G_i(t)/g_i\}$ and $\{G_i^*(t)/g_i\}$ are, in fact, ordered bases of the $\mathrm{GF}[q, x]$-module of polynomials over $\mathrm{GF}(q, x)$ that map $\mathrm{GF}[q, x]$ into itself.

Moreover, since $\mathrm{GF}[q, x]$ is dense in I_π and the polynomials $G_i(t)/g_i$ and $G_i^*(t)/g_i$ are, by an earlier observation, continuous functions, it follows that $\alpha \in I_\pi$ implies that $G_i(\alpha)/g_i$ and $G_i^*(\alpha)/g_i \in I_\pi$. With (5.12) and (5.13) this implies that $\{G_i(t)/g_i\}$ and $\{G_i^*(t)/g_i\}$ are ordered bases of the I_π-module of polynomials over F_π that map I_π into itself.

Hence, by Theorem 4.5, for every continuous function $f\colon I_\pi \to I_\pi$, there exist null sequences $\{B_i\}$ and $\{B_i^*\}$ in I_π such that

$$(5.14)\qquad f(t) = \sum_{i=0}^{\infty} B_i \frac{G_i(t)}{g_i}$$

and

$$(5.15)\qquad f(t) = \sum_{i=0}^{\infty} B_i^* \frac{G_i^*(t)}{g_i},$$

where (5.14) and (5.15) converge uniformly on I_π.

The coefficients B_i in (5.14) are uniquely determined by f. For if n is any positive integer, the finite sum

$$(5.16)\qquad \sum_{i=0}^{q^n-1} B_i \frac{G_i(t)}{g_i}$$

is the unique polynomial of degree $\leqslant q^n-1$ which takes the same values as f on the set of all polynomials in $\mathrm{GF}[q, x]$ of degree $< n$. Hence, by (5.12),

$$(5.17)\qquad B_i = (-1)^r \sum_{\deg m<r} \frac{G^*_{q^r-1-i}(m)}{g_{q^r-1-i}} f(m) \qquad (i < q^r).$$

The question of the unconditional uniqueness of the coefficients B_i^* remains open.

Interpolation series of the type which appears in (5.14) may be used to characterize continuous linear operators on the $\mathrm{GF}(q)$-vector space I_π.

THEOREM 5.1. *Let $f\colon I_\pi \to I_\pi$ be continuous. If the (unique) interpolation series for f constructed from the Carlitz polynomials is given by*

$$(5.18)\qquad f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i},$$

then f is a linear operator on the $\mathrm{GF}(q)$-vector space I_π if and only if $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$.

Proof. Sufficiency. If $A_i = 0$ for $i \neq q^k$, where $k \geqslant 0$, (5.18) becomes

$$(5.19)\qquad f(t) = \sum_{k=0}^{\infty} A_{q^k} \frac{\psi_k(t)}{F_k}.$$

Since, by (5.2), the partial sums of (5.19) are linear operators, it follows immediately that f is a linear operator.

Necessity. We require the following identities [4]:

$$(5.20)\qquad G_i(\lambda t) = \lambda^i G_i(t) \qquad (\lambda \in \mathrm{GF}(q)),$$

$$(5.21)\qquad G_i(t_1+t_2) = \sum_{j=0}^{i} \binom{i}{j} G_j(t_1) G_{i-j}(t_2).$$

Let $\lambda \in \mathrm{GF}(q)$ be a primitive root of unity. Then (5.18), (5.20), and $f(\lambda t) = \lambda f(t)$, yield

$$(5.22)\qquad \sum_{i=0}^{\infty} \lambda A_i \frac{G_i(t)}{g_i} = \sum_{i=0}^{\infty} \lambda^i A_i \frac{G_i(t)}{g_i},$$

and so $A_i = 0$, unless $i \equiv 1 \pmod{q-1}$.

