

On the Kusmin-Landau inequality for exponential sums

by

L. J. MORDELL (Cambridge)

Let throughout this paper a_1, a_2, \dots, a_n be n real numbers. Put $e^{ix} = e(x)$, and

$$(1) \quad S = e(2a_1) + e(2a_2) + \dots + e(2a_n).$$

A very important question in both analysis and number theory is to find estimates for $|S|$. These of course depend on the nature of the a 's; and simple results have been found when the a 's satisfy the conditions

$$(2) \quad 0 < \theta \leq a_2 - a_1 \leq \dots \leq a_n - a_{n-1} \leq \varphi < \pi.$$

The question has been of interest to quite a few mathematicians including van der Corput, Kusmin, Landau, Jarník, Popken, Karamata and Tomic⁽¹⁾.

When $\varphi = \pi - \theta$ in (2), we have

$$(3) \quad |S| \leq \cot \frac{1}{2} \theta.$$

Previous estimates for this S had been found by van der Corput and Kusmin, but (3) is due to Landau who showed that it was a best possible result. When φ is not specialized in (2), estimates for $|S|$ have been found by Karamata and Tomic. They give results of various kinds *e. g.*,

$$(4) \quad 2|S| \leq \cot \frac{1}{2} \theta + \tan \frac{1}{2} \varphi,$$

$$(5) \quad 2 \left| S - \frac{ie(-\theta)e(2a_1)}{2\sin\theta} \right| \leq \cot\theta + \tan\frac{1}{2}\varphi.$$

(¹) References to the first five are given by Koksma in [2]; the remaining two have written a joint paper [1].

4

L. J. Mordell

Other results were found by Popken. Write $\Delta a_r = a_{r+1} - a_r$, $\Delta^2 a_r = \Delta(\Delta a_r), \dots$. He imposed the conditions

$$(6) \quad \begin{aligned} 0 < \theta \leq \Delta a_r \leq \pi/4 & \quad (r = 1, 2, \dots, n-1), \\ \Delta^2 a_r \geq 0 & \quad (r = 1, 2, \dots, n-2), \end{aligned}$$

$$(7) \quad \Delta^3 a_r \geq 0 \quad (r = 1, 2, \dots, n-3).$$

The last condition is dropped if $n \leq 3$, and both the last are dropped if $n \leq 2$. He then proved

$$(8) \quad |S| \leq 1/\sin \theta,$$

a best possible result.

All these authors prove their results geometrically except that Landau translates the geometrical argument into a transformation of the series. Simple as his method is, it does not really reveal what underlies these results, and it might not be easy to deduce Popken's result in this way. I notice another simple method depending upon a well known and obvious transformation of a series. From this all, the estimates above and some new ones, follow in a natural manner.

Let $\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \mu_2, \dots, \mu_n$ be two sets of numbers. Then evidently, if $\lambda_0 = 0, \mu_{n+1} = 0$,

$$\sum_{r=1}^n \lambda_r (\mu_r - \mu_{r+1}) = \sum_{r=1}^n \mu_r (\lambda_r - \lambda_{r-1}).$$

Estimates for either sum follow easily if either $|\mu_r - \mu_{r+1}|$ or $|\lambda_r - \lambda_{r-1}|$ is a monotone sequence. Replace λ_r by $\lambda_r/(\mu_r - \mu_{r+1})$, then

$$(9) \quad \sum_{r=1}^n \lambda_r = \sum_{r=1}^n \mu_r \left(\frac{\lambda_r}{\mu_r - \mu_{r+1}} - \frac{\lambda_{r-1}}{\mu_{r-1} - \mu_r} \right).$$

Put $\mu_r = \lambda_r$ in (9) which then becomes

$$\sum_{r=1}^n \lambda_r = \sum_{r=1}^n \lambda_r \left(\frac{\lambda_r}{\lambda_r - \lambda_{r+1}} - \frac{\lambda_{r-1}}{\lambda_{r-1} - \lambda_r} \right).$$

