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Real zeros of the Dedekind zeta function of an imaginary quadratic field

by

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Introduction. It is known that the Dedekind zeta function of an imaginary quadratic field can be expressed as the product of the Riemann zeta function and a Dirichlet L-series and also as a sum of Epstein zeta functions. In symbols, if $\zeta_{-d}(s)$ denotes the Dedekind zeta function of the imaginary quadratic field having discriminant $-d$, $d>4$, then

$$\zeta_{-d}(s) = \zeta(s)L_{-d}(s) = \sum_{Q} Z(s;\, a,b,c),$$

where

$$L_{-d}(s) = \sum_{n=1}^{\infty} (-d\,|\,n)n^{-s} \qquad (\operatorname{Re} s>0)$$

and

$$Z(s;\, a,b,c) = Z(s) = \tfrac{1}{2}\sum (am^2+bmn+cn^2)^{-s} \qquad (\operatorname{Re} s>1).$$

The summation in $Z(s)$ is over all pairs of integers (m,n) except $(0,0)$, $(-d\,|\,n)$ is the Kronecker symbol, and Q is the set of reduced integral solutions of $-d = b^2-4ac$. In symbols,

$$Q = \{(a,b,c)\,|\,-d = b^2-4ac,\ -a<b\leqslant a<c \text{ or } 0\leqslant b\leqslant a=c;\ a,b,c \text{ are integers}\}.$$

The purpose of this paper is to investigate the behavior of $L_{-d}(s)$ for real s between 0 and 1 by working with the analytic continuations of the Epstein zeta functions. Now P. T. Bateman and E. Grosswald [1] found a very accurate approximation to $Z(s)$ in the critical strip near the real line. They showed that

$$k^{-1/2}(ak/\pi)^s\Gamma(s)Z(s) = f(s)+f(1-s)+H(s;a,b,c),$$

where

$$f(s) = k^{-1/2}(k/\pi)^s\Gamma(s)\zeta(2s), \quad k = d^{1/2}/(2a),$$

$\Gamma(s)$ is the gamma function, and $H(s;a,b,c)$ is an "error" function. Let us define

$$\alpha(a) = \log a+\log(8\pi e^{-\gamma})-\tfrac{1}{2}\log d$$

and

$$b_n = (2^n-1)\zeta(n)/n+2^n\beta_n,$$

where

$$\log\{(s-1)\zeta(s)\} = \sum_{n=1}^{\infty}(-1)^n\beta_n(s-1)^n \quad (|s|<3).$$

We then show that

$$f(s) = -\tfrac{1}{2}(\tfrac{1}{2}-s)^{-1}\exp\{(\tfrac{1}{2}-s)\alpha(a)+\sum_{n=2}^{\infty}b_n(\tfrac{1}{2}-s)^n\}.$$

Now $\{f(s)+f(1-s)\}$ has a removable singularity at $s=\frac{1}{2}$ and it is easily seen that the value there is $-\alpha(a)$. We prove the following theorems:

THEOREM 1. *If $n\geqslant 2$, then $b_n>0$.*

THEOREM 2. *If $-d$ is a fixed fundamental discriminant, and if*

(i) $$\sum_Q a^{-1/2}\alpha(a)\geqslant 0,$$

(ii) $$\sum_Q a^{-1/2}\{\alpha(a)-H(s;a,b,c)\}>0 \quad (0<s<1),$$

(iii) $$\sum_Q a^{-1/2}\{b_{2n+1}+\alpha^{2n+1}(a)/(2n+1)!\}\geqslant 0 \quad (n\geqslant 1),$$

then $L_{-d}(s)>0$ for $s>0$.

THEOREM 3. *If $d<593000$ and condition* (i) *of Theorem 2 holds, then* (iii) *holds.*

THEOREM 4. *If $0<s<1$, then*

$$H(s;a,b,c)\leqslant 2k^{-1/2}e^{-2\pi k}\cos(\pi b/a)+(0.04)k^{-1/2}e^{-2\pi k}|\cos(\pi b/a)|.$$

In particular, we find that $H(s;a,b,c)<0.005$. Thus when $d<593000$, Theorems 2 and 3 essentially tell us that if $L_{-d}(\frac{1}{2})>0$, then $L_{-d}(s)>0$ for all $s>0$.

We used the IBM 7094 at the University of Illinois to investigate conditions (i) and (ii) of Theorem 2, using Theorem 4 whenever necessary. We found that condition (ii) held for all $d<1000000$, and condition (i) held for all $d<1000000$ with the exceptions of $d=115147$ and $d=636184$. Thus our main result is

THEOREM 5. *If $-d$ is a fundamental discriminant and $d<593000$, then $L_{-d}(s)$ (and hence $\zeta_{-d}(s)$) has no real zeros for real $s>0$ with the possible exception of $L_{-115147}(s)$.*

It is possible to verify condition (iii) for individual $d>593000$. If T means true, F means false, and U means unknown, we have the following tabulation for fundamental discriminants $-d$.

d	(i)	(ii)	(iii)
$1<d<115147$	T	T	T
$d=115147$	F	T	T
$115147<d\leqslant 593000$	T	T	T
$593000<d<636184$	T	T	U
$d=636184$	F	T	T
$636184<d<1000000$	T	T	U

We give a detailed description of the programs written for the IBM 7094 and the machine analysis of one of the two exceptional cases that do not satisfy condition (i).

The previously best known results analogous to Theorem 5 are due to J. Barkley Rosser who showed in two published papers [2], [3] and in two other unpublished papers that if $d<986$, then $L_{-d}(s)$ has no real zeros. However, Rosser also considered positive discriminants as well.

1. Positivity of certain power series coefficients. We define

(1) $$b_n = (2^n-1)\zeta(n)/n+2^n\beta_n,$$

where

(2) $$\log\{(s-1)\zeta(s)\} = \sum_{n=1}^{\infty}(-1)^n\beta_n(s-1)^n \quad (|s|<3).$$

Since $\{(s-1)\zeta(s)\}$ has $s=-2$ as the zero closest to $s=1$, the radius of convergence of the above series is 3.

THEOREM 1. *$b_n>0$ $(n\geqslant 2)$.*

Proof. We need to show that $|\beta_n|$ does not get too large. It is more convenient to differentiate (2) to get

(3) $$\frac{(s-1)\zeta'(s)+\zeta(s)}{(s-1)\zeta(s)} = \sum_{n=1}^{\infty}(-1)^n n\beta_n(s-1)^{n-1}.$$

Now Cauchy's integral theorem tells us that

$$|\beta_n| \leqslant \frac{M}{nR^{n-1}}, \tag{4}$$

where M is an upper bound for the absolute value of the function in (3) on a circle of radius R with center $s=1$. We wish to select a suitable R and to calculate a numerical value for M corresponding to that R. Of course, by looking at (4), R should be greater than 1, but far enough away from 3 to retain control of M.