From (5.18), (5.21), and $f(t_1+t_2) = f(t_1)+f(t_2)$, we infer that

$$(5.23)\qquad \sum_{i=0}^{\infty} A_i \frac{G_i(t_1)}{g_i} + \sum_{i=0}^{\infty} A_i \frac{G_i(t_2)}{g_i} = \sum_{i=0}^{\infty} \frac{G_i(t_1)}{g_i} \sum_{j=i}^{\infty} \frac{g_i}{g_j} \binom{j}{i} A_j G_{j-i}(t_2).$$

Equating coefficients of $G_0(t_1)$, we see that $A_0 = 0$. Equating coefficients of $G_i(t_1)/g_i$ for $i > 0$, and subtracting A_i, we get

$$(5.24)\qquad \sum_{j=i+1}^{\infty} \frac{g_i}{g_j} \binom{j}{i} A_j G_{j-i}(t_2) = 0.$$

Hence, for all i, j with $1 \leqslant i < j$,

$$\binom{i}{j} A_j = 0. \tag{5.25}$$

It follows that $A_j = 0$ unless $j = p^t$, where p is the characteristic of $\mathrm{GF}(q)$. Since $p^t \equiv 1 (\mathrm{mod}\, q-1)$, we must have $p^t = q^k$, where $k \geqslant 0$.

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On the changes of sign of a certain class of error functions

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§ 1. Introduction. Since its introduction by Euler in the eighteenth century, $\varphi(n)$ and its behavior have been of great interest in number theory [1]. During the next century G. L. Dirichlet [2] proved that $\sum_{1}^{N}\varphi(n) \sim 3N^2/\pi^2$, and F. Mertens ([6]; [4], p. 268) showed the error to be $O(N\log N)$; this has only recently been improved, to $O\big(N\log^{2/3}N(\log\log N)^{4/3}\big)$, by A. Walfisz [11]. The average order of $\varphi(n)$ is thus $6n/\pi^2$, and it is well known ([4], p. 267) that $\limsup \varphi(n)/n = 1$ and that $n^{\delta-1}\varphi(n) \to \infty$ for all positive δ; there is also the theorem due to Landau [5] that $\sum_{1}^{N}\big(1/\varphi(n)\big) \sim \big(315\zeta(3)/2\pi^4\big)\log N$.

These results all support the assertion that $\varphi(n)$ behaves asymptotically very much like n. It is then reasonable to look at $\sum_{1}^{N} n - \frac{1}{2}N^2$ and $\sum_{1}^{N} 1 - N$ (which are $\frac{1}{2}N$ and 0) for qualitative information about the errors $E(N) = \sum_{1}^{N}\varphi(n) - 3N^2/\pi^2$ and $H(N) = \sum_{1}^{N}\varphi(n)/n - 6N/\pi^2$, on which basis one would expect $E(x) \nearrow +\infty$ and $H(x)$ very small. Sylvester ([9], [10]) conjectured that $E(x) > 0$ for all x. Between 1930 and 1950 it was shown that in each of these respects $\varphi(n)$ differs radically from n. Pillai and Chowla [7] proved that the average order of $H(n)$ is $3/\pi^2$ and that of $E(n)$ is $3n/2\pi^2$, which comes up to expectation; but they also proved that $E(x) = \Omega(x\log\log\log\log x)$. It follows that $H(x) = \Omega(\log\log\log\log x)$ refuting the conjecture that $H(x)$ is small. Subsequently M. L. N. Sarma [8] showed that $H(820)$ is negative; in 1950 P. Erdös and H. N. Shapiro [3] proved that $H(x) = \Omega_{\pm}(\log\log\log\log x)$.

The purpose of this paper is to show that this behavior is not peculiar to $\varphi(n)$, but is shared by a large class of functions $f(n) = n\sum_{c|n}\mu(c)p(c)/c$, where $p(n)$ satisfies certain admissibility conditions given below. The method is based on an extension of that used by Erdös