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On the coefficients of the zeta function of an imaginary quadratic field*

by

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§ 1. Introduction. Let $K = Q(\sqrt{D})$, $D < 0$ be an imaginary quadratic field of discriminant d and let $|d| = k$.

Let

$$\zeta_K(s) = \sum \frac{1}{N(\mathfrak{A})^s} = \sum_{n=1}^{\infty} \frac{F(n)}{n^s} \tag{1}$$

be the Dedekind zeta function of K where

$$F(n) = \sum_{N(\mathfrak{A})=n} 1. \tag{2}$$

It is known (see e.g. [1], Chap. V) that

$$\zeta_K(s) = \zeta(s)L(s, \chi_d) \tag{3}$$

and that

$$F(n) = \sum_{l|n} \chi_d(l)$$

where $\chi_d(n) = \left(\frac{d}{n}\right)$ = Kronecker symbol.

Let

$$H(x) = \sum_{n\leqslant x} F(n). \tag{4}$$

It is known [3] that

$$H(x) = ax + \Delta_k(x) \tag{5}$$

where a is the residue of $\zeta_K(s)$ at $s = 1$ and where $\Delta_k(x) = O(x^{1/3})$ with the constant implied by the O depending on k.

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In a brief note to be submitted, the author has observed a connection between imaginary quadratic fields of class number 1 and the function $\Delta_k(x)$. In particular, it is shown that if it could be established that there is a numerically calculable constant c such that

$$|\Delta_p(p)| \leqslant c\frac{p^{1/2}}{\log p} \tag{6}$$

where p, a prime, is the absolute value of the discriminant of an imaginary quadratic field, then we should have a numerical control on imaginary quadratic fields of class number 1.

It is therefore the object of this note to examine the behavior of $\Delta_k(x)$ with regard to uniformity in k.

Classical analytic methods are applicable coupled with a result due to H. Rademacher [4], [5]. We are unfortunately not able to establish (6) but we regard the following analysis as a first attempt. The referee has kindly pointed out to the author that E. Fogels [2] has also considered this question in a more general setting.

§ 2. Statement and proof of the theorem. Specifically, we shall here prove the following

THEOREM. *If $a = h\frac{\pi}{\sqrt{k}}$ with h the class number of K, then for any $\varepsilon > 0$,*

$$H(x) = ax + O(x^{1/3+\varepsilon}k^{1/3+\varepsilon}) + O(k^{1/2+\varepsilon}x^{\varepsilon}) \tag{7}$$

where the constants implied by the O depend only on ε and not on k.

Proof. We begin with an integral representation of $H(x)$, viz. if $\eta > 0$,

$$H(x) = \frac{1}{2\pi i}\int_{1+\eta-i\infty}^{1+\eta+i\infty} \zeta_K(s)\frac{x^s}{s}\,ds = I. \tag{8}$$

We split the path of integration into 3 parts, viz. $[1+\eta-i\infty, 1+\eta-iT]$, $[1+\eta-iT, 1+\eta+iT]$, $[1+\eta+iT, 1+\eta+i\infty]$ and get in this way

$$I = I_1 + J + I_2. \tag{9}$$

We estimate I_1; I_2 yields a similar result

$$I_1 = \frac{1}{2\pi i}\int_{1+\eta+iT}^{1+\eta+i\infty} \zeta_k(s)\frac{x^s}{s}\,ds = \frac{1}{2\pi i}\sum_{n=1}^{\infty} F(n)\int_{1+\eta+iT}^{1+\eta+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

We integrate the integral by parts which gives

$$I_1 = O\left\{\sum_{n=1}^{\infty}\frac{F(n)}{\left|\log\frac{x}{n}\right|}\cdot\frac{(x/n)^s}{s}\Big]_{1+\eta+iT}^{1+\eta+i\infty} + \sum_{n=1}^{\infty}\frac{F(n)}{\left|\log\frac{x}{n}\right|}\int_{1+\eta+iT}^{1+\eta+i\infty}\left(\frac{x}{n}\right)^s\frac{ds}{s}\right\}$$

$$= O\left\{\frac{1}{T}\sum_{n=1}^{\infty}\frac{F(n)}{\left|\log\frac{x}{n}\right|}\left(\frac{x}{n}\right)^{1+\eta}\right\}.$$