Put

$$B_1(x) = x-\tfrac{1}{2},$$
$$B_2(x) = x^2-x+\tfrac{1}{6},$$
$$B_3(x) = x^3-(\tfrac{3}{2})x^2+\tfrac{1}{2}x.$$

Then from the familiar formula

$$\zeta(s) = (s-1)^{-1}+\tfrac{1}{2}-s\int_1^\infty B_1(x-[x])x^{-s-1}dx \qquad (\mathrm{Re}\, s>0),$$

we get by two integrations by parts that

$$\zeta(s) = (s-1)^{-1}+\tfrac{1}{2}+s/12-\tfrac{1}{2}s(s+1)\int_1^\infty B_2(x-[x])x^{-s-2}dx \qquad (\mathrm{Re}\, s>-1),$$

$$\zeta(s) = (s-1)^{-1}+\tfrac{1}{2}+\frac{s}{12}-\frac{s(s+1)(s+2)}{6}\int_1^\infty \frac{B_3(x-[x])}{x^{s+3}}dx \qquad (\mathrm{Re}\, s>-2).$$

Multiplying by $(s-1)$ and collecting terms gives

(5) $(s-1)\zeta(s)$

$$= \{(s+3)-2(s-1)s(s+1)\int_1^\infty B_3(x-[x])x^{-s-3}dx\}(s+2)/12 \qquad (\mathrm{Re}\, s>-2).$$

To maximize the left hand side of (3), we want to minimize $|(s-1)\zeta(s)|$ on an appropriate circle about $s=1$. We therefore select $R=3/2$ and our circle is

$$|s-1| = 3/2.$$

Since

$$|s+u| = |s-1+u+1| \leqslant |s-1|+|u+1|$$

and

$$|s+u| = |s-1+u+1| \geqslant \big||s-1|-|u+1|\big|,$$

we find that

$$1/2 \leqslant |s| \leqslant 5/2,$$
$$1/2 \leqslant |s+1| \leqslant 7/2,$$
$$3/2 \leqslant |s+2| \leqslant 9/2,$$
$$5/2 \leqslant |s+3| \leqslant 11/2.$$

Let $s=\sigma+it$. Since

$$|B_3(u)| \leqslant \sqrt{3}/36 \qquad (0\leqslant u\leqslant 1),$$

we have

$$\left|\int_1^\infty B_3(x-[x])x^{-s-3}dx\right| \leqslant \frac{\sqrt{3}}{36(\sigma+2)} \leqslant \sqrt{3}/54$$

because $-1/2\leqslant\sigma\leqslant 5/2$. Thus

$$\left|2(s-1)s(s+1)\int_1^\infty B_3(x-[x])x^{-s-3}dx\right| \leqslant 35\sqrt{3}/72,$$

so

$$|(s-1)\zeta(s)| \geqslant (1/8)(5/2-35\sqrt{3}/72) > .2072$$

on the circle $|s-1|=3/2$. To maximize the numerator on the left hand side of (3), we differentiate (5) to get

$$(s-1)\zeta'(s)+\zeta(s) = \frac{d\big((s-1)\zeta(s)\big)}{ds} = (s+2)/12+(s+3)/12-$$

$$-\frac{(s-1)(s+1)(s+2)}{6}\int_1^\infty \frac{B_3(x-[x])}{x^{s+3}}dx-\frac{s(s+1)(s+2)}{6}\int_1^\infty \frac{B_3(x-[x])}{x^{s+3}}dx-$$

$$-\frac{(s-1)s(s+2)}{6}\int_1^\infty \frac{B_3(x-[x])}{x^{s+3}}dx-\frac{(s-1)s(s+1)}{6}\int_1^\infty \frac{B_3(x-[x])}{x^{s+3}}dx+$$

$$+\frac{(s-1)s(s+1)(s+2)}{6}\int_1^\infty \frac{B_3(x-[x])\log x}{x^{s+3}}dx \qquad (\sigma>-2).$$

If $|s-1|=3/2$, we have

$$\left|\int_1^\infty B_3(x-[x])x^{-s-3}\log x\,dx\right| \leqslant (\sqrt{3}/36)\int_1^\infty x^{-\sigma-3}\log x\,dx$$
$$= (\sqrt{3}/36)(\sigma+2)^{-2} \leqslant \sqrt{3}/81.$$

Thus

$$|(s-1)\zeta'(s)+\zeta(s)| \leqslant 9/24+11/24+(3/2)(7/2)(9/2)(\sqrt{3}/324)+$$
$$+(5/2)(7/2)(9/2)(\sqrt{3}/324)+$$
$$+(3/2)(5/2)(9/2)(\sqrt{3}/324)+$$
$$+(3/2)(5/2)(7/2)(\sqrt{3}/324)+$$
$$+(3/2)(5/2)(7/2)(9/2)(\sqrt{3}/486)$$
$$< 1.541.$$

So we may take

$$M = 7.44 > 1.541/.2072.$$

It is easy to show that if $n \geqslant 2$, then

$$(2^n-1)\zeta(n)/n > 2^n/n.$$

Thus for $n \geqslant 6$,

$$b_n = (2^n-1)\zeta(n)/n+2^n\beta_n$$
$$> 2^n/n-2^n(M/n)(2/3)^{n-1} = (2^n/n)(1-2^{n-1}M/3^{n-1})$$
$$\geqslant (64/6)\{1-32(7.44)/243\} > (64/6)(1-.98) > 0.$$

To show that b_2, b_3, b_4, and b_5 are positive, we will find numbers bounding $\beta_2, \beta_3, \beta_4$, and β_5. We know that $\{(s-1)\zeta(s)\}$ is an entire function of s, so we define

$$(s-1)\zeta(s) = 1+\sum_{n=0}^{\infty}(-1)^n\gamma_n(s-1)^{n+1}/n! = 1+y(s).$$

We know $\gamma_0 = \gamma = .577\ldots$ is Euler's constant, and Briggs [4] has computed $\gamma_1, \gamma_2, \gamma_3$, and γ_4 and asserted them to be approximately $-.073$, $-.516$, $-.147$, and $.002$ respectively. Now

$$y(s) = \gamma_0(s-1)-\gamma_1(s-1)^2+\tfrac{1}{2}\gamma_2(s-1)^3-\tfrac{1}{6}\gamma_3(s-1)^4+\tfrac{1}{24}\gamma_4(s-1)^5+\ldots,$$
$$y^2(s) = \gamma_0^2(s-1)^2-2\gamma_0\gamma_1(s-1)^3+(\gamma_0\gamma_2+\gamma_1^2)(s-1)^4-$$
$$-(\gamma_0\gamma_3/3+\gamma_1\gamma_2)(s-1)^5+\ldots,$$
$$y^3(s) = \gamma_0^3(s-1)^3-3\gamma_0^2\gamma_1(s-1)^4+(3\gamma_0^2\gamma_2/2+3\gamma_0\gamma_1^2)(s-1)^5+\ldots,$$
$$y^4(s) = \gamma_0^4(s-1)^4-4\gamma_0^3\gamma_1(s-1)^5+\ldots,$$
$$y^5(s) = \gamma_0^5(s-1)^5+\ldots$$

