n	a	b	c	d	$(a+b)(c+d)$
239	29	-28	13	1	$1\cdot 14$
247	10	4	-16	17	$14\cdot 1$
254	6	4	3	-1	$10\cdot 2$
265	7	55	53	26	$62\cdot 79$
266	11	-10	1	14	$1\cdot 15$
268	9	2	-12	13	$11\cdot 1$
274	7	4	2	5	$11\cdot 7$
275	11	2	10	4	$13\cdot 14$
283	7	1	-4	5	$8\cdot 1$
286	7	2	4	1	$9\cdot 5$
290	7	1	3	3	$8\cdot 6$
292	52	52	-151	153	$104\cdot 4$
293	8	5	1	7	$13\cdot 8$

Or, on a $232 = 2^2\cdot 29$, $248 = 2^3\cdot 31$, $250 = 5^3\cdot 2$, $256 = 4^3\cdot 4$, $272 = 2^3\cdot 34$.

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On the so-called density-hypothesis in the theory of the zeta-function of Riemann

by

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§ 1. Introduction

1. If $w = u+iv$, then the zeta-function of Riemann is for $u>1$ defined by

$$(1.1.1)\qquad \zeta(w) = \sum_{n=1}^{\infty} n^{-w}.$$

As is well known, $(w-1)\zeta(w)$ is an integral-function, which has an infinity of zeros, called "non-trivial" zeros, in the vertical strip $0<u<1$, which according to Riemann have a mysterious connection with the prime-numbers. Denoting by $N(T)$ the number of these non-trivial zeros in the parallelogram

$$(1.1.2)\qquad 0<u<1,\quad 0<v\leqslant T,$$

we have according to Riemann-Mangoldt[1] for $T\geqslant 2$

$$(1.1.3)\qquad \left|N(T)-\frac{T}{2\pi}\log\frac{T}{2\pi e}\right| \leqslant c_1\log T,$$

where c_1 and later $c_2, \ldots$ are positive numerical constants (if some of them depend upon small parameters ε or $\varkappa$, δ, the dependence will always be explicitly stated). The famous unproved conjecture of Riemann asserts that $\zeta(w)\neq 0$ for $u>\frac{1}{2}$. Recently it has been realized that many of its consequences in the number-theory could have been deduced from "density-theorems" which assert that in parallelograms

$$(1.1.4)\qquad u\geqslant\alpha,\quad 0<v\leqslant T,\quad \tfrac{1}{2}\leqslant\alpha\leqslant 1,$$

the number of zeros of $\zeta(w)$ is "not too large". More exactly, $N(\alpha, T)$

(1) See *e. g.* [3], p. 181, the name of Mangoldt not being mentioned.

denoting the number of zeros in the parallelogram (1.1.4), this sort of theorems assert that

(1.1.5) $$N(\alpha, T) < T^{c_2(1-\alpha)}\log^{c_3} T$$

for $T \geqslant c_4$ uniformly for $\frac{1}{2} \leqslant \alpha \leqslant 1$. Comparing (1.1.5) with (1.1.3) we get at once

(1.1.6) $$c_2 \geqslant 2.$$

After the pioneering works of Bohr-Landau (see [3], p. 197) and Carlson (see [3], p. 197) the best results were achieved by Ingham (see [3], p. 197) who proved (1.1.5) with

$$c_2 = \tfrac{61}{23}.$$

Moreover he proved the inequality

(1.1.7) $$N(\alpha, T) < c_5 T^{\lambda(\alpha)(1-\alpha)}\log^5 T,$$

where $T \geqslant 2$ and

(1.1.8) $$\lambda(\alpha) = \begin{cases} \frac{61}{23}, & \text{if } \frac{53}{61} \leqslant \alpha \leqslant 1, \\ \dfrac{3}{2-\alpha}, & \text{if } \dfrac{1}{2} \leqslant \alpha \leqslant \frac{53}{61}. \end{cases}$$

These results are superseded only in the neighbourhood of $\alpha = 1$; I have proved (²) that for a certain (small) c_6 we have for

(1.1.9) $$1 - c_6 \leqslant \alpha \leqslant 1, \quad T \geqslant c_7$$

the inequality

(1.1.10) $$N(\alpha, T) < T^{2(1-\alpha)+600(1-\alpha)^{1,01}}\log^6 T.$$

2. The proof of the inequality

(1.2.1) $$N(\alpha, T) < T^{2(1+\varepsilon)(1-\alpha)}, \quad T \geqslant c_8(\varepsilon), \quad \tfrac{1}{2} \leqslant \alpha \leqslant 1$$

would be very important in the analytical theory of numbers; *e. g.* we could derive from it a proof of the longstanding conjecture

$$p_{n+1} - p_n < p_n^{1/2+\varepsilon}, \quad n > n_0(\varepsilon),$$

where p_n denotes the nth prime. This is called (³) "the density-hypothesis"

(²) See [4]. In an essentially enlarged Chinese version (1.1.10) is replaced by

$$N(\alpha, T) < T^{2(1-\alpha)+(1-\alpha)^{1,1}}$$

for (1.1.9).

(³) Sometimes the somewhat stronger inequality

$$N(\alpha, T) < c_9 T^{2(1-\alpha)}\log^2 T, \quad \tfrac{1}{2} \leqslant \alpha \leqslant 1, \quad T \geqslant 2$$

is called the density-hypothesis.

and is not proved so far. Ingham (see [3], p. 202) deduced it from the unproved Lindelöf-hypothesis, according to which

(1.2.2) $$|\zeta(u+iv)| \leqslant c_{10}(\varepsilon)|v|^{\varepsilon}$$

holds for $\frac{1}{2} \leqslant u \leqslant 1$, $|v| \geqslant 1$. Wanting to know whether or not my methods in [4] work also "far" from the line $u = 1$, I gave an alternative proof (⁴) of Ingham's result; this proof gave me the impression that they can work and that from (1.2.2) much stronger estimation of $N(\alpha, T)$ than (1.2.1) can be derived. In turn it seemed to me possible that the density-hypothesis (1.2.1) is deducible from results much weaker than (1.2.2). In what follows we shall show that this is indeed the case.

3. As Littlewood (see [3], p. 279) has shown, Lindelöf's conjecture (1.2.2) is equivalent to the inequality

(1.3.1) $$\lim_{T\to+\infty} \frac{N(\alpha_1, T+1) - N(\alpha_1, T)}{\log T} = 0$$

provided $\frac{1}{2} < \alpha_1 < 1$ and α_1 is fixed. Consider now the following

Conjecture A. *There is a function $g(x)$ positive and increasing for $x > 0$ with*

(1.3.2) $$\lim_{x\to+0} g(x) = 0$$

having the following property. Let

$$\tfrac{1}{2} < \varkappa < 1, \quad 0 < \delta \leqslant \tfrac{1}{10}(\varkappa - \tfrac{1}{2}) \quad \textit{and} \quad \varkappa \leqslant \alpha_2.$$

Then denoting by $M(\tau, \alpha_2, \delta)$ the number of zeros in the parallelogram

(1.3.3) $$\alpha_2 - \delta \leqslant u \leqslant \alpha_2, \quad |v - \tau| \leqslant \delta/2$$

we have for $\tau > c_{11}(\varkappa, \delta)$ the estimation

(1.3.4) $$M(\tau, \alpha_2, \delta) < \delta g(\delta)\log\frac{\tau}{2}.$$

The content of conjecture A may roughly be expressed by saying: "the concentration of zeta-zeros is not too big". If Lindelöf's conjecture (1.2.2) is true, then owing to (1.3.1) the truth of conjecture A trivially follows; the converse assertion, however, is not true: (1.3.1) does not follow, even with a $c_{12}(\alpha_1)$ instead 0.

The conjecture of Lindelöf in its form (1.3.1) is not proved for any $\alpha_1 < 1$. The conjecture A follows quite easily at least in the case $\alpha_2 = 1$

(⁴) Even in a little stronger form. See [5].

