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A remark on number-theoretical functions

by

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In the following $f(n)$, $g(n)$ denote additive number-theoretical functions, i.e.

$$f(nm) = f(n)+f(m), \quad g(nm) = g(n)+g(m),$$

whenever $(n, m) = 1$.

$p, q, p_1, p_2, \ldots$ denote prime numbers. $c, c_1, c_2, \ldots, K$ denote suitable positive constants not the same at every place.

We shall say that a sequence of natural numbers $n_1 < n_2 < \ldots$ has a *lower density* δ_1 and an *upper density* δ_2 if

$$\varliminf_{x\to\infty} \frac{\sum_{n_i<x} 1}{x} = \delta_1, \quad \varlimsup_{x\to\infty} \frac{\sum_{n_i<x} 1}{x} = \delta_2.$$

We say that a set of natural numbers *contains almost every natural number*, if its lower density is 1.

A well-known theorem due to P. Erdös states that:

If $f(n)$ is an additive number-theoretical function such that $f(n+1) \geqslant f(n)$ for every natural number n, then $f(n)$ is a constant multiple of $\log n$.

I believed that the following generalization is also true:

If

$$h(n) = \max\big(f(n), g(n)\big)$$

such that

$$h(n) \leqslant h(n+1), \quad n = 1, 2, \ldots,$$

then $h(n)$ is a constant multiple of $\log n$.

It is easy to prove that this assertion holds if both functions $f(n)$, $g(n)$ are totally additive. But it is not difficult to give a counter-example concerning the general case.

Let p_1 be a fixed prime number,

$$f(n) = \log n; \quad g(p_1^a) = \log p_1^a + \varepsilon(p_1^a),$$

where $0 < \varepsilon(p_1^a) < c_1/p_1^a$, and let $g(p^a) = 0$ for every prime power p^a, if p is different from p_1. Then

$$h(n) = \log n, \quad \text{if} \quad (n, p_1) = 1, \quad \text{and} \quad h(p_1^{a_1}) > \log p_1^{a_1}.$$

If we choose c_1 small enough, the corresponding $h(n)$ is a monotonically increasing function.

We can prove the following assertion.

THEOREM. *Let $f(n)$, $g(n)$ be additive number-theoretical functions for which*

$$h(n) = \max(f(n), g(n))$$

is a monotone non-decreasing function. Then the following three assertions hold:

1. *$h(n) = c\log n + r(n)$, where $r(n) \to 0$ as $n \to \infty$, and $r(n) = 0$, if every prime factor of n is greater than K. K is a suitable constant depending on $h(n)$.*

2. *If $f(n) \geqslant g(n)$ for almost every n, then*

$$f(n) = c\log n \quad \text{and} \quad g(n) = c\log n + \varepsilon(n),$$

where $\varepsilon(p^a) \leqslant 0$, if p is large enough.

Let $S = (p_1, \ldots, p_l)$ denote the set of all "irregular primes", i.e. those primes for which there exist exponents a_i such that

$$\varepsilon(p_i^{a_i}) > 0, \quad i = 1, \ldots, l.$$

There are three possibilities only:

(a) *S is an empty set,*

(b) *S consists of one element only,*

(c) *S consists of at least two but finitely many elements. In this case*

$$\varepsilon(p_i^{\beta}) \leqslant 0, \quad i = 1, \ldots, l$$

if β is large enough.

3. *If the set of the n's satisfying $f(n) \geqslant g(n)$ has a positive upper-density and a lower-density smaller than 1, then:*

$$h(n) = c\log n \quad \textit{for every } n.$$

Further we have

$$f(p^a) = g(p^a) = c\log p^a, \quad a = 1, 2, \ldots$$

for every prime p, with the exception of at most one.

Remark. The conditions in 2 and 3 are complementary.

2. For the proof of our theorem we need the following lemmas.

LEMMA 1 [1]. *If $f(n)$ is an additive number-theoretical function, non-decreasing monotonically, then $f(n)$ is a constant multiple of $\log n$.*

LEMMA 2. *Let us suppose that for a sequence of natural numbers $A = \{n_i,\ i = 1, 2, \ldots\}$*

$$n_1 < n_2 < \ldots$$

having density 1

$$f(n_i) \leqslant f(n_{i+1}), \quad i = 1, 2, \ldots$$

holds. Then $f(n)$ is a constant multiple of $\log n$ (¹).