Thus

$$\sum_{n=1}^{\infty}(-1)^n\beta_n(s-1)^n = \log\{(s-1)\zeta(s)\} = \log\{1+y(s)\}$$
$$= y(s)-y^2(s)/2+y^3(s)/3-y^4(s)/4+y^5(s)/5+\ldots$$
$$= \gamma_0(s-1)-(\gamma_0^2/2+\gamma_1)(s-1)^2+(\gamma_0^3/3+\gamma_0\gamma_1+\gamma_2/2)(s-1)^3-$$
$$-(\gamma_0^4/4+\gamma_0^2\gamma_1+\gamma_0\gamma_2/2+\gamma_1^2/2+\gamma_3/6)(s-1)^4+$$
$$+(\gamma_0^5/5+\gamma_0^3\gamma_1+\gamma_0\gamma_1^2+\gamma_0^2\gamma_2/2+\gamma_1\gamma_2/2+\gamma_0\gamma_3/6+\gamma_4/24)(s-1)^5+\ldots$$

Unfortunately it has been found that Briggs' estimates for the γ_n's are incorrect. Using the Euler-Maclaurin sum formula and an IBM 1620, we found that

$$-.0729 < \gamma_1 < -.0728,$$
$$-.0098 < \gamma_2 < -.00968,$$
$$.000 < \gamma_3 < .003,$$
$$.000 < \gamma_4 < .003.$$

Recalling that

$$b_n = (2^n-1)\zeta(n)/n+2^n\beta_n,$$

we see easily that b_2, b_3, b_4, and b_5 are positive. However, we will need later a lower bound on b_3 and b_5, so we find that

$$\beta_3 > -.017244,$$
$$\beta_5 > -.00106,$$

which gives

$$b_3 > 2.6668 > 8/3,$$

and

$$b_5 > 6.395.$$

A trivial upper bound on b_n is also needed later, but clearly

$$b_n < 2^{n+1}.$$

2. The Epstein zeta function. Bateman and Grosswald showed that

(6) $$k^{-1/2}(ak/\pi)^s\Gamma(s)Z(s) = f(s)+f(1-s)+H(s;\,a,b,c),$$

where

$$f(s) = k^{-1/2}(k/\pi)^s\Gamma(s)\zeta(2s)$$

and

$$H(s;\,a,b,c) = 4\sum_{n=1}^{\infty}n^{s-1/2}\sigma_{1-2s}(n)\cos(\pi bn/a)K_{s-1/2}(2\pi kn).$$

Here

$$k = d^{1/2}/(2a),$$

$K_\nu(z)$ is a Bessel function, and $\sigma_\nu(n) = \sum_{m|n} m^\nu$. We are interested in these functions primarily for $0 < s < 1$. In this interval $f(s)+f(1-s)$ has a removable singularity at $s = \frac{1}{2}$ and takes the value $-\alpha(a, d)$ there. We shall ignore the error term $H(s; a, b, c)$ for the moment and concentrate on the main term $f(s)+f(1-s)$. Clearly

$$f(s) = \zeta(2s)\exp\left\{(s-\tfrac{1}{2})\log(k/\pi)+\log\frac{\Gamma(s)}{\Gamma(\frac{1}{2})}\right\}$$

$$= \tfrac{1}{2}(s-\tfrac{1}{2})^{-1}\exp\left\{\log\big((2s-1)\zeta(2s)\big)+(s-\tfrac{1}{2})\log(k/\pi)+\log\frac{\Gamma(s)}{\Gamma(\frac{1}{2})}\right\}.$$

If $|s-\frac{1}{2}| < \frac{1}{2}$, then using the partial fraction decomposition for the derivatives of $\Gamma'(z)/\Gamma(z)$, we have

$$\log\frac{\Gamma(s)}{\Gamma(\frac{1}{2})} = \frac{\Gamma'(\frac{1}{2})(s-\frac{1}{2})}{\Gamma(\frac{1}{2})}+\sum_{n=2}^{\infty}\frac{(s-\frac{1}{2})^n}{n!}\sum_{j=0}^{\infty}\frac{(-1)^n(n-1)!}{(j+\frac{1}{2})^n}$$

$$= (\gamma+\log 4)(\tfrac{1}{2}-s)+\sum_{n=2}^{\infty}(2^n-1)\zeta(n)(\tfrac{1}{2}-s)^n/n.$$

Since we defined

$$\log\{(s-1)\zeta(s)\} = \sum_{n=1}^{\infty}(-1)^n\beta_n(s-1)^n,$$

we have

$$\log\{(2s-1)\zeta(2s)\} = \sum_{n=1}^{\infty}2^n\beta_n(\tfrac{1}{2}-s)^n.$$

Thus

$$f(s) = -\tfrac{1}{2}(\tfrac{1}{2}-s)^{-1}\exp\left((\tfrac{1}{2}-s)\log\frac{4\pi}{e^\gamma k}+\sum_{n=2}^{\infty}\{(2^n-1)\zeta(n)/n+2^n\beta_n\}(\tfrac{1}{2}-s)^n\right)$$

because $\beta_1 = -\gamma$. Recalling that we defined

$$b_n = (2^n-1)\zeta(n)/n+2^n\beta_n$$

and

$$\alpha(a) = \alpha(a, d) = \log a+\log(8\pi e^{-\gamma})-\tfrac{1}{2}\log d = \log\frac{4\pi}{e^\gamma k},$$

if we put

$$r = \tfrac{1}{2}-s,$$

we get

$$f(s) = -\tfrac{1}{2}r^{-1}\exp\left\{r\alpha(a)+\sum_{n=2}^{\infty}b_nr^n\right\}$$

and

$$f(1-s) = \tfrac{1}{2}r^{-1}\exp\left\{-r\alpha(a)+\sum_{n=2}^{\infty}(-1)^nb_nr^n\right\}.$$

Setting

$$m(r) = \sum_{n=1}^{\infty}b_{2n+1}r^{2n},\qquad l(r) = \exp\Big(\sum_{n=1}^{\infty}b_{2n}r^{2n}\Big),$$

and

$$g(r) = g(r; a, d) = r\{\alpha(a, d)+m(r)\},$$

we have

$$f(s) = -\tfrac{1}{2}r^{-1}\exp\left\{r\alpha(a)+rm(r)+\sum_{n=1}^{\infty}b_{2n}r^{2n}\right\} = -\tfrac{1}{2}r^{-1}l(r)\exp\{g(r)\}$$

and

$$f(1-s) = \tfrac{1}{2}r^{-1}\exp\left(-r\alpha(a)-rm(r)+\sum_{n=1}^{\infty}b_{2n}r^{2n}\right) = \tfrac{1}{2}r^{-1}l(r)\exp\{-g(r)\}.$$

Thus

$$f(s)+f(1-s) = -\tfrac{1}{2}r^{-1}l(r)\{e^{g(r)}-e^{-g(r)}\} = -r^{-1}l(r)\sinh\{g(r)\}.$$

Define

$$h(r) = h(r; a, d) = 1+\frac{g^2(r)}{3!}+\frac{g^4(r)}{5!}+\ldots = 1+\sum_{n=1}^{\infty}\frac{g^{2n}(r)}{(2n+1)!}.$$

Then

$$f(s)+f(1-s) = -r^{-1}l(r)\{g(r)h(r; a, d)\} = -l(r)\{\alpha(a, d)+m(r)\}h(r; a, d). \tag{7}$$

From Theorem 1 we conclude that $l(r) \geqslant 1$, $m(r) \geqslant 0$, and $h(r; a, d) \geqslant 1$.