(by Jensen's inequality) from the known(5) fact that for $\frac{63}{64} \leqslant u \leqslant 1$, $|v| \geqslant 2$ the inequality

$$(1.3.5) \qquad |\zeta(u+iv)| \leqslant c_{13}|v|^{4(1-u)/\log(1-u)^{-1}}\log|v|$$

holds; even the weaker fact

$$(1.3.6) \qquad \overline{\lim_{u\to 1-0}} \frac{1}{1-u} \overline{\lim_{v\to+\infty}} \frac{\log|\zeta(u+iv)|}{\log v} = 0$$

would suffice. But even a proof that *e. g.* for all squares (1.3.3) the much weaker inequality

$$M(\tau, a_2, \delta) < \delta^{0,01}\log\frac{\tau}{2}$$

holds, seems to be almost as difficult as Lindelöf's conjecture itself. Therefore I should not consider it a particularly great advance to deduce the density hypothesis from conjecture A.

4. However if we could restrict ourselves only to squares (1.3.3) which "are affixed from the left to big zero-free parallelograms", the situation changes. More exactly we consider the

Conjecture B. *There is a function $g(x)$, positive and increasing for $x > 0$ with*

$$(1.4.1) \qquad \lim_{x\to+0} g(x) = 0,$$

having the following property. Let

$$\tfrac{1}{2} < \varkappa < 1, \quad 0 < \delta \leqslant \tfrac{1}{10}(\varkappa - \tfrac{1}{2}) \quad \textit{and} \quad \varkappa \leqslant a_3,$$

and suppose that $\zeta(w)$ does not vanish in the parallelogram(6)

$$(1.4.2) \qquad a_3 \leqslant u \leqslant 1, \quad |v-\tau| \leqslant \left[\log\frac{\tau}{2}\right].$$

Denoting by $M(\tau, a_3, \delta)$ the number of zeros in the parallelogram

$$(1.4.3) \qquad a_3 - \delta \leqslant u \leqslant a_3, \quad |v-\tau| \leqslant \delta/2,$$

we have for $\tau > c_{14}(\varkappa, \delta)$ the estimation

$$(1.4.4) \qquad M(\tau, a_3, \delta) < \delta g(\delta)\log\frac{\tau}{2}.$$

(5) Theorem of Hardy and Littlewood. In this form see [1], p. 40.
(6) Throughout this paper $[x]$ denotes as usual, the greatest integer $\leqslant x$.

As we shall see in the Appendix, in this case the proof of the inequality

$$(1.4.5) \qquad M(\tau, a_3, \delta) < 0{,}71\delta\log\frac{\tau}{2}$$

is easy: it requires only Jensen's formula, the three-circle theorem and the classical inequality of Hadamard-Carathéodory.

5. In what follows we shall prove the following

Theorem. *The truth of conjecture* B *implies the truth of the density-hypothesis* (1.2.1).

A sketch of the proof of this theorem and a number of remarks to the conjecture B I gave in a lecture (see [6]). As to the proof of the theorem it is based not only on the methods of my book (the knowledge of which, however, is not assumed) but also on two new ideas. The starting point of my papers on the zeta-function was always the forming of appropriate identities, which connected zeta-roots with prime-numbers. But in these identities the complex variable was restricted to the half-plane $u > 1$ and therefore their full force could work only in the neighbourhood of the line $u = 1$. In this paper the starting point is another identity, which can be applied also *in* the critical strip. This identity (which is written, to save a step, in the form of the inequality (4.1.4)) seems to me applicable also to other aims in the theory of the zeta-function; to these I shall return elsewhere. The second idea is a simple reduction-process which enables me to replace the conjecture A by conjecture B; a more detailed description must be postponed to the first few lines of § 5.

The methods of this paper could also give the best-known estimations for $N(\alpha, T)$ without any conjectures. *E. g.*, before finding the new starting inequality and using as a new tool only the reduction-process I proved that for $T > c_{16}$ and $1-2^{-15} \leqslant \alpha \leqslant 1$ the estimation

$$(1.5.1) \qquad N(\alpha, T) < T^{(5/2)(1-\alpha)}$$

holds without any conjectures. Slight changes in the proofs of this paper and in particular in the Appendix would certainly enlarge the α-interval in which (1.5.1) holds; we do not go into details here. I mean in particular such a change in (1.4.5) which would replace in it the constant 0,71 by another one depending on a_3 and tending to 0 with $(1-a_3)$.

§ 2. Preliminaries to the proof

1. We shall make use of the following three inequalities, which are easy consequences of (1.1.3). For $\tau > c_{17}$ we have

$$(2.1.1) \qquad N(\tau) < \tau\log\tau,$$

further very roughly

$$(2.1.2)\qquad N(\tau+[\log\tfrac{1}{2}\tau])-N(\tau)>0$$

and for all real τ-values

$$(2.1.3)\qquad N(\tau+1)-N(\tau)<c_{18}\log(2+|\tau|).$$

2. Let ε be an arbitrarily small positive number $<\frac{1}{40}$ and fixed. With $g(x)$ of conjecture B we determine uniquely $\delta_1>0$ by

$$(2.2.1)\qquad g(\delta_1)\leqslant\varepsilon^5,\qquad \delta_1\leqslant\tfrac{1}{11}\varepsilon^2$$

and let

$$(2.2.2)\qquad N_1=\big(N_1(\varepsilon)=\big)\frac{12}{\varepsilon^2\delta_1}.$$

Taking ε sufficiently small we have

$$(2.2.3)\qquad 0<\delta_1<\tfrac{1}{240},$$

i. e.

$$(2.2.4)\qquad \varepsilon^2N_1>2000,\qquad N_1>40000.$$

Further let the α of $N(\alpha,T)$ be restricted by

$$(2.2.5)\qquad \frac{1}{2}+\frac{125}{\varepsilon^2N_1}\leqslant\alpha\leqslant1-\max\left(3\varepsilon+3\varepsilon^2,\ \frac{6}{\varepsilon^2N_1}+3\varepsilon^2\right)$$

and fixed. Let T be so large that

$$(2.2.6)\qquad T>\max(e^{100},e^{1/\varepsilon^{200}}),\qquad \frac{1}{[\log^4T]}<\min\left(\frac{1}{N_1},\varepsilon^2\right);$$

later we shall have some more restrictions upon T, but all of the type $T>c(\varepsilon)$. Further let k be an even integer $\geqslant20$, to be determined later; at present we require only

$$(2.2.7)\qquad \log T\leqslant kN_1\leqslant(1+\varepsilon^2)\log T.$$

Finally the complex parameter

$$(2.2.8)\qquad s=\sigma+it$$

is restricted by

$$(2.2.9)\qquad \frac{1}{2}+\frac{2}{[\log^2T]}\leqslant\sigma\leqslant1-\frac{2}{[\log^2T]},\qquad T\leqslant t\leqslant2T.$$

§ 3. Distribution of the values of an almost periodical polynomial

1. Let $J_k(s)$ be defined by

$$(3.1.1)\qquad J_k(s)=-\frac{1}{2\pi i}\int\limits_{(2)}\left\{e^{N_1w}\frac{e^{\varepsilon^2N_1w}-e^{-\varepsilon^2N_1w}}{2\varepsilon^2N_1w}\right\}^k\frac{\zeta'}{\zeta}(s+w)\,dw.$$

If $\Lambda(n)$ stands for the usual Dirichlet-symbol we have the

LEMMA I. *$J_k(s)$ has also the representation*

$$(3.1.2)\qquad J_k(s)=\sum_{e^{kN_1(1-\varepsilon^2)}\leqslant n\leqslant e^{kN_1(1+\varepsilon^2)}}\frac{\Lambda(n)R_k(n,\varepsilon)}{n^s}$$

where for the $R_k(n,\varepsilon)$-numbers the estimation

$$(3.1.3)\qquad |R_k(n,\varepsilon)|<\delta_1$$

holds.