Proof. We can easily deduce this assertion from Lemma 1. Let n be a given integer. The positive integer solutions x, y of the equation

$$(n+1)x - ny = 1, \quad (x, n+1) = 1, \quad (y, n) = 1$$

have positive density. Hence it follows that there exist infinitely many solutions x, y for which $x, y, (n+1)x, ny \in S$. Then

$$f\big((n+1)x\big) \geqslant f(ny), \quad \text{i.e.} \quad f(n+1) - f(n) \geqslant f(y) - f(x) \geqslant 0,$$

because $y \geqslant x$, $y, x \in S$.

Hence it follows that $f(n)$ is non-decreasing on the whole set of natural numbers. Using Lemma 1 we infer Lemma 2.

The following lemma plays a principal role in the proof.

LEMMA 3 [1]. *If there exist two positive constants c_1 and c_2 and an infinite sequence $x_k \to \infty$ so that for every x_k there are at least $c_1 x_k$ integers:*

$$1 \leqslant a_1 < a_2 < \ldots a_l \leqslant x_k, \quad l \geqslant c_1 x_k,$$

for which

$$|f(a_i) - f(a_j)| < c_2, \quad 1 \leqslant i < j \leqslant l,$$

then

$$f(n) = c\log n + l(n),$$

where

$$\sum_p \frac{\|l(p)\|^2}{p} < \infty.$$

As usual,

$$\|l(p)\| = \begin{cases} l(p), & \text{when} \quad |l(p)| \leqslant 1, \\ 1, & \text{when} \quad |l(p)| > 1. \end{cases}$$

(¹) The assertion of this lemma was conjectured by P. Erdös in [2].

3. Proof of the Theorem. In the proof we distinguish two complementary cases according to the assertions 2 and 3.

A. $f(n) \geqslant g(n)$ for almost every n.

B. The set of n's for which $f(n) \geqslant g(n)$ has a positive upper-density and a lower-density smaller than one.

Case A. In this case $f(n)$ is non-decreasing on a set with density 1. Then $f(n) = c\log n$ by Lemma 2.

Suppose that there exists an n_1 for which $g(n_1) > f(n_1)$. Let

$$g(n_1) = f(n_1) + \Delta, \quad \Delta > 0.$$

We shall prove that there exist at most finitely many m, coprime to n_1 for which $g(m) \geqslant f(m)$. In the opposite case there would exist $m_1 < m_2 < \ldots$, $m_i \to \infty$, $(m_i, n_1) = 1$ for which

$$h(m_i n_1) = g(m_i n_1) \geqslant f(m_i) + g(n_1) = f(m_i n_1) + \Delta.$$

Hence it would follow that

$$g(n) > c\log n = f(n)$$

in the interval $m_i n_1 \leqslant n \leqslant (1+c_3) m_i n_1$, with a suitable constant $c_3 > 0$. But this cannot occur in the case A.

Hence it follows immediately that the set of irregular primes S has at most finitely many elements.

Suppose now that there exist infinitely many exponents β for which

$$\varepsilon(p_1^\beta) > 0,$$

and let

$$q_\beta = p_1^\beta p_2^{\alpha_2} \ldots p_l^{\alpha_l},$$

where

$$0 < \varepsilon(p_i^{\alpha_i}), \quad i = 2, \ldots, l.$$

Since $q_\beta + 1$ has no irregular prime factors we have

$$h(q_\beta) = g(q_\beta) = f(q_\beta) + \sum_{i=2}^{l} \varepsilon(p_i^{\alpha_i}) + \varepsilon(p_1^\beta) \leqslant h(q_\beta + 1) = f(q_\beta + 1),$$

whence

$$0 < \varepsilon(p_1^\beta) + \sum_{i=2}^{l} \varepsilon(p_i^{\alpha_i}) \leqslant c/q_\beta.$$

Let now $\beta \to \infty$ in a suitable set. Hence

$$\sum_{i=2}^{l} \varepsilon(p_i^{\alpha_i}) = 0$$

follows. So there exists only one irregular prime. Further we have

$$\overline{\lim_{\beta\to\infty}}\, \varepsilon(p_1^\beta) \leqslant 0.$$

Let us suppose that there exists more than one irregular prime. Let $\alpha_1, \ldots, \alpha_l$ be the greatest powers for which $\varepsilon(p_i^{\alpha_i}) > 0$. Let $P = \prod p_i^{\alpha_i}$. Every integer n can be represented in the form $n = dn_1$, where $d | P$, $(n_1, d) = 1$ and d contains the p's with the power zero or greater than α_i. Repeating the arguments used above we have $g(n_1 d) \leqslant f(n_1 d)$ for fixed d, if n_1 is large enough. So $h(n) = f(n)$, if n is large enough, because the set of the d's is finite.