Since $k = d^{1/2}/(2a)$, we have

$$k^{-1/2}(ak/\pi)^s\Gamma(s) = k^{s-1/2}a^s\pi^{-s}\Gamma(s) = a^{1/2}\{d^{s/2-1/4}2^{1/2-s}\pi^{-s}\Gamma(s)\} = a^{1/2}C(d, s). \tag{8}$$

Now $C(d, s) > 0$ if $0 < s < 1$ and depends only on d and s and not on a, b, or c. Furthermore $C(d, \frac{1}{2}) = 1$.

Putting (8) back into (6), we have

(9) $$C(d,s)Z(s) = a^{-1/2}\{f(s)+f(1-s)+H(s;a,b,c)\}.$$

We are now ready to prove

THEOREM 2. *For a fixed fundamental discriminant $-d$, if*

(i) $$\sum_Q a^{-1/2}\alpha(a) \geqslant 0,$$

(ii) $$\sum_Q a^{-1/2}\{\alpha(a)-H(s;a,b,c)\} > 0 \quad (0<s<1),$$

(iii) $$\sum_Q a^{-1/2}\left\{b_{2n+1}+\frac{\alpha^{2n+1}(a)}{(2n+1)!}\right\} \geqslant 0 \quad (n\geqslant 1),$$

then

$$L_{-d}(s) > 0 \quad (s>0).$$

Proof. Suppose (iii) holds. Then

$$\sum_{n=1}^{\infty} r^{2n}\sum_Q a^{-1/2}\left\{b_{2n+1}+\frac{\alpha^{2n+1}(a)}{(2n+1)!}\right\} \geqslant 0$$

or

$$\sum_Q a^{-1/2}\sum_{n=1}^{\infty} r^{2n}\left\{b_{2n+1}+\frac{\alpha^{2n+1}(a)}{(2n+1)!}\right\} \geqslant 0.$$

If $|r|<\frac{1}{2}$, then the series converge absolutely, since $b_n < 2^{n+1}$. Hence

$$0 \leqslant \sum_Q a^{-1/2}\left(m(r)+\sum_{n=1}^{\infty}\frac{r^{2n}\alpha^{2n+1}(a)}{(2n+1)!}\right)$$

$$\leqslant \sum_Q a^{-1/2}\left(m(r)+\sum_{n=1}^{\infty}\frac{r^{2n}\{\alpha(a)+m(r)\}^{2n+1}}{(2n+1)!}\right)$$

because $m(r)\geqslant 0$ and y^{2n+1} increases with y. Thus

$$0 \leqslant \sum_Q a^{-1/2}\left(\sum_{n=0}^{\infty}\frac{r^{2n}\{\alpha(a)+m(r)\}^{2n+1}}{(2n+1)!}-\alpha(a)\right)$$

$$= \sum_Q a^{-1/2}\left(\{\alpha(a)+m(r)\}\left\{1+\sum_{n=1}^{\infty}\frac{r^{2n}(\alpha(a)+m(r))^{2n}}{(2n+1)!}\right\}-\alpha(a)\right).$$

Combining this with (i), we obtain

$$\sum_Q a^{-1/2}\{\alpha(a)+m(r)\}h(r;a,d) \geqslant \sum_Q a^{-1/2}\alpha(a) \geqslant 0,$$

so that

$$\sum_Q a^{-1/2}l(r)\{\alpha(a)+m(r)\}h(r;a,d) \geqslant \sum_Q a^{-1/2}\alpha(a)$$

because $l(r)\geqslant 1$ and does not depend on Q. Note that to multiply only the left hand side by $l(r)$, it must be non-negative; hence, the requirement of (i). (The exceptional cases listed later in Theorem 5 do not satisfy (i).) Thus, from (7),

$$\sum_Q a^{-1/2}\{f(s)+f(1-s)\} \leqslant -\sum_Q a^{-1/2}\alpha(a),$$

so that

$$\sum_Q a^{-1/2}\{f(s)+f(1-s)+H(s;a,b,c)\} \leqslant -\sum_Q a^{-1/2}\{\alpha(a)-H(s;a,b,c)\}.$$

This is negative by (ii), so by (9)

$$C(d,s)\sum_Q Z(s;a,b,c) < 0.$$

Since $C(d,s)>0$ and $\zeta(s)<0$ in the unit interval, we have

$$\sum_Q Z(s;a,b,c) < 0,$$

and from the introduction

$$\zeta(s)L_{-d}(s) = \sum_Q Z(s;a,b,c) < 0$$

or

$$L_{-d}(s) > 0 \quad (0<s<1).$$

It is well known that $L_{-d}(s)>0$ if $s\geqslant 1$, so

$$L_{-d}(s) > 0 \quad (s>0).$$

Since $C(d,\frac{1}{2})=1$ and $\zeta(\frac{1}{2})$ is about -1.460, we can approximate $L_{-d}(\frac{1}{2})$ by evaluating

$$\sum_Q a^{-1/2}\{\alpha(a)-H(\tfrac{1}{2};a,b,c)\}.$$

Because $L_{-d}(\frac{1}{2})$ can be estimated by directly using the definition of $L_{-d}(\frac{1}{2})$, for any particular d we can find an independent verification for our calculations.

THEOREM 3. *If* $d < 593000$ *and condition* (i) *of Theorem* 2 *holds, then* (iii) *holds.*

Proof. We will divide the proof into cases depending on n.

Case I. $n = 1$. Here we want to show that

$$\sum_Q a^{-1/2}\{b_3+\alpha^3(a)/6\} \geqslant 0.$$

First let us work with a cubic equation.

LEMMA 1. *If*

$$F_1(t) = t^3/6-2t+b_3$$

and $t \geqslant -4$, *then* $F_1(t) > 0$.

Proof. The local minimum of $F_1(t)$ is taken at $t = 2$, where the value is

$$b_3-8/3.$$

This is positive since $b_3 > 8/3$. Thus, $F_1(t)$ has only one real root, and since

$$F_1(-4) = -32/3+8+b_3 > 0,$$

the lemma follows.

If $d \leqslant 593000$, then

$$\alpha(a, d) = \log a+\log(8\pi e^{-\gamma})-\tfrac{1}{2}\log d > 0+2.6469-6.6466 > -4.$$

Thus, for $d \leqslant 593000$,

$$b_3+\alpha^3(a, d)/6-2\alpha(a, d) > 0$$

by Lemma 1. Multiplying by $a^{-1/2}$, summing over Q, and transposing gives

$$\sum_Q a^{-1/2}\{b_3+\alpha^3(a, d)/6\} > 2\sum_Q a^{-1/2}\alpha(a, d).$$

But this last quantity is non-negative by condition (i), so the desired result follows.

Case II. $n = 2$. We need to show that

$$\sum_Q a^{-1/2}\{b_5+\alpha^5(a, d)/120\} \geqslant 0.$$

LEMMA 2. *If*

$$F_2(t) = t^5/120-2.5t+b_5$$

and $t \geqslant -4.6$, *then* $F_2(t) > 0$.

Proof. The local minimum of $F_2(t)$ occurs at $t = (60)^{1/4}$. This is positive since $b_5 > 6.395$. So $F_2(t)$ has only one real root, and since

$$F_2(-4.6) > -17.164+11.5+6.395 > 0,$$

the lemma follows.