Proof. Obviously we may insert in (3.1.1) the Dirichlet-series of ζ'/ζ. This gives

$$(3.1.4)\qquad J_k(s)=\sum_n\frac{\Lambda(n)}{n^s}\cdot\frac{1}{2\pi i}\int\limits_{(2)}\left\{e^{N_1w}\frac{e^{\varepsilon^2N_1w}-e^{-\varepsilon^2N_1w}}{2\varepsilon^2N_1w}\right\}^k\frac{dw}{n^w}.$$

Writing the integral in the form

$$(3.1.5)\qquad \sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{1}{(2\varepsilon^2N_1)^k}\cdot\frac{1}{2\pi i}\int\limits_{(2)}\left(\frac{e^{kN_1+(k-2j)\varepsilon^2N_1}}{n}\right)^w\frac{dw}{w^k}$$

we see at once, owing to the well-known integral-formula

$$\frac{1}{2\pi i}\int\limits_{(2)}\frac{a^w}{w^k}dw=0,\qquad 0\leqslant a\leqslant1,$$

that each term in (3.1.5) vanishes if

$$n\geqslant e^{kN_1(1+\varepsilon^2)}.$$

Further an easy application of Cauchy's integral-theorem gives

$$\frac{1}{2\pi i}\int\limits_{(2)}\left\{e^{N_1w}\frac{e^{\varepsilon^2N_1w}-e^{-\varepsilon^2N_1w}}{2\varepsilon^2N_1w}\right\}^k\frac{dw}{n^w}=\frac{1}{2\pi i}\int\limits_{(-1)}\left\{e^{N_1w}\frac{e^{\varepsilon^2N_1w}-e^{-\varepsilon^2N_1w}}{2\varepsilon^2N_1w}\right\}^k\frac{dw}{n^w},$$

and thus expanding the binom on the right we see at once that each term vanishes again for

$$n\leqslant e^{kN_1(1-\varepsilon^2)}$$

owing to the well-known integral-formula

$$\frac{1}{2\pi i}\int\limits_{(-1)}\frac{a^w}{w^k}dw=0,\qquad a\geqslant1.$$

Thus (3.1.2) is proved with

$$R_k(n,\varepsilon) = \frac{1}{2\pi i}\int\limits_{(2)} \left\{ e^{N_1 w}\frac{e^{\varepsilon^2 N_1 w} - e^{-\varepsilon^2 N_1 w}}{2\varepsilon^2 N_1 w}\right\}^k \frac{dw}{n^w}.$$

Cauchy's integral-theorem gives again

$$(3.1.6)\qquad R_k(n,\varepsilon) = \frac{1}{2\pi i}\int\limits_{(0)} \left\{ e^{N_1 w}\frac{e^{\varepsilon^2 N_1 w} - e^{-\varepsilon^2 N_1 w}}{2\varepsilon^2 N_1 w}\right\}^k \frac{dw}{n^w}$$

$$= \frac{1}{\pi}\int\limits_0^\infty \left(\frac{\sin \varepsilon^2 N_1 v}{\varepsilon^2 N_1 v}\right)^k \cos(kN_1 - \log n)v\,dv.$$

Hence we have

$$|R_k(n,\varepsilon)| \leqslant \frac{1}{\pi\varepsilon^2 N_1}\int\limits_0^\infty \left(\frac{\sin v}{v}\right)^k dv,$$

and thus very roughly

$$|R_k(n,\varepsilon)| \leqslant \frac{1}{\pi N_1\varepsilon^2}\left\{\int\limits_0^1 \left(\frac{\sin v}{v}\right)^k dv + \frac{1}{k-1}\right\} < \frac{1}{N_1\varepsilon^2} < \delta_1$$

using (2.2.2).

2. Writing

$$U_k(\sigma) = \int\limits_T^{2T} |J_k(\sigma+it)|^2 dt$$

we have the

LEMMA II. *The inequality* (⁷)

$$U_k(\sigma) < c_{19}(\varepsilon) T^{2(1-\sigma)+3\varepsilon^2}\log^7 T$$

holds.

Proof. If we write shortly

$$J_k(s) = \sum_{e^{kN_1(1-\varepsilon^2)}\leqslant n\leqslant e^{kN_1(1+\varepsilon^2)}} \frac{a_n}{n^s},$$

we have, by Lemma I,

$$(3.2.1)\qquad |a_n| \leqslant \delta_1 \log n.$$

(⁷) It would be easy to diminish the exponent of the logarithms but it is of no importance.

The usual technique gives

$$U_k(\sigma) < T\sum_n \frac{|a_n|^2}{n^{2\sigma}} + 4\sum_{n<m}\frac{|a_m||a_n|}{(mn)^\sigma \log(m/n)},$$

i. e., using (3.2.1) and (2.2.7), we obtain

$$U_k(\sigma)$$

$$< T\delta_1^2 \sum_{e^{kN_1(1-\varepsilon^2)}\leqslant n\leqslant e^{kN_1(1+\varepsilon^2)}} \log^2 n\cdot n^{-2\sigma} + 4\delta_1^2 \sum_{n<m\leqslant e^{kN_1(1+\varepsilon^2)}} \frac{\log m\log n}{(mn)^\sigma\log(m/n)}$$

$$< 2T\delta_1^2(kN_1)^2 \sum_{n\geqslant e^{kN_1(1-\varepsilon^2)}} n^{-2\sigma} + 16\delta_1^2(kN_1)^2 \sum_{n<m\leqslant e^{kN_1(1+\varepsilon^2)}} \frac{1}{(mn)^\sigma\log(m/n)}$$

$$< 32\delta_1^2\log^2 T\left\{T\sum_{n>T^{1-\varepsilon^2}} n^{-2\sigma} + \sum_{n<m\leqslant T^{(1+\varepsilon^2)^2}} \frac{(mn)^{-\sigma}}{\log(m/n)}\right\}.$$

Owing to (2.2.9), this evidently gives

$$U_k(\sigma)$$

$$< c_{20}\delta_1^2\log^2 T\left\{T\cdot\frac{T^{(1-2\sigma)(1-\varepsilon^2)}}{2\sigma-1} + \sum_{2n\leqslant m\leqslant T^{(1+\varepsilon^2)^2}} (mn)^{-\sigma} + \sum_{\substack{m\leqslant T^{(1+\varepsilon^2)^2}\\ n<m<2n}} \frac{(mn)^{-\sigma}}{\log(m/n)}\right\}$$

$$< c_{21}\delta_1^2\log^2 T\left\{\frac{T^{2(1-\sigma)+\varepsilon^2}}{2\sigma-1} + \left(\sum_{m\leqslant T^{(1+\varepsilon^2)^2}} m^{-\sigma}\right)^2 + \sum_{n\leqslant T^{(1+\varepsilon^2)^2}} n^{-2\sigma}\sum_{n<m<2n} 2\frac{n}{m-n}\right\}$$

$$< c_{22}(\varepsilon)\log^3 T\left\{\frac{T^{2(1-\sigma)+\varepsilon^2}}{2\sigma-1} + \frac{T^{2(1+\varepsilon^2)^2(1-\sigma)}}{(1-\sigma)^2} + \sum_{n\leqslant T^{(1+\varepsilon^2)^2}} \frac{\log n}{n^{2\sigma-1}}\right\}$$

$$< c_{19}(\varepsilon)T^{2(1-\sigma)+3\varepsilon^2}\log^7 T, \qquad \text{q. e. d.}$$

3. We denote by $A_k(\sigma, T)$ the set of t-values in $T\leqslant t\leqslant 2T$, where at fixed σ and k with a fixed α from (2.2.5) the inequality

$$(3.3.1)\qquad |J_k(\sigma+it)| \geqslant T^{\alpha-1}U_k(\sigma)^{1/2}\log^{10}T$$

holds. This is a finite set of disjoint closed intervals. Denoting its measure by $M(A_k(\sigma,T))$, we have

$$U_k(\sigma) \geqslant \int\limits_{A_k(\sigma,T)} |J_k(\sigma+it)|^2 dt > M(A_k(\sigma,T))T^{2(\alpha-1)}U_k(\sigma)\log^{20}T,$$

i. e.,

$$(3.3.2)\qquad M(A_k(\sigma,T)) < \frac{T^{2(1-\alpha)}}{\log^{20}T}.$$

Owing to (2.2.7) the number of possible values of k is $< 2\log T$. If we restrict the σ-values to

$$(3.3.3)\qquad \sigma_\nu = 1-\frac{\nu}{[\log^2 T]},\qquad 2\leqslant \nu \leqslant \frac{1}{2}[\log^2 T]-2,$$

then (2.2.9) is fulfilled; the number of these σ_ν-values is $< \log^2 T$. Hence from (3.3.2)

$$\sum_k \sum_\nu M\big(A_k(\sigma_\nu, T)\big) < 2\frac{T^{2(1-a)}}{\log^{17} T}$$

and we find that if $A(T)$ is the union of all $A_k(\sigma_\nu, T)$-sets ([8]) then

$$(3.3.4)\qquad M\big(A(T)\big) < 2\frac{T^{2(1-a)}}{\log^{17} T}$$

and if $\overline{A(T)}$ is the set complementary to $A(T)$ with respect to $(T, 2T)$, we have owing to (3.3.1) and lemma II

$$(3.3.5)\qquad |J_k(\sigma_\nu+it)| < \sqrt{c_{19}(\varepsilon)}\,T^{a-\sigma_\nu+2\varepsilon^2}\log^{14} T,\qquad t\,\epsilon\,\overline{A(T)}$$

for *each* permitted values of k and ν from (2.2.7) resp. (3.3.3).