For the proof of 1 we note that if

$$\overline{\lim_{n\to\infty}} \{h(n) - c\log n\} = \Delta > 0,$$

then $g(n) > f(n)$ on a set having positive upper-density, which was excluded now.

Case B. It is enough to prove assertion 3 only. Using the monotonicity and subadditivity of $h(n)$ we have

$$h(n) \leqslant h(n+j) \leqslant h(2n) \leqslant h(n) + h(2)$$

if n is an odd integer and $j = 1, 2, \ldots, n$. In our case for $f(n)$ and $g(n)$ there exist infinite sequences x_k, x_k' tending to infinity, such that the equation

$$h(n) = f(n), \; n \in (x_k/2, x_k) \quad \text{resp.} \quad h(n) = g(n), \; n \in (x_k'/2, x_k')$$

has at least δx_k resp. $\delta x_k'$ solutions, where δ is a positive constant. Hence it follows by Lemma 3 that

$$f(n) = c\log n + g_1(n), \quad g(n) = c'\log n + g_2(n),$$

where

$$\sum_p \frac{\|g_i(p)\|^2}{p} < \infty, \quad i = 1, 2. \tag{3.1}$$

We prove that $c = c'$. If we suppose the contrary, say $c > c'$, then $f(n) \geqslant g(n)$ for almost every n, because from (3.1) it follows, that $g_i(n) = o(\log n)$ for almost every n. But this is not our case. So $c = c'$.

Let

$$h'(n) = \max\big(g_1(n), g_2(n)\big);$$

so

$$h(n) = c\log n + h'(n).$$

From the monotonicity of $h(n)$ we obtain

$$h'(n+1)-h'(n) \geqslant -K/n, \tag{3.2}$$

where K is a suitable constant.

From (3.1) it follows that there exists a suitable sequence of positive numbers ε_p tending to zero monotonically, such that

$$\sum_{\substack{\max\limits_{i=1,2} |g_i(p)|>\varepsilon_p}} \frac{1}{p} < \infty.$$

Let P' denote those primes which occur in this sum. Let P denote the set of primes not belonging to P'. In view of the prime number theorem it is almost evident that there exists a monotonically decreasing positive function $\varepsilon(x)$ tending to zero, and an $\omega(x)$ tending to infinity as $x \to \infty$, such that there exist two primes $p_1, p_2 \epsilon P$ such that

$$x \leqslant p_1 p_2 \leqslant x\big(1+\varepsilon(x)\big), \quad p_1, p_2 > \omega(x). \tag{3.3}$$

It is evident that

$$h'(p_1 p_2) \to 0 \quad \text{as} \quad x \to \infty.$$

Using the relation (3.2) and (3.3) we have

$$h'(n) \to 0.$$

Considering that

$$h'(np) \leqslant h'(n)+h'(p) \quad \text{if} \quad p > n,$$

and tends to infinity with p in the set $p \epsilon P$ we have

$$h'(n) \geqslant 0 \tag{3.4}$$

for every n. Further

$$g_i(n) = \overline{\lim_{p\to\infty}}\, g_i(np) \leqslant \overline{\lim_{p\to\infty}}\, h'(np) = 0 \quad \text{for} \quad i = 1, 2,$$

whence $g_i(n) \leqslant 0$ $(i = 1, 2)$; consequently $h'(n) \leqslant 0$.

Hence it follows, that

$$h'(n) = 0 \tag{3.5}$$

for every n. Suppose now that there exist prime powers $p_1^{a_1}, p_2^{a_2}, p_1 \neq p_2$, such that

$$g_1(p_1^{a_1}) < 0, \quad g_2(p_2^{a_2}) < 0.$$

Hence it would follow that

$$h'(p_1^{a_1} p_2^{a_2}) < 0,$$

which is a contradiction of (3.5).

So we have obtained

$$g_1(p^a) = g_2(p^a) = 0$$

for every power of primes p, except for at most the powers of one exceptional prime p_1.

Hence the assertion 3 follows.

I am indebted to Professors P. Turán and P. Erdös for their valuable remarks.

References

[1] P. Erdös, *On the distribution function of additive functions*, Ann. of Math. 47 (1946), pp. 4-20.

[2] — *On the distribution function of additive arithmetical functions*, Rend. Sem. Mat. Fis. Milano 27 (1958), pp. 3-7.

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