If $d \leqslant 1970000$ then

$$\alpha(a, d) > 0+2.6469-7.2468 > -4.6.$$

Thus, for $d \leqslant 1970000$, we have

$$b_5+\alpha^5(a, d)/120-2.5\,\alpha(a, d) > 0$$

by Lemma 2. So we see, as in Case I, that

$$\sum_Q a^{-1/2}\{b_5+\alpha^5(a, d)/120\} > 2.5\sum_Q a^{-1/2}\alpha(a, d) > 0,$$

which completes Case II.

Case III. $n \geqslant 3$. We must show that

$$\sum_Q a^{-1/2}\{b_{2n+1}+\alpha^{2n+1}(a, d)/(2n+1)!\} \geqslant 0 \qquad (n \geqslant 3).$$

Now we find as in the proof of Theorem 1 that

$$b_{2n+1} > \frac{2^{2n+1}}{2n+1}\{1-2^{2n}M/3^{2n}\} \geqslant (2^7/7)\{1-2^6M/3^6\}$$
$$= (128/7)\{1-64(7.44)/729\} > 6.34.$$

If $d < 1320000$ then

$$\alpha(a, d) > 0+2.6469-7.0466 > -4.4.$$

If $\alpha(a, d) \geqslant 0$, then

$$b_{2n+1}+\alpha^{2n+1}(a, d)/(2n+1)!$$

is trivially positive, and if $-4.4 < \alpha(a, d) < 0$, then

$$b_{2n+1}+\alpha^{2n+1}(a, d)/(2n+1)! \geqslant b_7+\alpha^7(a, d)/7! \geqslant 6.34-6.335 > 0,$$

which completes Case III and Theorem 3.

We see that Cases II and III in the proof of Theorem 3 hold for $d < 1320000$. Thus, if condition (i) of Theorem 2 holds and if $593000 < d < 1320000$, to verify that condition (iii) holds, we need only to show that

$$\sum_Q a^{-1/2}\{b_3+\alpha^3(a, d)/6\} \geqslant 0.$$

We did this in Case I by showing that every summand was non-negative. But it is the sum that we want to be positive, and if we know the set Q for a particular d, we can verify that the sum is positive. Thus, for $d = 115147$ and $d = 636184$, we show that condition (iii) holds in spite of the fact that condition (i) does not hold for these d's.

3. The error term $H(s; a, b, c)$. We recall that the error term was defined to be

$$H(s; a, b, c) = 4\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(\pi nb/a)K_{s-1/2}(2\pi kn).$$

Now Bateman and Grosswald proved that if $0 < s < 1$, then

$$1-\frac{1-4(s-\frac{1}{2})^2}{16\pi nk} \leqslant 2(kn)^{1/2}e^{2\pi kn}K_{s-1/2}(2\pi kn) \leqslant 1,$$

$$n^{s-1/2}\sigma_{1-2s}(n) \leqslant n^{1/2}\sigma_{-1}(n),$$

and

$$\sigma_{-1}(n) \leqslant 3(n-1)/2 \qquad (n \geqslant 2).$$

From this we will show

THEOREM 4. *If* $0 < s < 1$, *then*

$$H(s; a, b, c) \leqslant \frac{2\cos(\pi b/a)}{k^{1/2}e^{2\pi k}} + 0.04\frac{|2\cos(\pi b/a)|}{k^{1/2}e^{2\pi k}}.$$

Proof. If we write out the first two terms in the series for H and estimate the rest, using the above upper bounds, we get

$$H(s) = 4\cos(\pi b/a)K_{s-1/2}(2\pi k) + 4(2^{s-1/2}+2^{1/2-s})\cos(2\pi b/a)K_{s-1/2}(4\pi k) + \\ + 3\theta k^{-1/2}e^{-6\pi k}(1-e^{-2\pi k})^{-2}(2-e^{-2\pi k}),$$

where $|\theta| < 1$. Clearly

$$\begin{aligned} &4\cos(\pi b/a)K_{s-1/2}(2\pi k) \\ &\quad \leqslant 2\cos(\pi b/a)k^{-1/2}e^{-2\pi k} + |2\cos(\pi b/a)|\,|2K_{s-1/2}(2\pi k) - k^{-1/2}e^{-2\pi k}| \\ &\quad \leqslant \frac{2\cos(\pi b/a)}{k^{1/2}e^{2\pi k}} + \frac{|2\cos(\pi b/a)|}{k^{1/2}e^{2\pi k}}\cdot\frac{1-4(s-\frac{1}{2})^2}{16\pi k} \\ &\quad \leqslant \frac{2\cos(\pi b/a)}{k^{1/2}e^{2\pi k}} + \frac{1}{40}\cdot\frac{|2\cos(\pi b/a)|}{k^{1/2}e^{2\pi k}}, \end{aligned}$$

because $k = d^{1/2}/(2a) \geqslant \sqrt{3}/2$. To treat the second and succeeding terms consider two cases.

First suppose $0 \leqslant |b|/a \leqslant 1/3$ or $2/3 \leqslant |b|/a \leqslant 1$. Then

$$|\cos(2\pi b/a)| \leqslant |\cos(\pi b/a)|$$

and so

$$\begin{aligned} |4(2^{s-1/2}+2^{1/2-s})\cos(2\pi b/a)K_{s-1/2}(4\pi k)| &\leqslant 6\sqrt{2}|\cos(\pi b/a)|e^{-4\pi k}(8k)^{-1/2} \\ &= (3/2)e^{-2\pi k}|2\cos(\pi b/a)|k^{-1/2}e^{-2\pi k} < (1/100)|2\cos(\pi b/a)|k^{-1/2}e^{-2\pi k}. \end{aligned}$$

Also

$$\begin{aligned} &3\theta k^{-1/2}e^{-6\pi k}(1-e^{-2\pi k})^{-2}(2-e^{-2\pi k}) \\ &\quad < 7e^{-4\pi k}k^{-1/2}e^{-2\pi k} \leqslant 7e^{-4\pi k}|2\cos(\pi b/a)|k^{-1/2}e^{-2\pi k} \\ &\quad < (1/200)|2\cos(\pi b/a)|k^{-1/2}e^{-2\pi k}, \end{aligned}$$

since $|2\cos(\pi b/a)| \geqslant 1$. Thus the desired result follows in this case.

If $1/3 \leqslant |b|/a \leqslant 2/3$, we have $\cos(2\pi b/a) \geqslant -\frac{1}{2}$, so

$$\begin{aligned} &4(2^{s-1/2}+2^{1/2-s})\cos(2\pi b/a)K_{s-1/2}(4\pi k) \\ &\quad \leqslant -4K_{s-1/2}(4\pi k) \leqslant -4\left(1-\frac{1}{32k\pi}\right)\frac{e^{-4\pi k}}{(8k)^{1/2}} < -k^{-1/2}e^{-4\pi k}. \end{aligned}$$

Here

$$\begin{aligned} &3\theta k^{-1/2}e^{-6\pi k}(1-e^{-2\pi k})^{-2}(2-e^{-2\pi k}) \\ &\quad < (7e^{-2\pi k})(k^{-1/2}e^{-4\pi k}) < (1/10)k^{-1/2}e^{-4\pi k}. \end{aligned}$$

Thus the desired result follows in this case since the sum of the second and succeeding terms is negative. This completes the proof of Theorem 4.