4. We cover the horizontal strip

$$T\leqslant t\leqslant 2T$$

of the $s=\sigma+it$-plane by the horizontal l_μ-strips, defined by

$$(3.4.1)\qquad T+\frac{\mu}{[\log^3 T]}\leqslant t < T+\frac{\mu+1}{[\log^3 T]},$$

$$(3.4.2)\qquad 0\leqslant \mu < T[\log^3 T];$$

the last one may be thinner. We shall call μ the index of the strip l_μ. Some l_{μ_1}-strips contain horizontal lines $t=\tau_{\mu_1}$ with

$$(3.4.3)\qquad \tau_{\mu_1}\,\epsilon\,\overline{A(T)};$$

we shall call such strips "good" ones. If the strip l_{μ_2} does not contain such a horizontal line, we shall call l_{μ_2} a "bad" strip. We fix the τ_{μ_1}-values in the "good" strips. Owing to (3.3.4) the number of the "bad" strips for $T > c_{23}$ is at most

$$3\frac{T^{2(1-a)}}{\log^{14} T}.$$

Summing up the results of this paragraph we obtain the following

([8]) The set $A(T)$ depends of course also upon a; but this is fixed.

Lemma III. *For $T > c_{24}(\varepsilon)$ with the exception of at most*

$$(3.4.4)\qquad 3\frac{T^{2(1-a)}}{\log^{14} T}$$

"bad" strips to each "good" l_μ-strips for the points

$$(3.4.5)\qquad \sigma_\nu+i\tau_\mu \equiv s_{\nu\mu},\qquad 2\leqslant\nu\leqslant \tfrac{1}{2}[\log^2 T]-3$$

the inequality

$$(3.4.6)\qquad |J_k(s_{\nu\mu})|\leqslant c_{25}(\varepsilon)\,T^{a-\sigma_\nu+2\varepsilon^2}\log^{14} T$$

holds for each permitted k-value in (2.2.7).

This is the assertion to which the title of this paragraph refers.

§ 4. Connection of $J_k(s)$ with the zeros of $\zeta(s)$

1. The connection that we are going to show will be proved for all s-values with

$$\tfrac{1}{2}\leqslant\sigma\leqslant 1,\qquad T\leqslant t\leqslant 2T,$$

but actually used only for the $s_{\nu\mu}$-numbers defined above. We start from the representation (3.1.1). The usual contour-integration gives at once

$$(4.1.1)\qquad J_k(s) = \left\{e^{N_1(1-s)}\frac{e^{\varepsilon^2 N_1(1-s)}-e^{-\varepsilon^2 N_1(1-s)}}{2\varepsilon^2 N_1(1-s)}\right\}^k -$$

$$-\sum_\varrho \left\{e^{N_1(\varrho-s)}\frac{e^{\varepsilon^2 N_1(\varrho-s)}-e^{-\varepsilon^2 N_1(\varrho-s)}}{2\varepsilon^2 N_1(\varrho-s)}\right\}^k -$$

$$-\frac{1}{2\pi i}\int_{(-1-\sigma)}\left\{e^{N_1 w}\frac{e^{\varepsilon^2 N_1 w}-e^{-\varepsilon^2 N_1 w}}{2\varepsilon^2 N_1 w}\right\}^k \frac{\zeta'}{\zeta}(s+w)\,dw.$$

Denoting the last integral by D we get by (2.2.4)

$$(4.1.2)\qquad |D|\leqslant \frac{e^{-kN_1(1+\sigma)}}{2\pi}\int_{-\infty}^{\infty}\left(\frac{e^{\varepsilon^2 N_1(1+\sigma)}+e^{-\varepsilon^2 N_1(1+\sigma)}}{2\varepsilon^2 N_1\sqrt{1+v^2}}\right)^k\cdot\left|\frac{\zeta'}{\zeta}\big(-1+i(t+v)\big)\right|dv$$

$$\leqslant \frac{e^{-kN_1(1-\varepsilon^2)(1+\sigma)}}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(1+v^2)^{k/2}}\left|\frac{\zeta'}{\zeta}\big(-1+i(t+v)\big)\right|dv.$$

As is well known,

$$\left|\frac{\zeta'}{\zeta}\big(-1+i(t+v)\big)\right|\leqslant c_{26}\log(2+|t+v|);$$

from this, (4.1.2) and (2.2.7) we get

$$(4.1.3)\qquad |D| \leqslant c_{27} T^{(-3/2)(1-\varepsilon^2)} \log T.$$

Further we have roughly

$$\left| e^{N_1(1-s)} \frac{e^{\varepsilon^2 N_1(1-s)} - e^{-\varepsilon^2 N_1(1-s)}}{2\varepsilon^2 N_1(1-s)} \right|^k \leqslant \frac{e^{kN_1(1+\varepsilon^2)(1-\sigma)}}{T^k} \leqslant T^{-k+(1+\varepsilon^2)^2(1-\sigma)} < T^{-2},$$

using again (2.2.7). From this, (4.1.3) and (4.1.1) we get

$$(4.1.4)\qquad \left| J_k(s) + \sum_{\varrho} \left\{ e^{N_1(\varrho-s)} \frac{e^{\varepsilon^2 N_1(\varrho-s)} - e^{-\varepsilon^2 N_1(\varrho-s)}}{2\varepsilon^2 N_1(\varrho-s)} \right\}^k \right| \leqslant c_{28} T^{(-3/2)(1-\varepsilon^2)} \log T.$$

Comparing this with lemma III we obtain

LEMMA IV. *For $T > c_{29}(\varepsilon)$ we have for each "good" l_μ-strips and $s_{\nu\mu}$-numbers in* (3.4.5) *the inequality*

$$(4.1.5)\qquad \left| \sum_{\varrho} \left\{ e^{N_1(\varrho-s_{\nu\mu})} \frac{e^{\varepsilon^2 N_1(\varrho-s_{\nu\mu})} - e^{-\varepsilon^2 N_1(\varrho-s_{\nu\mu})}}{2\varepsilon^2 N_1(\varrho-s_{\nu\mu})} \right\}^k \right| \leqslant c_{30}(\varepsilon) T^{\alpha-\sigma_\nu+2\varepsilon^2} \log^{14} T.$$

§ 5. A reduction-process

1. Let

$$(5.1.1)\qquad \varrho_j^* = \sigma_j^* + it_j^*$$

stand for a zero of $\zeta(w)$ in l_j with the greatest real part if there exists a zero in l_j at all; in this case we may call it an "extreme righthand" zero in l_j. If there are more zeros in l_j with the same maximal real parts, choose an arbitrary one as ϱ_j^* and fix it. We may vaguely call a "neighbourhood" of an l_j-strip those strips whose indices differ "not much" from j. The aim of this paragraph is to show that omitting "not too many" "good" strips, we may reduce the study of the zeros to the study of zeros in such "good" l_μ-strips, which contain indeed zeros and for which the ϱ_μ^*-zero is "essentially the extreme righthand in a big neighbourhood of l_μ".