We split the sum into 3 parts $n < x/2$, $x/2 \leqslant n \leqslant 3x/2$, $n > 3x/2$ getting

$$I_1 = O(1/T)(\Sigma_1 + \Sigma_2 + \Sigma_3).$$

In Σ_1 we have $\left|\log\frac{x}{n}\right| > \log 2$ and hence

$$\Sigma_1 = O\big(x^{1+\eta}\zeta_K(1+\eta)\big)$$

a similar estimate holding for Σ_3.

In Σ_2, let $\bar{F} = \max_{x/2\leqslant n\leqslant 3x/2} F(n)$; then

$$\Sigma_2 = \sum_{x/2\leqslant n\leqslant 3x/2}\frac{F(n)}{\left|\log\frac{x}{n}\right|}\left(\frac{x}{n}\right)^{1+\eta} = O\left(\bar{F}\sum_{x/2\leqslant n\leqslant 3x/2}\frac{1}{\left|\log\frac{x}{n}\right|}\right).$$

The inner sum however is readily shown to be $O(x\log x)$. Consequently,

$$I_1 = O\left(\frac{x^{1+\eta}}{T}\zeta_K(1+\eta)\right) + O\left(\frac{\bar{F}x\log x}{T}\right)$$

where $\bar{F} = \max_{x/2\leqslant n\leqslant 3x/2} F(n)$. This gives

$$H(x) = \frac{1}{2\pi i}\int_{1+\eta-iT}^{1+\eta+iT}\zeta_K(s)\frac{x^s}{s}\,ds + O\left(\frac{x^{1+\eta}}{T}\zeta_K(1+\eta)\right) + O\left(\frac{\bar{F}x\log x}{T}\right). \tag{10}$$

The errors are uniform in k. On the other hand, for any $\varepsilon > 0$, $F(n) = O(n^{\varepsilon})$ uniformly in k. Hence from (10)

$$H(x) = \frac{1}{2\pi i}\int_{1+\eta-iT}^{1+\eta+iT}\zeta_K(s)\frac{x^s}{s}\,ds + O\left(\frac{x^{1+\eta}\zeta_K(1+\eta)}{T}\right) + O\left(\frac{x^{1+\varepsilon}}{T}\right). \tag{11}$$

We now move the path of integration in the integral to the line $\sigma = -\eta$. We get the principal term from the pole of $\zeta_K(s)$ at $s = 1$. This now results, from (9), in

$$(12)\qquad J = ax + \frac{1}{2\pi i}\Big(\int_{1-\eta+iT}^{-\eta+iT} + \int_{-\eta+iT}^{-\eta-iT} + \int_{-\eta-iT}^{1+\eta-iT}\Big)\zeta_K(s)\,\frac{x^s}{s}\,ds$$
$$= ax + J_1 + L + J_2 + \zeta_K(0).$$

We now estimate J_1 and J_2. As these are similar, we concentrate on J_1. For this estimate, we need a uniform estimate for $\zeta_K(s)$ along the horizontal path in the strip $-\eta \leqslant \sigma \leqslant 1+\eta$. Fortunately this is provided by H. Rademacher [4], [5] (see also E. Fogels [2]) using the theory of subharmonic functions coupled with Phragmen-Lindelöf Theorems. Specifically, he proves that if $0 < \eta \leqslant 1/2$, and $-\eta \leqslant \sigma \leqslant 1+\eta$, then uniformly in k, we have

$$(13)\qquad \zeta_K(\sigma+it) = O\bigg(k^{(1+\eta-\sigma)/2}\Big(\frac{|1+\sigma+it|}{2\pi}\Big)^{1+\eta-\sigma}\zeta^2(1+\eta)\bigg).$$

Choosing $\eta = 1/\log k$, we find that $\zeta_K(0) = O(k^{1/2}\log^2 k)$.