We can now apply Theorem 4 to find a numerical bound for $H(s; a, b, c)$. We really have condition (ii) of Theorem 2 in mind when we prove

COROLLARY. *For a fixed* d,

$$\sum_{Q} a^{-1/2}\{\alpha(a) - H(s; a, b, c)\} > \sum_{Q} a^{-1/2}\{\alpha(a) - .005\}.$$

Proof. From Theorem 4 we have

$$-H(s; a, b, c) \geqslant -\frac{2\cos(\pi b/a)}{k^{1/2}e^{2\pi k}} - \frac{.04|2\cos(\pi b/a)|}{k^{1/2}e^{2\pi k}}.$$

If $|b| \geqslant a/2$, then $-\cos(\pi b/a) > 0$ and so $-H(s; a, b, c) > 0$. Thus

$$\alpha(a) - H(s; a, b, c) > \alpha(a) > \alpha(a) - .005.$$

If $|b| \leqslant a/2$, then

$$d = 4ac - b^2 \geqslant 4a^2 - a^2/4 = 15a^2/4.$$

Thus

$$a \leqslant 2d^{1/2}(15)^{-1/2}.$$

Since $k = d^{1/2}/(2a)$, we would have

$$k \geqslant (15)^{1/2}/4.$$

Hence

$$-H(s; a, b, c) \geqslant -(1.04)2\big((15)^{1/2}/4\big)^{-1/2}\exp\big(-2\pi(15)^{1/2}/4\big) > -.005.$$

So

$$\alpha(a) - H(s; a, b, c) > \alpha(a) - .005.$$

In either case

$$\alpha(a) - H(s) > \alpha(a) - .005.$$

Multiplying by $a^{-1/2}$ and summing over Q then gives the desired result.

Unfortunately the programs for the IBM 7094 used to verify conditions (i) and (ii) of Theorem 2 were written before Theorem 4 and its corollary were proved in the present form. The program was based on a previous weaker form of the corollary in which .009 played the role now occupied by .005. The number .009 was derived from an estimate of $H(s; a, b, c)$ in Bateman and Grosswald's paper. However the lower bound $-.005 < -H(s; a, b, c)$ is still not sufficient for all d values. The more precise estimation of Theorem 4 is still required.

4. The main program. The programs written for the IBM 7094 at the University of Illinois were to check that conditions (i) and (ii) of Theorem 2 held if $1 < d < 1000000$. We want to convince you that the programs were adequate and that the IBM 7094 — hereafter referred to as the machine — did its job.

The main program was designed so that the machine could verify that

$$\sum_Q a^{-1/2}\{\alpha(a, d) - .009\} \geqslant 0. \tag{10}$$

From the corollary of Theorem 4 we see that if (10) is valid for a particular d, then conditions (i) and (ii) of Theorem 2 hold and thus, by Theorem 3, $L_{-d}(s) > 0$. Clearly it would not be practical to print out the sum in (10) for all the d's we work with, so if the machine found (10) was valid for a particular d, then it printed out nothing. If it found that (10) was false, then that d value and its associated sum from (10) were printed out.

Let us see how the program was written to allow the machine to verify (10). The heart of the problem here is to find the elements of the set Q to sum over. We need to digress here to explain how this may be done and the problems that arise from it pertaining to the writing of the program.

Now Q is the set of reduced integral solutions of $d = 4ac - b^2$. Furthermore, $-d$ is a fundamental discriminant. This means that d is not divisible by the square of an odd prime, and if d is even, then

$$d \equiv 4 \text{ or } 8 \pmod{16}.$$

Since $|b| \leqslant a \leqslant (d/3)^{1/2}$, an easy way to look for Q is to write

$$x(b, d) = (d + b^2)/4 = ac.$$

We fix d and then factor $x(b, d) = x(b)$ into a product of two integers, taking the smaller factor as a if it is as large as $|b|$ and ignoring the factorization otherwise. Since $|b|$ has the same parity as d, $|b|$ would range over the even or the odd integers from 0 to $[(d/3)^{1/2}]$. This method, while fine for hand computation, unhappily requires completely factoring about $\frac{1}{2}(d/3)^{1/2}$ integers for each d, which is very time-consuming for the machine. To best utilize the machine, we need to use the fact that x is a function of both b and d.

The basic idea is to consider many d's simultaneously so we can reduce the number of factorizations needed. How many d's we can consider at one time depends on the number of words the core memory of the machine can hold. The machine can store $2^{15} = 32768$ words, so we decided to consider $6144 = (3/8)2^{14}$ d's simultaneously. With each d we stored $\frac{1}{2}\log d$ for computational purposes and a partial sum of $\sum a^{-1/2}\{\alpha(a) - .009\}$. If we think of these forming tables in the memory, then these three tables alone took 18432 positions of memory. Now d is such that

$$d \equiv 3, 4, 7, 8, 11, \text{ or } 15 \pmod{16}.$$

In the machine computation we consider all d satisfying the above congruence whether or not $-d$ is a fundamental discriminant. The value computed for (10) in the case $-d$ is not a fundamental discriminant is irrelevant. Thus the numerical difference between the first and the last d's stored in the table in memory was less than $2^{14} = 16384$. Since $x = (d + b^2)/4$, we want the d's in the table to depend on x. In particular, for a given x, we want the d's in the table to be such that

$$4x - 2^{14} < d \leqslant 4x,$$

or

$$4ac - (2^7)^2 < d \leqslant 4ac - 0^2.$$

Suppose we let

$$\sum_Q a^{-1/2}\{\alpha(a) - .009\} = \sum_{Q_1} a^{-1/2}\{\alpha(a) - .009\} + \sum_{Q_2} a^{-1/2}\{\alpha(a) - .009\},$$

where

$$Q_1 = \{(a, b, c) |\ (a, b, c) \in Q \text{ and } |b| < 128 = 2^7\}$$

and $Q_2 = Q - Q_1$. As x runs through consecutive integers, any particular d will eventually enter and leave the table of d's in the memory. While it is in the table, the set Q_1 will be found as solutions of $d = 4ac - b^2$, and for each triple found as a solution, $a^{-1/2}\{a(a) - .009\}$ will be added to the partial sum of (10) associated with this d. Thus by the time a particular d is ready to leave the table, the associated partial sum is

$$\sum_{Q_1} a^{-1/2}\{a(a) - .009\}.$$

As x moves to $x^* = x+1$, those d's ready to drop out of the table are $d = 4x^* - 2^{14} - 1$ and (if it is in the table) $d = 4x^* - 2^{14}$. The new d's ready to go into the table are $d = 4x^* - 1$ and (possibly) $d = 4x^*$. Since $4x^* \equiv 4x^* - 2^{14} \pmod{16}$, the same number of d's go into the table as come out. The incoming d's were now checked for divisibility by 3^2, 5^2, or 7^2. If they were divisible by one of these numbers, then we stored 0 in place of them. Hence about 98% of the d values actually checked by the machine were negatives of fundamental discriminants.