In order to give an exact meaning to what has been said above, we call a "block" a maximal sequence of "good" strips with consecutive indices. We speak of a "short block" if the "block" contains at most

$$(5.1.2)\qquad 24\log^8 T + 4$$

consecutive strips (all of which are of course "good" strips); in the opposite case we speak about "long blocks". Since a block is bordered

on both sides by "bad" strips, (3.4.4) implies at once that the total number of "blocks" and thus *a fortiori* that of the "short blocks" is

$$\leqslant 3\frac{T^{2(1-\alpha)}}{\log^{14} T}.$$

Thus the number of "good" strips contained in "short blocks" is

$$(5.1.3)\qquad < 3\frac{T^{2(1-\alpha)}}{\log^{14} T}(24\log^8 T + 4) < 84\frac{T^{2(1-\alpha)}}{\log^6 T}.$$

Considering (2.1.3), (5.1.3) and (3.4.4) imply at once that the contribution of the "bad" strips and of the "good" strips in "short blocks" to the total number of zeros in

$$T \leqslant v \leqslant 2T,$$

i. e., a fortiori to the total number of zeros in the parallelogram

$$(5.1.4)\qquad \alpha \leqslant u \leqslant 1 \quad (T \leqslant v \leqslant 2T),$$

is at most

$$(5.1.5)\qquad c_{31}\frac{T^{2(1-\alpha)}}{\log^5 T}.$$

2. Hence we have to consider only the contribution of the "good" strips belonging to "long blocks" to the number of zeros in the domain (5.1.4), *i. e.*, to

$$N(\alpha, 2T) - N(\alpha, T).$$

Consider an A „long block" and divide it starting from the l_μ-strip with the smallest index into "sub-blocks" each containing

$$(5.2.1)\qquad 2[3\log^8 T] + 1$$

consecutive strips; the last "sub-block" may contain less. The contribution of these "incomplete sub-blocks" to $N(\alpha, 2T) - N(\alpha, T)$ is obviously at most

$$(5.2.2)\qquad 3\frac{T^{2(1-\alpha)}}{\log^{14} T}(2[3\log^8 T])c_{18}\log(2+2T) < c_{32}\frac{T^{2(1-\alpha)}}{\log^5 T}.$$

Thus it remains to consider the contribution of those "long blocks" to $N(\alpha, 2T) - N(\alpha, T)$ which consist of "sub-blocks", each containing exactly $2[3\log^8 T]+1$ consecutive "good" strips. We remark that obviously each "long block" contains owing to (5.1.2) and (5.2.1) at least four "sub-blocks".

3. We consider the "extreme righthand" zeros ϱ_j^* defined in (5.1.1) belonging to the different good l_j-strips. We shall call such a ϱ_j^*-zero belonging to a "good" l_j-strip an "outstanding" one, if $\zeta(w) \neq 0$ in the parallelogram

$$u \geqslant \sigma_j^* + \frac{1}{[\log^4 T]}, \tag{5.3.1}$$
$$|v - t_j^*| \leqslant [\log T].$$

We assert the simple

Lemma V. *Each "sub-block" of a "long block" contains at least one "outstanding" ϱ_j^*-zero.*

Proof. We consider an "arbitrary sub-block" E of our "long block" and consider first the strip l_j in the middle of E (E consists of an *odd* number of strips). Perhaps this l_j-strip does not contain any zeta-zeros; but owing to (2.1.2) there is a j_0-index such that the l_{j_0}-strip indeed contains zeta-roots and for this j_0-index we have

$$|j - j_0| \leqslant [\log^3 T][\log T]. \tag{5.3.2}$$

If $\varrho_{j_0}^*$ is not an "outstanding" zero, this means that there is a j_1-index with

$$|j_1 - j_0| \leqslant [\log^3 T][\log T],$$

i. e., by (5.3.2), with

$$|j_1 - j| \leqslant 2[\log^3 T][\log T] \tag{5.3.3}$$

such that

$$\sigma_{j_1}^* \geqslant \sigma_{j_0}^* + \frac{1}{[\log^4 T]}. \tag{5.3.4}$$

If also $\varrho_{j_1}^*$ were not an "outstanding" zero, then there would be a j_2-index with

$$|j_2 - j_1| \leqslant [\log^3 T][\log T],$$

i. e., by (5.3.3), with

$$|j_2 - j| \leqslant 3[\log^3 T][\log T] \tag{5.3.5}$$

such that

$$\sigma_{j_2}^* \geqslant \sigma_{j_1}^* + \frac{1}{[\log^4 T]} \geqslant \sigma_{j_0}^* + \frac{2}{[\log^4 T]};$$

taking (5.3.4) in account. Since $\zeta(w) \neq 0$ for $u \geqslant 1$, this process necessarily stops within $[\log^4 T]$ such steps. Hence the index of the last strip is at most

$$j + ([\log^4 T]+1)[\log^3 T][\log T] < j + 2\log^8 T < j + [3\log^8 T],$$

i. e., the corresponding l-strip still belongs to our sub-block E, q. e. d.

4. As has been said, each "long block" A consists of at least four "sub-blocks". Let us call the two "sub-blocks" containing the l_j-strips of A with the greatest, or the smallest, indices the "wings" of our "long block" A, and A without the "wings" — the "kernel of A". According to the introductory remark of this section 4 the "kernel" of A is not empty. The contribution of all "wings" to $N(\alpha, 2T) - N(\alpha, T)$ is obviously

$$< 2\cdot 3\frac{T^{2(1-\alpha)}}{\log^{14} T}(2[3\log^8 T]+1)\cdot c_{18}\log(2+2T) < c_{33}\frac{T^{2(1-\alpha)}}{\log^5 T}. \tag{5.4.1}$$

As to the contribution of all "kernels" we assert

Lemma VI. *If λ is an upper bound of the real parts of the "outstanding" zeros lying in the ("good") l_j-strips of all "long blocks" then in all "kernels" $\zeta(w)$ does not vanish for*

$$u > \lambda.$$

Proof. Let γ be the maximum of the real parts of zeros lying in the "kernels". If γ is at the same time the real part of an "outstanding" zero, the proof is finished. If not, then applying the process used in the proof of lemma V we shall find that γ is majorized by an "outstanding" zero of a "wing"; but this means that again

$$\gamma \leqslant \lambda,$$

from which our lemma follows.

5. It follows immediately from this lemma that if for $T > c_{34}(\varepsilon)$ and for the above λ the inequality

$$\lambda \leqslant \alpha + 3\varepsilon^2 \tag{5.5.1}$$

holds, then owing to (5.1.5), (5.2.2), (5.4.1) and lemma VI the estimation

$$N(\alpha+3\varepsilon^2, 2T) - N(\alpha+3\varepsilon^2, T) < c_{37}\frac{T^{2(1-\alpha)}}{\log^5 T} \tag{5.5.2}$$

holds. We are going to prove in the next chapter that (5.5.1) holds for all "outstanding" zeros (*i. e.*, *not only* for those belonging the „long blocks"). Here we shall see why the reduction to the „outstanding" zeros has been so essential.

§ 6. The upper estimation of the real parts of the "outstanding" zeros

1. We consider an arbitrary "good" l_j-strip where the "extreme righthand" zero is an "outstanding" one, *i. e.*, with

$$\varrho_j^* = \sigma_j^* + it_j^* \tag{6.1.1}$$

the zeta-function does not vanish for

$$u \geqslant \sigma_j^* + \frac{1}{[\log^4 T]}, \qquad |v - t_j^*| \leqslant [\log T]. \tag{6.1.2}$$

The index j is fixed. Then we have the crucial

Lemma VII. *We have*

$$\sigma_j^* \leqslant \alpha + 3\varepsilon^2.$$

Proof. If this lemma would false, then we have

$$\alpha + 3\varepsilon^2 < \sigma_j^* \left(\leqslant 1 - \frac{10}{\log T}\right). \tag{6.1.3}$$

We shall apply lemma IV with

$$\mu = j; \tag{6.1.4}$$

the ν-index is uniquely determined by requiring

$$\sigma_\nu \equiv 1 - \frac{\nu}{[\log^2 T]} \geqslant \sigma_j^* > 1 - \frac{\nu+1}{[\log^2 T]} \equiv \sigma_\nu - \frac{1}{[\log^2 T]}. \tag{6.1.5}$$

Owing to (6.1.3) and (2.2.5) the condition

$$2 \leqslant \nu \leqslant \tfrac{1}{2}[\log^2 T] - 2$$

is fulfilled for $T > c_{35}(\varepsilon)$. Owing to the definition we have

$$\left| t_j^* - T - \frac{j}{[\log^3 T]} \right| < \frac{1}{[\log^3 T]}. \tag{6.1.6}$$