Subtract $\frac{1}{2}$ from each of the terms in the bracket. Then

$$(10) \quad \begin{aligned} 2 \sum_{r=1}^n \lambda_r &= \sum_{r=1}^n \lambda_r \left(\frac{\lambda_r + \lambda_{r+1}}{\lambda_r - \lambda_{r+1}} - \frac{\lambda_{r-1} + \lambda_r}{\lambda_{r-1} - \lambda_r} \right) \\ &= \lambda_1 \left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + 1 \right) + \lambda_n \left(1 - \frac{\lambda_{n-1} + \lambda_n}{\lambda_{n-1} - \lambda_n} \right) + \\ &\quad + \sum_{r=2}^{n-1} \lambda_r \left(\frac{\lambda_r + \lambda_{r+1}}{\lambda_r - \lambda_{r+1}} - \frac{\lambda_{r-1} + \lambda_r}{\lambda_{r-1} - \lambda_r} \right). \end{aligned}$$

Put now $\lambda_r = e(2a_r)$. Then

$$(11) \quad \begin{aligned} 2S &= e(2a_1)(1 + i \cot(a_2 - a_1)) + e(2a_n)(1 - i \cot(a_n - a_{n-1})) + \\ &\quad + \sum_{r=2}^{n-1} ie(2a_r)(\cot(a_{r+1} - a_r) - \cot(a_r - a_{r-1})). \end{aligned}$$

It is now easy to deduce inequalities for $|S|$ by imposing conditions on the a 's. Thus with the condition (2)

$$0 < \theta \leq a_2 - a_1 \leq \dots \leq a_n - a_{n-1} \leq \varphi < \pi,$$

we have at once the results of Karamata and Tomic. Since $\cot x$ decreases steadily in $0 < x < \pi$,

$$\begin{aligned} 2|S| &\leq \frac{1}{\sin(a_2 - a_1)} + \frac{1}{\sin(a_n - a_{n-1})} + \cot(a_2 - a_1) - \cot(a_n - a_{n-1}) \\ &\leq \cot\left(\frac{a_2 - a_1}{2}\right) + \tan\left(\frac{a_n - a_{n-1}}{2}\right), \end{aligned}$$

and so

$$2|S| \leq \cot \frac{\theta}{2} + \tan \frac{\varphi}{2}.$$

For a slight modification of this result, put

$$(12) \quad S' = \frac{1}{2}e(2a_1) + \sum_{r=2}^{n-1} e(2a_r) + \frac{1}{2}e(2a_n).$$

Then from (11),

$$2|S'| \leq |\cot(a_2 - a_1)| + |\cot(a_n - a_{n-1})| + \cot(a_2 - a_1) - \cot(a_n - a_{n-1}).$$

If we now impose the condition $\varphi \leq \pi/2$, then

$$(13) \quad |S'| \leq \cot \theta.$$

This is a sharper result than (4) with any $\varphi \leq \pi/2$ since $\cot \theta < \frac{1}{2} \cot(\theta/2)$. We also can improve on (5). Since

$$1 + i \cot x = \frac{\sin x + i \cos x}{\sin x} = \frac{ie(-x)}{\sin x},$$

we have from (11) that if $\varphi \leq \pi/2$,

$$\begin{aligned} \left| 2S - \frac{ie(2a_1)e(a_1 - a_2)}{\sin(a_2 - a_1)} + \frac{ie(2a_n)e(a_n - a_{n-1})}{\sin(a_n - a_{n-1})} \right| &\leq \cot(a_2 - a_1) - \cot(a_n - a_{n-1}) \\ &\leq \cot \theta - \cot \varphi. \end{aligned}$$

We give in (14) below a new result which can be an improvement on (13). Let λ be any number. Write (10) as

$$\begin{aligned} 2 \sum_{r=1}^n \lambda_r &= \lambda_1 \left(\frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} + 1 \right) + \lambda_n \left(1 - \frac{\lambda_{n-1} + \lambda_n}{\lambda_{n-1} - \lambda_n} \right) + \\ &+ \sum_{r=2}^{n-1} (\lambda_r - \lambda) \left(\frac{\lambda_r + \lambda_{r+1}}{\lambda_r - \lambda_{r+1}} - \frac{\lambda_{r-1} + \lambda_r}{\lambda_{r-1} - \lambda_r} \right) + \lambda \left(\frac{\lambda_{n-1} + \lambda_n}{\lambda_{n-1} - \lambda_n} - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2} \right). \end{aligned}$$

Then with $\lambda_r = e(2a_r)$, $\lambda = e(2a)$, a real, and with the conditions (2),

$$\begin{aligned} 2|S'| &\leq 2|\sin(a_1 - a)| |\cot(a_2 - a_1)| + 2|\sin(a_n - a)| |\cot(a_n - a_{n-1})| + \\ &+ 2 \max_{r=2, \dots, n-1} |\sin(a_r - a)| |\cot(a_2 - a_1) - \cot(a_n - a_{n-1})|. \end{aligned}$$