When a particular d is ready to be dropped out of the table we look at its associated partial sum, which is the sum over Q_1. If this was negative, we found the triples in Q_2 and added $a^{-1/2}\{a(a) - .009\}$ to the partial sum for each triple found. If the full sum (i.e., the sum over Q_1 and Q_2) was still negative, then d and its associated full sum were printed out. If the partial sum was positive, or became positive upon adding summands associated with Q_2, then d was dropped from the table with no printout.

It is important to verify that indeed, (10) holds if d is dropped with no printout. This is so because the summands associated with the elements of Q_2 are all positive if $d < 3000000$. To see this we note that since $a \geqslant |b| \geqslant 128$, we have

$$\begin{aligned} a(a) - .009 &= \log a + \log(8\pi e^{-\gamma}) - \tfrac{1}{2}\log d - .009 \\ &> 4.852 + 2.646 - 7.039 - .009 = .450 > 0. \end{aligned}$$

The reason that factoring is so time-consuming for the machine is because it requires a large number of divisions. The machine factors by a sieve process; it divides an integer x by successive primes from 2 to $x^{1/2}$. In the example given later, we list the number of divisions required by the machine to factor each x. Once we have factored x, it is easy to find the divisors of x not exceeding $x^{1/2}$ which will serve as our possible a values. Suppose we have found a reduced solution (a, b, c) of $d = 4ac - b^2$. Then $(a, -b, c)$ is also a reduced solution if $0 < |b| < a < c$. If $b = 0$ or $b = a$ or $a = c$, then $(a, -b, c)$ is not a distinct reduced solution. The divisors of x which will give rise to possible reduced solutions are those numbers a such that $b \leqslant a \leqslant x^{1/2}$. Furthermore, each a value in this range gives two solutions unless $a = b$ or $a = x^{1/2} = c$, in which case there is only one solution. Also if $b = 0$ there is only one solution. Thus we were careful to handle these cases separately.

Checks on the program and the machine calculations were made in various ways. We followed through the program by hand. We got memory dumps on the debugging runs and we could actually see the factors, divisors, and d tables in the memory. Since the program is entirely accurate for d's $\leqslant 49152$ (i.e., a full sum of the form in (10) is computed), we had every d and its associated sum printed out if $d < 32000$. Thus for various small d's, the computations were done by hand and checked against the machine's sums. There were also internal checks. For example, an integer x is a perfect square if and only if the number of its divisors is odd. In this case the largest value of a will be $x^{1/2}$, and this was checked internally against $x^{1/2}$ computed in another way. This was a useful check because we found a bug in the program that did not show up until the machine was processing d's > 100000.

About two hours of machine time was used in debugging runs until we were convinced the program ran perfectly. On production runs, the machine took about eight minutes to process all d's < 100000, but by the time the d values got up around 400000, it was only processing an interval of about 40000 every twenty minutes. This was because for nearly every d of this size or larger, extra solutions had to be found. In all, around ten hours of machine time were spent on production runs and we eventually ran the main program for all $d < 1200000$.

The machine carried eight-place accuracy at all times and this was more than enough; six-place accuracy would have been sufficient. The vital constant

$$\log(8\pi e^{-\gamma}) = 2.64695576\ldots$$

was found by tables and read into the machine. The program with its tables used over 32000 words in the memory, so we utilized the machine to full advantage.

Example: $d = 17923$. We have constructed an example which we thought to be interesting and instructive because $d = 17923$ and $(d+1)/4 = 4481$ are primes. Our basic idea is to consider $x = ac = (d+b^2)/4$ as a function of b to find all possible a values. Since d is odd, then b must be odd. Then to get from one x value to the next we simply add $b+1$ to the first value because

$$\frac{d+(b+2)^2}{4} - \frac{d+b^2}{4} = b+1.$$

Since $b^2 \leqslant a^2 \leqslant (d/3)$, our x values should range from $\frac{1}{4}(d+1)$ to $[\frac{1}{4}(d+d/3)] = [d/3]$, or in this case from 4481 to 5974. Although this example is atypical from a factoring viewpoint, it emphasises how the factorization problem affected the writing of the program. Hence the first column gives the number of divisions required for the machine to factor the number x. The second column gives the appropriate b value; the third column x and its factorization; and the fourth column the a values if there are any. Here p is a prime.

Number of divisions	b	x	a
19	1	$4481 = \mathrm{p}$	1
19	3	$4483 = \mathrm{p}$	
11	5	$4487 = 7\cdot 641$	7
20	7	$4493 = \mathrm{p}$	
11	9	$4501 = 7\cdot 643$	
9	11	$4511 = 13\cdot 347$	13
20	13	$4523 = \mathrm{p}$	
9	15	$4537 = 13\cdot 349$	
11	17	$4553 = 29\cdot 157$	29
11	19	$4571 = 7\cdot 653$	
20	21	$4591 = \mathrm{p}$	
11	23	$4613 = 7\cdot 659$	
20	25	$4637 = \mathrm{p}$	
20	27	$4663 = \mathrm{p}$	
20	29	$4691 = \mathrm{p}$	
20	31	$4721 = \mathrm{p}$	
7	33	$4753 = 7^2\cdot 97$	49
20	35	$4787 = \mathrm{p}$	
8	37	$4823 = 7\cdot 13\cdot 53$	53
20	39	$4861 = \mathrm{p}$	
8	41	$4901 = 13^2\cdot 29$	
20	43	$4943 = \mathrm{p}$	
20	45	$4987 = \mathrm{p}$	
11	47	$5033 = 7\cdot 719$	
21	49	$5081 = \mathrm{p}$	
11	51	$5131 = 7\cdot 733$	
21	53	$5183 = 71\cdot 73$	71
21	55	$5237 = \mathrm{p}$	
20	57	$5293 = 67\cdot 79$	67
22	59	$5351 = \mathrm{p}$	
11	61	$5411 = 7\cdot 773$	
10	63	$5473 = 13\cdot 421$	
7	65	$5537 = 7^2\cdot 113$	
10	67	$5603 = 13\cdot 431$	
17	69	$5671 = 53\cdot 107$	
22	71	$5741 = \mathrm{p}$	
22	73	$5813 = \mathrm{p}$	
12	75	$5887 = 7\cdot 29^2$	
20	77	$5963 = 67\cdot 89$	

For the machine to check just this one d value would involve 612 divisions to perform the necessary factorizations. Now $a = b$ only when $a = 1$, so we have further analysis as follows:

a	b	$2a^{-1/2}\alpha(a)$
1	1	-2.2500
7	± 5	$-.2299$
13	± 11	.1747
29	± 17	.4150
49	± 33	.4689
53	± 37	.4724
71	± 53	.4778
67	± 57	.4773
		.0062

The heading of the third column is $2a^{-1/2}\alpha(a)$, where it is understood that the 2 is used only if the corresponding number in the b column has a double sign. Thus we see in this case that

$$\sum_{Q} a^{-1/2}\alpha(a) = .0062.$$

We have given no error terms because we see that $|b| > \frac{1}{2}a$ in every case, so by Theorem 4 all the error terms are positive.