Here $\zeta(s)$ is the Riemann zeta function. Using the estimate (13) in J_1, we get

$$(14)\qquad J_1 = O\bigg(\int_{-\eta}^{1+\eta}|\zeta_K(\sigma+iT)|\,\frac{x^\sigma}{|\sigma+iT|}\,d\sigma\bigg)$$
$$= O\bigg(\int_{-\eta}^{1+\eta}k^{(1+\eta-\sigma)/2}\Big(\frac{|1+\sigma+iT|}{2\pi}\Big)^{1+\eta-\sigma}\frac{x^\sigma\zeta^2(1+\eta)}{|\sigma+iT|}\,d\sigma\bigg)$$
$$= O\Big(\zeta^2(1+\eta)\,\frac{x^{1+\eta}}{T}\Big) + O\Big(\frac{x^{-\eta}}{T}(\sqrt{k}T)^{1+2\eta}\zeta^2(1+\eta)\Big).$$

J_2 yields the same estimate.

To L we apply the functional equation. This is given by

$$(15)\qquad \zeta_K(s) = g_k(s)\,\zeta_K(1-s)$$

where

$$(16)\qquad g_k(s) = \Big(\frac{\sqrt{k}}{2\pi}\Big)^{1-2s}\frac{\pi\operatorname{cosec}\pi s}{\Gamma^2(s)}.$$

Since (see e.g. [3], Satz 160)

$$(17)\qquad \Gamma(\sigma+iT) = \sqrt{2\pi}\,t^{\sigma-1/2+it}e^{-\frac{\pi t}{2}-it+\frac{\pi i}{2}(\sigma-1/2)}\big(1+O(1/t)\big)$$

as $t\to\infty$ and likewise

$$(18)\qquad \operatorname{cosec}\pi(\sigma+it) = 2ie^{\pi i(\sigma+it)}\big(1+O(1/t)\big),$$

it follows from (16), (17) and (18) that for $1\leqslant t\leqslant T$, we have

$$(19)\qquad g(-\eta+it) = \Big(\frac{\sqrt{k}}{2\pi}\Big)^{1+2\eta-2it}t^{2\eta+1-2it}e^{2it}\big(1+O(1/t)\big)$$
$$= e^{-2it\log t+2it+2it\log 2\pi}\Big(\frac{t\sqrt{k}}{2\pi}\Big)^{1+2\eta} + O\bigg(\Big(\frac{\sqrt{k}}{2\pi}\Big)^{1+2\eta}t^{2\eta}\bigg).$$

Moreover

$$(20)\qquad \frac{1}{-\eta+it} = \frac{1}{it} + O\Big(\frac{\eta}{t^2}\Big).$$

Now

$$L = \frac{1}{2\pi i}\Big(\int_{-T}^{-1} + \int_{-1}^{1} + \int_{1}^{T}\Big)g(-\eta+it)\,\zeta_K(1+\eta-it)\,\frac{x^{-\eta+it}}{-\eta+it}\,dt$$
$$= L_1 + L_2 + L_3.$$