If we impose again the condition $\varphi \leq \pi/2$, we have

$$(14) \quad |S'| \leq 2 \max_{r=1, 2, \dots, n} |\sin(a_r - a)| \cot \theta.$$

We come now to Popken's result. Write (9) as

$$\sum_{r=1}^n \lambda_r = \mu_1 \left(\frac{\lambda_1}{\mu_1 - \mu_2} \right) + \mu_n \left(\frac{\lambda_n}{\mu_n - \mu_{n-1}} \right) + \sum_{r=2}^{n-1} \mu_r \left(\frac{\lambda_r}{\mu_r - \mu_{r+1}} - \frac{\lambda_{r-1}}{\mu_{r-1} - \mu_r} \right).$$

Put $\lambda_r = e(2a_r)$, $\mu_r = e(2\theta_r)$ with real θ 's. Then

$$\begin{aligned} \sum_{r=1}^{n-1} e(2a_r) &= \frac{e(2a_1)}{1 - e(2\theta_2 - 2\theta_1)} + \frac{e(2a_{n-1})}{1 - e(2\theta_{n-1} - 2\theta_n)} + \\ &+ \sum_{r=2}^{n-1} e(2\theta_r) \left(\frac{e(2a_r)}{e(2\theta_r) - e(2\theta_{r+1})} - \frac{e(2a_{r-1})}{e(2\theta_{r-1}) - e(2\theta_r)} \right). \end{aligned}$$

Put now

$$(15) \quad 2a_r = \theta_r + \theta_{r+1},$$

and so

$$\begin{aligned} 2 \left| \sum_{r=1}^{n-1} e(2a_r) \right| &\leq \frac{1}{|\sin(\theta_2 - \theta_1)|} + \frac{1}{|\sin(\theta_n - \theta_{n-1})|} + \\ &+ \sum_{r=2}^{n-1} \left| \frac{1}{\sin(\theta_r - \theta_{r-1})} - \frac{1}{\sin(\theta_{r+1} - \theta_r)} \right|. \end{aligned}$$

We impose now the conditions

$$(16) \quad 0 < \theta_2 - \theta_1 \leq \dots \leq \theta_n - \theta_{n-1} \leq \pi/2.$$

Then

$$2 \left| \sum_{r=1}^{n-1} e(2a_r) \right| \leq \frac{1}{\sin(\theta_2 - \theta_1)} + \frac{1}{\sin(\theta_n - \theta_{n-1})} + \frac{1}{\sin(\theta_2 - \theta_1)} - \frac{1}{\sin(\theta_n - \theta_{n-1})};$$

and so changing n into $n+1$

$$(17) \quad \left| \sum_{r=1}^n e(\theta_r + \theta_{r+1}) \right| \leq 1/\sin(\theta_2 - \theta_1),$$

provided

$$(18) \quad 0 < \theta_2 - \theta_1 \leq \dots \leq \theta_{n+1} - \theta_n \leq \pi/2.$$

This result (17) is best possible. For take $\theta_r = (r-1)\theta$. Then (17) becomes

$$\left| \sum_{r=1}^n e(2r-1)\theta \right| \leq 1/\sin \theta,$$

or

$$\left| \frac{e(\theta) - e(2n+1)\theta}{1 - e(2\theta)} \right| \leq \frac{1}{\sin \theta},$$

or

$$\left| \frac{\sin n\theta}{\sin \theta} \right| \leq \frac{1}{\sin \theta},$$

and equality holds if $n\theta = \pi/2$.

We now investigate what (17) becomes when expressed in terms of the a 's and the conditions that (18) imposes on the a 's defined in (15).