The secondary program. At first we believed that the crude estimate for the error term, $-H(s; a, b, c) > -.009$, would be sufficient because we had checked all d's < 1700 by hand and it was quite good enough. When we later put it on the machine, 17923 was the first d value for which this estimate was not sufficient. Thus we wrote a second program to check further the d's printed out by the main program. The loop to do the factoring and to find the divisors was lifted intact from the main program. We checked the d value to make sure $-d$ was a fundamental discriminant. With each summand $a^{-1/2}\alpha(a)$, we also computed

$$-a^{-1/2}\cos(\pi b/a)k^{-1/2}e^{-2\pi k}. \tag{11}$$

There were five columns in the printout of the secondary program. These showed a, $|b|$, $\delta a^{-1/2}\alpha(a)$, δ times the quantity in (11), and $\delta a^{-1/2}$ (.009), where $\delta = 1$ if $a = b$ or $a = c$ or $b = 0$, and $\delta = 2$ otherwise. The sums of the last three columns were also given, so by subtracting the sum of the fifth column from the sum of the third column, we could check the results of the small program with the results of the main program. Unfortunately we did not carry along the sum of the absolute values of the quantities (11), so we had to compute this by hand when it was necessary. This sum of absolute values was needed to apply Theorem 4 to show that condition (ii) holds. Thus we show that

$$\sum_Q a^{-1/2}\{\alpha(a)-H(s;a,b,c)\}$$
$$> \sum_Q a^{-1/2}\left\{\alpha(a)-\frac{2\cos(\pi b/a)}{k^{1/2}e^{2\pi k}}-\frac{.04\,|2\cos(\pi b/a)|}{k^{1/2}e^{2\pi k}}\right\} > 0$$

by a numerical evaluation of the last sum. This method shows that condition (ii) holds in all cases left in doubt by the main program for all $d < 1000000$.

Since the factor loop was the same in both programs, another check was provided on the accuracy of the programs and the machine, because the secondary program printed out all the a values and the corresponding values of $a^{-1/2}\alpha(a)$ which could then be checked by hand.

5. Conclusion. When we apply the results of the programs — using Theorem 4 whenever necessary — to Theorem 2, we find that we have proved

THEOREM 5. *If $-d$ is a fundamental discriminant and $d < 593000$, then $L_{-d}(s)$ has no real zeros for real $s > 0$ with the possible exception of $L_{-115147}(s)$.*

Now Theorem 2 fails to apply to $d = 115147$ and $d = 636184$ because condition (i) does not hold. However, condition (ii) does hold, as we shall see in the appendix. Now $\{f(s)+f(1-s)\}$ has a power series expansion in the even powers of $(\frac{1}{2}-s)$. Thus

$$f(s)+f(1-s) = -\alpha(a)-\{\alpha^3(a)/6+b_2\alpha(a)+b_3\}(\tfrac{1}{2}-s)^2-\ldots$$

For these exceptional cases we find that $\sum_Q a^{-1/2}\{f(s)+f(1-s)\}$ has a local maximum at $s=\frac{1}{2}$. Thus we can suspect strongly that Theorem 5 holds for these d's. Dr. Norman Hamilton of the University of Illinois wrote a program for the Iliac II to find $L_{-115147}(\frac{1}{2})$ by the definition of the L-series. Our value of .000071 compares favorably with his value of .000067.

For d's other than 636184 between 593000 and 1000000, condition (iii) undoubtedly holds, but we were not able to prove it by a general theorem.

Appendix. We analyze in this appendix the exceptions to Theorem 5, namely $d = 115147$ and $d = 636184$. As an item of interest we give $h(-d)$ which is the number of triples in Q and is the class number of the imaginary quadratic field with discriminant $-d$. We note from our example that $h(17923) = 15$. For comparison purposes, we state that $h(-571267) = 71$ and $h(-636307) = 75$. We recall that $\delta = 1$ if $a = b$ or $a = c$ or $b = 0$, and $\delta = 2$ otherwise, and that $k = d^{1/2}/(2a)$. The cases where $\delta = 1$ are indicated by asterisks.

$d = 115147$

a	$\|b\|$	$\delta a^{-1/2}\alpha(a)$	$-\delta a^{-1/2}k^{-1/2}e^{-2\pi k}2\cos(\pi b/a)$
1	1	−3.18002666*	.00000000*
11	1	−.47164297	.00000000
31	7	.09122532	.00000000
127	13	.29534072	−.00006590
67	19	.25036574	.00000002
37	21	.14167617	.00000000
71	21	.25697460	−.00000005
121	23	.29377525	−.00003788
103	25	.28667216	−.00000710
59	27	.23369190	.00000000
107	29	.28862927	−.00000953
47	35	.19549437	.00000000
97	63	.28321748	.00000234
109	75	.28951664	.00000967
149	103	.29884264	.00013564
113	113	.14556349*	.00001228*
179	147	.30007413	.00067354
		−.00060976	.00071297

Since $\zeta(\frac{1}{2}) \doteq -1.460355$, we see that the approximate value of $L_{-115147}(\frac{1}{2})$ is

$$(.00071297-.00060976)/1.460355 \doteq .000070675$$

while our possible error is

$$(1/1.460355)(.04)\sum_Q a^{-1/2}k^{-1/2}e^{-2\pi k}|2\cos(\pi b/a)| \doteq .00002613.$$

Thus $L_{-115147}(\frac{1}{2}) > 0$ and $h(-115147) = 32$.

For $d = 636184$, the sum of the main terms is $-.00023609$ while the sum of the error terms is .00285593. Thus

$$L_{-636184}(\tfrac{1}{2}) \doteq .00179398$$

with a potential error of .00022416 and therefore is positive. Also, $h(-636184) = 224$.

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On the number of integer points in the displaced circles

by

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Let a lattice of points with integer coordinates be given in the plane. Take a circle with the radius $\lambda^{1/2}$. Without loss of generality one can assume the centre of the circle to be a point $(u, v)\,\epsilon\, G$, where G is the domain defined by the inequalities:

$$0 \leqslant u < 1, \qquad 0 \leqslant v < 1.$$

Let $A(\lambda, u, v)$ denote the number of integer points inside the circle of the radius $\lambda^{1/2}$ with the centre in the point (u, v). Then it is easy to show that

$$A(\lambda, u, v) = \pi\lambda + P(\lambda, u, v),$$

where $P(\lambda, u, v) = O(\lambda^{\theta})$, $0 < \theta \leqslant 1/3$.

Kendall [4] proved that

$$\int_0^1\int_0^1 P^2(\lambda, u, v)\,du\,dv = \lambda\sum_{n=1}^{\infty}\frac{r(n)}{n}I_1^2(2\pi\sqrt{n\lambda}), \tag{1}$$

where $r(n)$ is the number of representations of the number n as the sum of two squares, $I_1(z)$ being Bessel's function.

By well known asymptotic behaviour of $I_1(z)$ it follows from (1) that

$$\int_0^1\int_0^1 P^2(\lambda, u, v)\,du\,dv = O(\lambda^{1/2}).$$

We shall show below that

$$\overline{\lim_{\lambda\to\infty}}\frac{P(\lambda, u, v)}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} > c > 0, \qquad \underline{\lim_{\lambda\to\infty}}\frac{P(\lambda, u, v)}{\lambda^{1/4}(\ln\ln\lambda)^{1/4-\varepsilon}} < -c < 0 \tag{2}$$

where ε is an arbitrarily small positive number, c is an absolute constant.