Owing to (6.1.2) the inequality (4.1.5) can be written in the form

$$\Bigg| \sum_{\substack{\varrho \\ |t_\varrho - t_j^*| > [\log T]}} \left\{ e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right\}^k + \tag{6.1.7}$$

$$+ \sum_{\substack{\varrho \\ |t_\varrho - t_j^*| \leqslant [\log T] \\ \sigma_\varrho \leqslant \sigma_j^* + 1/[\log^4 T]}} \left\{ e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right\}^k \Bigg| < c_{30}(\varepsilon) T^{\alpha - \sigma_\nu + 2\varepsilon^2} \log^{14} T,$$

where $\varrho = \sigma_\varrho + it_\varrho$ stand for the zeros of $\zeta(w)$. By (6.1.5) the expression on the right of (6.1.7) is a fortiori

$$\leqslant c_{30}(\varepsilon) T^{\alpha - \sigma_j^* + 2\varepsilon^2} \log^{14} T. \tag{6.1.8}$$

2. We estimate the first sum on the left of (6.1.7) roughly by (2.1.3). This implies at once that this sum is absolutely

$$\leqslant 2c_{18} \sum_{n \geqslant [\log T]} \log(2 + t_j^* + n) e^{kN_1(1-\sigma_\nu)} \frac{e^{\varepsilon^2 k N_1(1-\sigma_\nu)}}{(\varepsilon^2 N_1)^k n^k}$$

i. e., owing to (2.2.4) and (2.2.7) for $T > c_{38}(\varepsilon)$,

$$< c_{39} e^{kN_1(1+\varepsilon^2)} \frac{\log^2 T}{[\log T]^k} < c_{39} \frac{T^2 \log^2 T}{(\log T)^{(1/2N_1)\log T}} < \frac{1}{T^2}.$$

This, (6.1.7) and (6.1.8) give together for $T > c_{40}(\varepsilon)$

$$\Bigg| \sum_{\substack{\varrho \\ |t_\varrho - t_j^*| \leqslant [\log T] \\ \sigma_\varrho \leqslant \sigma_j^* + 1/[\log^4 T]}} \left\{ e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right\}^k \Bigg| \leqslant c_{41}(\varepsilon) T^{\alpha - \sigma_j^* + 2\varepsilon^2} \log^{14} T. \tag{6.2.1}$$

Next we consider the contribution of the zeros with

$$\sigma_\varrho \leqslant \sigma_j^* - \frac{6}{\varepsilon^2 N_1}, \qquad |t_\varrho - t_j^*| \leqslant [\log T]. \tag{6.2.2}$$

Since from (6.2.2) and (6.1.5)

$$\operatorname{Re} N_1(\varrho - s_{\nu j}) \leqslant N_1 \left(\left(\sigma_j^* - \frac{6}{\varepsilon^2 N_1} \right) - \sigma_j^* \right) = -\frac{6}{\varepsilon^2},$$

we have owing to (2.2.7)

$$\left| e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right|^k \leqslant \left(\frac{e^{-(1-\varepsilon^2)6/\varepsilon^2}}{6} \right)^k < T^{-2/N_1};$$

hence the use of (2.1.3) implies at once that the contribution of these ϱ's is absolutely

$$< c_{42} T^{-2/N_1} \log^2 T. \tag{6.2.3}$$

Next we consider the contribution of the zeros with

$$\sigma_j^* - \frac{6}{\varepsilon^2 N_1} < \sigma_\varrho \leqslant \sigma_j^* + \frac{1}{[\log^4 T]}, \qquad \frac{6}{\varepsilon^2 N_1} < |t_\varrho - t_j^*| \leqslant [\log T]. \tag{6.2.4}$$

Since owing to (6.1.5)

$$\left| e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right|^k \leqslant \left(\frac{e^{(1+\varepsilon^2)N_1(\sigma_j^* + 1/[\log^4 T] - \sigma_\nu)}}{\varepsilon^2 N_1(6/\varepsilon^2 N_1)} \right)^k$$

$$< \frac{e^{2kN_1/[\log^4 T]}}{6^k} < \frac{2}{6^k},$$

we have using (2.2.7) for the absolute value of this contribution the upper bound

$$\frac{c_{41}\log^2 T}{6^k} < c_{43} T^{-3/2N_1} \log^2 T. \tag{6.2.5}$$

Combining (6.2.1), (6.2.3) and (6.2.5) we obtain

$$Z \equiv \left| \sum_{\substack{\varrho \\ |t_\varrho - t_j^*| \leqslant 6/\varepsilon^2 N_1 \\ \sigma_j^* - 6/\varepsilon^2 N_1 < \sigma_\varrho \leqslant \sigma_j^* + 1/[\log^4 T]}} \left\{ e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})} \right\}^k \right| \tag{6.2.6}$$

$$\leqslant c_{44}(\varepsilon) \log^{14} T \{ T^{\alpha - \sigma_j^* + 2\varepsilon^2} + T^{-3/2N_1} \}$$

for $T > c_{45}(\varepsilon)$.

3. Until now we have not used the conjecture B. We shall use it in estimating the number ν_0 of terms in Z. Owing to (2.2.6) this number is not *decreased when we replace it by the number of zeros in the square*

$$\sigma_j^* - \frac{6}{\varepsilon^2 N_1} \leqslant \sigma \leqslant \sigma_j^* + \frac{6}{\varepsilon^2 N_1}, \quad |t - t_j^*| \leqslant \frac{6}{\varepsilon^2 N_1}. \tag{6.3.1}$$

We apply the conjecture B with

$$a_3 = \sigma_j^* + \frac{6}{\varepsilon^2 N_1}, \quad \delta = \frac{12}{\varepsilon^2 N_1} (= \delta_1), \quad \varkappa = \frac{1}{2} + \frac{120}{\varepsilon^2 N_1}, \quad \tau = t_j^*. \tag{6.3.2}$$

Then owing to (2.2.6) the domain (1.4.2) is contained in the parallelogram

$$\sigma_j^* + \frac{1}{[\log^4 T]} \leqslant u \leqslant 1, \quad |v - t_j^*| \leqslant [\log T],$$

which is indeed free of zeros of $\zeta(w)$ owing to the definition of the "outstanding" zeros. The squares (6.3.1) and (1.4.3) are then identical, further by (2.2.4) $0 < \delta \leqslant \frac{1}{10}(\varkappa - \frac{1}{2})$ is fulfilled as well as $\frac{1}{2} < \varkappa < 1$. Finally by (6.1.3) and (2.2.5)

$$\sigma_j^* > a \geqslant \frac{1}{2} + \frac{125}{\varepsilon^2 N_1}$$

i. e.,

$$\varkappa \leqslant a_3$$

is also satisfied and thus conjecture B is indeed applicable. Formula (1.4.4) gives for ν_0 — by (2.2.1) and (2.2.2) — the upper bound

$$\delta_1 g(\delta_1) \log T = \frac{12}{N_1 \varepsilon^2} \varepsilon^5 \log T < \frac{\varepsilon^2}{N_1} \log T. \tag{6.3.3}$$

Let us denote this last quantity by L_1 in the sequel.

4. Up to this point the integer k has only been restricted by the inequality (2.2.7). We shall now estimate Z from below by an appropriate choice of k within the given limits (2.2.7). This will be done by the following theorem.

If [9]

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n| \tag{6.4.1}$$

are arbitrary complex numbers, $m > 0$ arbitrary real and $n \leqslant L_1$, then there is an integer ν with

$$m \leqslant \nu \leqslant m + L_1 \tag{6.4.2}$$

such that

$$|z_1^\nu + z_2^\nu + \ldots + z_n^\nu| \geqslant \left(\frac{L_1}{23(m + L_1)} \right)^{L_1} |z_1|^\nu. \tag{6.4.3}$$

If we choose as z_ν-vectors the quantities

$$e^{N_1(\varrho - s_{\nu j})} \frac{e^{\varepsilon^2 N_1(\varrho - s_{\nu j})} - e^{-\varepsilon^2 N_1(\varrho - s_{\nu j})}}{2\varepsilon^2 N_1(\varrho - s_{\nu j})}, \tag{6.4.4}$$

these vectors and the domain of summation are independent of k, and hence the number of terms in the sum of (6.2.6) is independent of k. Hence the sum in (6.2.6) is a power-sum of fixed complex numbers. We choose as m of (6.4.2)

$$m = \frac{1}{N_1} \log T. \tag{6.4.5}$$

Owing to (6.3.3) the number of z_ν's is at most L_1, *i. e.*, the interval $(m, m + L_1)$ is identical with the interval given for k in (2.2.7); thus k can be chosen as the ν of theorem (6.4.1), (6.4.2), (6.4.3). The factor $|z_1|^\nu$