We put $\theta_1 = 0$ and then show that it suffices to take the conditions

$$(19) \quad 0 < a_1 \leq \pi/4, \quad 0 < a_r - a_{r-1} \leq \pi/4 \quad (r = 2, 3, \dots, n),$$

$$(20) \quad a_2 - 3a_1 \geq 0, \quad a_3 - 3a_2 + 4a_1 \geq 0,$$

$$(21) \quad 4^3 a_r = a_{r+3} - 3a_{r+2} + 3a_{r+1} - a_r \geq 0 \quad (r = 1, 2, \dots, n-3).$$

The condition $a_1 \leq \pi/4$ is redundant since (20) and (19) give $3a_1 - a_1 \leq \pi/4$. Next from (15) with $\theta_1 = 0$,

$$(22) \quad \begin{aligned} \theta_2 &= 2a_1, \quad \theta_3 = 2a_2 - 2a_1, \quad \theta_4 = 2a_3 - 2a_2 + 2a_1, \dots, \\ \theta_n &= 2a_{n-1} - 2a_{n-2} + \dots + (-1)^n 2a_1. \end{aligned}$$

The condition $0 < a_1 \leq \pi/4$ is obvious from (18). The other conditions in (18) become

$$(23) \quad \begin{aligned} 0 < a_1 \leq a_2 - 2a_1 \leq a_3 - 2a_2 + 2a_1 \leq \dots \\ \leq a_n - 2a_{n-1} + \dots + (-1)^{n-1} 2a_1 \leq \pi/4. \end{aligned}$$

It is easy to see from (19) that each of these terms is $\leq \pi/4$. Thus

$$a_2 - 2a_1 = a_2 - a_1 - a_1 \leq \pi/4, \quad a_3 - 2a_2 + 2a_1 \leq a_3 - a_2 - (a_2 - 2a_1) \leq \pi/4.$$

Generally by induction

$$\begin{aligned} a_r - 2a_{r-1} + \dots + (-1)^{r-1} 2a_1 \\ = a_r - a_{r-1} - (a_{r-1} - 2a_{r-2} + \dots + (-1)^{r-3} 2a_1) \leq \pi/4. \end{aligned}$$

Also the terms in (22) form a monotone sequence since

$$\begin{aligned} a_r - 2a_{r-1} + \dots + (-1)^{r-1} 2a_1 - (a_{r-1} - 2a_{r-2} + \dots + (-1)^{r-2} 2a_1) \\ = a_r - 3a_{r-1} + 4a_{r-2} - 4a_{r-3} + \dots + 4(-1)^{r-1} a_1 \\ \geq a_{r-2} - 3a_{r-3} + 4a_{r-4} + \dots + 4(-1)^{r-3} a_1 \end{aligned}$$

on substituting for a_r from (21) with $r-3$ in place of r . On continuing the process, it suffices to show that both

$$a_3 - 3a_2 + 4a_1 \geq 0, \quad a_2 - 3a_1 \geq 0,$$

and these are given in (20). Hence we have proved that subject to (19), (20), (21),

$$(24) \quad \left| \sum_{r=1}^n e(2a_r) \right| \leq 1/\sin 2a_1.$$

We shall now deduce Popken's result. We put $a_1 = a + A_1$, $a_2 = a + A_2$, ..., $a_n = a + A_n$, and take $2a = A_2 - 3A_1$. The conditions (19) become

$$(25) \quad 0 < A_r - A_{r-1} \leq \pi/4 \quad (r = 2, 3, \dots, n).$$

The conditions (20) become

$$A_2 - 3A_1 - 2a \geq 0, \quad A_3 - 3A_2 + 4A_1 + 2a \geq 0.$$

The first here holds with equality sign. The second becomes

$$(26) \quad \Delta^2 A_1 = A_3 - 2A_2 + A_1 \geq 0.$$

The condition (21) becomes

$$(27) \quad \Delta^3 A_2 \geq 0 \quad (r = 1, 2, \dots, n-3).$$

The result (17) now becomes Popken's result

$$\left| \sum_{r=1}^n e(2A_r) \right| \leq 1/\sin(A_2 - A_1),$$

and (25), (26), (27) are Popken's conditions. It may be noted that Popken's conditions $\Delta^2 A_r > 0$ are redundant. Thus if we add $\Delta^2 A_1 \geq 0$ and $\Delta^3 A_1 \geq 0$, we get $\Delta^2 A_2 \geq 0$ etc.

References

- [1] J. Karamata and M. Tomic, *Sur une inégalité de Kusmin-Landau relative aux sommes trigonométriques et son application à la somme de Gauss*, Publications de l'Institut Mathématique de l'Académie Serbe des Sciences 3 (1950), p. 207-218.
[2] J. F. Koksma, *Diophantische Approximationen*, Berlin 1936.

ST. JOHN'S COLLEGE, CAMBRIDGE

Reçu par la Rédaction le 3. 9. 1956