We evaluate L_3. Indeed using (19) and (20) we get

$$(21)\qquad L_3 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{F(n)}{n^{1+\eta}}\int_1^T\frac{g_k(-\eta+it)}{-\eta+it}(nx)^{it}\,dt$$
$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{F(n)}{n^{1+\eta}}\int_1^T e^{-2it\log t+2it+2it\log 2\pi}\Big(\frac{t\sqrt{k}}{2\pi}\Big)^{1+2\eta}(nx)^{it}\frac{dt}{t} +$$
$$+ O\bigg(\sum_{n=1}^{\infty}\frac{F(n)}{n^{1+\eta}}\int_1^T(\sqrt{k})^{1+2\eta}t^{2\eta-1}\,dt\bigg)$$
$$= \frac{1}{2\pi}\Big(\frac{\sqrt{k}}{2\pi}\Big)^{1+2\eta}\sum_{n=1}^{\infty}\frac{F(n)}{n^{1+\eta}}\int_1^T e^{if(t)}t^{2\eta}\,dt + O\big((\sqrt{k})^{1+2\eta}T^{2\eta}x^{-\eta}\zeta_K(1+\eta)\big)$$

where

$$f(t) = -2t\log t + 2t + 2t\log 2\pi + t\log nx.$$

The integral is now of a well known type and since

$$f''(t) = -\frac{2}{t} \leqslant -\frac{2}{T}$$

we may apply a classical lemma (see Titchmarsh [6], Ch. IV, § 1, Lemma 4) to deduce that

$$(22)\qquad \int_1^T e^{if(t)}t^{1+2\eta}\,dt = O(T^{2\eta+1/2}).$$

Hence from (21), and (22),

$$L_3 = O\left((\sqrt{k})^{1+2\eta}T^{2\eta+1/2}\zeta_K(1+\eta)\right)+O\left((\sqrt{k})^{1+2\eta}T^{2\eta}x^{-\eta}\zeta_K(1+\eta)\right). \tag{23}$$

The same argument applies to L_1 and likewise to L_2, the former giving the same error as L_3 and the latter an error

$$O\left((\sqrt{k})^{1+2\eta}\zeta_K(1+\eta)x^{-\eta}\right). \tag{24}$$

Thus from (11), (12), (14), (23) and (24), we get

$$\begin{aligned} \Delta_k(x) &= H(x)-ax \\ &= O\left(\frac{x^{1+\eta}\zeta_K(1+\eta)}{T}\right)+O\left(\frac{x^{1+\varepsilon}}{T}\right)+O\left(\frac{\zeta^2(1+\eta)}{T}x^{1+\eta}\right)+ \\ &\quad+O\left(x^{-\eta}(\sqrt{k})^{1+2\eta}\zeta^2(1+\eta)T^{2\eta}\right)+O\left((\sqrt{k})^{1+2\eta}\zeta_K(1+\eta)T^{2\eta+1/2}\right)+ \\ &\quad+O\left((\sqrt{k})^{1+2\eta}T^{2\eta}x^{-\eta}\zeta_K(1+\eta)\right)+O(k^{1/2}\log^2 k). \end{aligned} \tag{25}$$

However, we also have

$$\zeta_K(s) = \zeta(s)L(s)$$

and since

$$|L(1+\eta)| \leqslant \sum_{n=1}^{\infty}\frac{1}{n^{1+\eta}} = \zeta(1+\eta)$$

if follows that $\zeta_K(1+\eta) = O\left(\zeta^2(1+\eta)\right)$ uniformly in k. Moreover as $\eta\to 0$, $\zeta(1+\eta) = O(1/\eta)$. In (25), we now choose $\eta=\varepsilon$ and $T = (x/\sqrt{k})^{2/3}$. Thus (25) then gives

$$\Delta_k(x) = O(x^{1/3+\varepsilon}k^{1/3+\varepsilon})+O(k^{1/2+\varepsilon}x^{\varepsilon})$$

where the constant implied by the O now depends upon ε but not on k. The restriction that x be half an odd integer is now unnecessary.

In particular

$$\Delta_k(k) = O(k^{2/3+\varepsilon})$$

an error far short of the desired result (6). If the class number of K is 1, then

$$H(k) = \pi\sqrt{k}+\Delta_k(k).$$

Is it plausible to conjecture that in this case at any rate $\Delta_k(k) = o(\sqrt{k})$?

It should be noted that the above argument may be modified to yield a result for general algebraic number fields.

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