(9) This is an improved form of the theorem X of my above mentioned book. See [2].

can be estimated from below by taking the term corresponding to $\varrho = \varrho_j^*$. Owing to (6.1.5), (2.2.7) and (2.2.6) we have

(6.4.6) $$|e^{N_1(\varrho_j^*-s_{\nu j})}|^k = e^{kN_1(\sigma_j^*-\sigma_\nu)} > e^{-(1+\varepsilon^2)\log T/[\log^2 T]} > \tfrac{1}{2}.$$

Further using the inequality valid for $|z| \leqslant \frac{1}{2}$

$$\left|\frac{e^z-e^{-z}}{2z}\right| = \left|1+\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\right| > 1-\frac{|z|^2}{3}$$

and observing that for $T > c_{46}(\varepsilon)$ owing to (6.1.5) and (6.1.6)

$$\varepsilon^2 N_1 |\varrho_j^* - s_{\nu j}| \leqslant \varepsilon^2 N_1 \sqrt{\frac{1}{[\log^2 T]^2}+\frac{1}{[\log^3 T]^2}} < \frac{1}{2},$$

we have by (2.2.7)

$$\left|\frac{e^{\varepsilon^2 N_1(\varrho_j^*-s_{\nu j})}-e^{-\varepsilon^2 N_1(\varrho_j^*-s_{\nu j})}}{2\varepsilon^2 N_1(\varrho_j^*-s_{\nu j})}\right|^k \geqslant \left(1-\frac{(\varepsilon^2 N_1)^2}{3[\log^2 T]^2}\right)^k > e^{(-\varepsilon^4 N_1/[\log^2 T]^2)kN_1}$$
$$> e^{(-\varepsilon^4 N_1(1+\varepsilon^2)/[\log^2 T]^2)\log T} > \tfrac{1}{2}.$$

This, (6.4.6.) and (6.4.1) give

$$|z_1|^k > \tfrac{1}{4}$$

and hence (6.4.3) gives the lower estimation

$$Z > \frac{1}{4}\left(\frac{L_1}{23(m+L_1)}\right)^{L_1} = \frac{1}{4}\left(\frac{\varepsilon^2 N_1^{-1}}{23(N_1^{-1}+\varepsilon^2 N_1^{-1})}\right)^{(\varepsilon^2/N_1)\log T}$$
$$= \frac{1}{4}T^{(-\varepsilon^2/N_1)\log(23(1+\varepsilon^2)/\varepsilon^2)} > \frac{1}{4}T^{(-\varepsilon^2/N_1)\log(25/\varepsilon^2)}.$$

Comparing this with (6.2.6) we get for $T > c_{47}(\varepsilon)$

$$\frac{1}{4}T^{(-\varepsilon^2/N_1)\log(25/\varepsilon^2)} < c_{44}(\varepsilon)\log^{14}T\{T^{a-\sigma_j^*+2\varepsilon^2}+T^{-3/2N_1}\},$$

i. e., since for all sufficiently small ε's

$$\varepsilon^2\log\frac{25}{\varepsilon^2} < 1,$$

then

$$T^{(-\varepsilon^2/N_1)\log(25/\varepsilon^2)} < c_{48}(\varepsilon)T^{a-\sigma_j^*+2\varepsilon^2}\log^{14}T,$$

and for $T > c_{49}(\varepsilon)$

$$T^{(-2\varepsilon^2/N_1)\log(25/\varepsilon^2)} < T^{a-\sigma_j^*+2\varepsilon^2}.$$

This means that owing to (2.2.2) for all sufficiently small positive ε's

$$\sigma_j^* \leqslant a+2\varepsilon^2+\frac{2\varepsilon^2}{N_1}\log\frac{25}{\varepsilon^2} = a+\varepsilon^2\left(2+\frac{1}{6}\varepsilon^2\delta_1\log\frac{25}{\varepsilon^2}\right) < a+3\varepsilon^2,$$

i. e., lemma VII is proved.

§ 7. Proof of the theorem

1. According to lemma VII and (5.5.2) we have for the a's in (2.2.5) for $T > c_{36}(\varepsilon)$ *a fortiori*

(7.1.1) $$N(a+3\varepsilon^2, 2T)-N(a+3\varepsilon^2, T) < T^{2(1-a)}.$$

Replacing T by $T/2, T/2^2, \ldots, T/2^{\nu_1}$, where

$$\frac{T}{2^{\nu_1}} > c_{36}(\varepsilon) \geqslant \frac{T}{2^{\nu_1+1}}$$

and summing, we get

$$N(a+3\varepsilon^2, T) < c_{50}(\varepsilon)T^{2(1-a)},$$

or owing to (2.2.5) for

(7.1.2) $$\frac{1}{2}+\frac{125}{\varepsilon^2 N_1}+3\varepsilon^2 \leqslant a \leqslant 1-\max\left(3\varepsilon, \frac{6}{\varepsilon^2 N_1}\right)$$

the inequality

(7.1.3) $$N(a, T) < c_{50}(\varepsilon)T^{2(1-a+3\varepsilon^2)}.$$

Since for

$$a \leqslant 1-3\varepsilon$$

we have

$$2(1-a)+6\varepsilon^2 < 2(1+\varepsilon)(1-a)$$

and by (2.2.2) and (2.2.3)

$$125/\varepsilon^2 N_1 < 11\delta_1 < \varepsilon^2,$$

we obtain for $T > c_{51}(\varepsilon)$ and

$$\frac{1}{2}+4\varepsilon^2 \leqslant a \leqslant 1-\max\left(3\varepsilon, \frac{6}{\varepsilon^2 N_1}\right)$$

the inequality

(7.1.4) $$N(a, T) < c_{52}(\varepsilon)T^{2(1+\varepsilon)(1-a)}.$$

The case

$$(7.1.5)\qquad 1-\max\left(3\varepsilon, \frac{6}{\varepsilon^2 N_1}\right) \leqslant a \leqslant 1$$

is, by (1.1.10) already settled, provided ε is so small that with the c_6 from (1.1.9)

$$\frac{6}{\varepsilon^2 N_1} < c_6 \quad \text{and} \quad 3\varepsilon < c_6;$$

the exponent of T in (7.1.4) becomes

$$(7.1.6)\qquad 2(1-a)\{1+300(1-a)^{0,01}\} \leqslant 2\left\{1+300\left(\max\left(3\varepsilon, \frac{6}{\varepsilon^2 N_1}\right)\right)^{1/100}\right\}(1-a).$$

Since the case

$$(7.1.7)\qquad \tfrac{1}{2} \leqslant a \leqslant \tfrac{1}{2}+4\varepsilon^2$$

is trivial, our theorem is proved.

§ 8. Appendix

1. As has been said, we shall outline for $\tau > \tau_{53}(\varkappa, \delta)$ a proof of the inequality

$$(8.1.1)\qquad M(\tau, a_3, \delta) \leqslant 0{,}71\delta \log\frac{\tau}{2}$$

for the number $M(\tau, a_3, \delta)$ of the zeros of $\zeta(w)$ in the parallelogram

$$(8.1.2)\qquad a_3-\delta \leqslant u \leqslant a_3, \quad |v-\tau| \leqslant \frac{\delta}{2},$$

when $\frac{1}{2} < \varkappa < 1$, $\varkappa \leqslant a_3 \leqslant 1$, $0 < \delta \leqslant \frac{1}{10}(\varkappa-\frac{1}{2})$ and $\zeta(w)$ does not vanish in the parallelogram

$$(8.1.3)\qquad a_3 \leqslant u \leqslant 1, \quad |v-\tau| \leqslant \left[\log\frac{\tau}{2}\right].$$

First we need the simple

Lemma VIII. *If $\tau > c_{54}$, then in the domain*

$$(8.1.4)\qquad u \geqslant a_3+48\cdot\frac{\log\log\log\tau}{\log\log\tau}, \quad |v-\tau| \leqslant \frac{3}{4}\log\tau$$

the inequality

$$(8.1.5)\qquad \left|\frac{\zeta'}{\zeta}(w)\right| \leqslant \frac{\log\tau}{(\log\log\tau)^2}$$

holds.

Proof. For the sake of brevity denote

$$(8.1.6)\qquad 16\frac{\log\log\log\tau}{\log\log\tau} = \Delta$$

and take τ so large that

$$(8.1.7)\qquad \frac{3}{4}\log\tau < \log\frac{\tau}{2}-2.$$

We apply the inequality of Hadamard-Carathéodory ([10]) to the circle

$$(8.1.8)\qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{3}{2}-a_3$$

with

$$|\tau-\tau_0| \leqslant \left[\log\frac{\tau}{2}\right]-1,$$

and to the function

$$\log\frac{\zeta(w)}{\zeta(\frac{3}{2}+i\tau_0)},$$

which is certainly regular in our circle. Since in the circle we have roughly for $\tau > c_{55}$

$$\left|\frac{\zeta(w)}{\zeta(\frac{3}{2}+i\tau_0)}\right| < \tau,$$

it follows that in the circle

$$(8.1.9)\qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{3}{2}-a_3-\Delta$$

the inequality

$$\left|\log\frac{\zeta(w)}{\zeta(\frac{3}{2}+i\tau_0)}\right| \leqslant c_{56}\frac{\log\tau}{\Delta},$$

i. e., also

$$(8.1.10)\qquad |\log\zeta(w)| \leqslant c_{57}\frac{\log\tau}{\Delta}$$

([10]) According to this theorem if $f(w)$ is regular for $|w-w_0| \leqslant R$ and here $\mathrm{Re} f(w) \leqslant M$, then for $|w-w_0| \leqslant r$ $(< R)$ we have

$$|f(w)-f(w_0)| \leqslant \frac{2r}{R-r}(M-\mathrm{Re} f(w_0)).$$

holds. Next we apply the three circle-theorem to $\log\zeta(w)$ and to the circles

$$K_3: \qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{3}{2}-a_3-\Delta,$$

$$K_2: \qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{3}{2}-a_3-2\Delta,$$

$$K_1: \qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{1}{4}.$$

This gives by (8.1.10)

$$\max_{w\epsilon K_2}|\log\zeta(w)| \leqslant c_{58}\left(c_{57}\frac{\log\tau}{\Delta}\right)^{\log(6-4a_3-8\Delta)/\log(6-4a_3-4\Delta)}. \tag{8.1.11}$$

The exponent for $\tau > c_{59}$ is

$$1+\frac{\log\left(1-\dfrac{4\Delta}{6-4a_3-4\Delta}\right)}{\log(6-4a_3-4\Delta)} < 1-\frac{4\Delta}{(6-4a_3-4\Delta)\log(6-4a_3-4\Delta)} < 1-\tfrac{1}{2}\Delta,$$

i. e., by (8.1.11), for $\tau > c_{60}$

$$\max_{w\epsilon K_2}|\log\zeta(w)| \leqslant (\log\tau\log\log\tau)^{1-\Delta/2}$$

$$< \frac{\log\tau\cdot\log\log\tau}{(\log\tau)^{8\log\log\log\tau/\log\log\tau}} = \frac{\log\tau}{(\log\log\tau)^7}.$$

Then we have in the circle

$$K_4: \qquad |w-\tfrac{3}{2}-i\tau_0| \leqslant \tfrac{3}{2}-a_3-3\Delta$$

the estimation

$$\left|\frac{\zeta'}{\zeta}(w)\right| \leqslant \frac{1}{\Delta}\cdot\frac{\log\tau}{(\log\log\tau)^7} < \frac{\log\tau}{(\log\log\tau)^2}.$$

Since the circles K_4 cover $\tau > c_{61}$ the parallelogram

$$u \geqslant a_3+3\Delta, \qquad |v-\tau| \leqslant \tfrac{3}{4}\log\tau,$$

when τ_0 varies continuously between

$$\tau \pm \left(\left[\log\frac{\tau}{2}\right]-1\right),$$

lemma VIII is proved owing to (8.1.7).

2. Next we turn to the proof of (8.1.1). As is well known (see [3], p. 31),

$$\frac{\zeta'}{\zeta}(w) = b-\frac{1}{w-1}+\sum_{\varrho}\left(\frac{1}{w-\varrho}+\frac{1}{\varrho}\right)-\frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}+1\right), \tag{8.2.1}$$

where b is a constant. Restricting w to the parallelogram

$$a_3+48\frac{\log\log\log\tau}{\log\log\tau} \leqslant u \leqslant 1, \qquad |v-\tau| \leqslant \tfrac{1}{2}\log\tau \tag{8.2.2}$$

and taking real parts we obtain owing to lemma VIII for $\tau > c_{62}$

$$\left|\sum_{\varrho}\frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2}-\frac{1}{2}\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}+1\right)\right| \leqslant 2\frac{\log\tau}{(\log\log\tau)^2}$$

or by Stirling's formula

$$\left|\sum_{\varrho}\frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2}-\frac{1}{2}\log\tau\right| \leqslant 3\frac{\log\tau}{(\log\log\tau)^2}. \tag{8.2.3}$$

The contribution of the ϱ-zeros with

$$|t_\varrho-\tau| > \left[\log\frac{\tau}{2}\right]$$

is by (2.1.3) absolutely

$$< c_{18}\sum_{n>[\log(\tau/2)]}\frac{\log(\tau+n+2)}{n^2} < c_{63},$$

i. e., by (8.2.3) and (8.1.3)

$$\left|\sum_{\substack{|t_\varrho-\tau|\leqslant[\log(\tau/2)]\\ \sigma_\varrho\leqslant a_3}}\frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2}-\frac{1}{2}\log\tau\right| \leqslant 4\frac{\log\tau}{(\log\log\tau)^2} \tag{8.2.4}$$

or

$$\sum_{\substack{|t_\varrho-\tau|\leqslant[\log(\tau/2)]\\ \sigma_\varrho\leqslant a_3}}\frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2} \leqslant \frac{1}{2}\log\tau+4\frac{\log\tau}{(\log\log\tau)^2}. \tag{8.2.5}$$

Owing to (8.2.2) the terms of the sum in (8.2.5) are non-negative. We choose τ so large that

$$\delta \geqslant 500\frac{\log\log\log\tau}{\log\log\tau} \tag{8.2.6}$$

and

$$\frac{500\log\log\log\tau}{\log\log\tau} < \frac{1}{20}\left(\varkappa-\frac{1}{2}\right).$$

Keeping on the left of (8.2.5) only the terms whose ϱ belongs to the square (8.1.2) and using (8.2.5) with

$$u = a_3 + \frac{\sqrt{2}-1}{2}\delta, \qquad v = \tau$$

we get

$$\sum_{\substack{\varrho \\ |t_\varrho - \tau| \leqslant [\log(\tau/2)] \\ \sigma_\varrho \leqslant a_3}} \frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2} \geqslant M_3(\tau, a_3, \delta) \min_{a_3-\delta \leqslant x \leqslant a_3} \frac{a_3 + \frac{\sqrt{2}-1}{2}\delta - x}{\left(a_3 + \frac{\sqrt{2}-1}{2}\delta - x\right)^2 + \delta^2/4}$$

$$= M_3(\tau, a_3, \delta) \min_{\frac{\sqrt{2}-1}{2}\delta \leqslant y \leqslant \frac{\sqrt{2}+1}{2}\delta} \frac{y}{y^2+\delta^2/4} = M_3(\tau, a_3\ \delta)\frac{1}{\sqrt{2}}\cdot\frac{1}{\delta}.$$

This and (8.2.5) give together

$$M_3(\tau, a_3, \delta) \leqslant \frac{\sqrt{2}}{2}\delta\log\tau + 4\sqrt{2}\,\delta\frac{\log\tau}{(\log\log\tau)^2} < 0{,}71\,\delta\log\frac{\tau}{2}$$

if $\tau > c_{64}(\delta)$, q. e. d.

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On prime numbers in an arithmetical progression

by

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1. Let $k \geqslant 3$, $0 < l < k$, $(l, k) = 1$ be integers ((l, k) denotes the greatest common divisor of l, k).

Throughout this paper p denotes prime numbers, $\pi(x, k, l)$ denotes the number of primes not exceeding x, belonging to the arithmetical progression

$$l,\ l+k,\ l+2k,\ \ldots,$$

$c, c_0, c_1, \ldots$ denote positive numerical constants, $\Lambda(n)$ denotes the Dirichlet symbol:

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^a,\ a = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

$\varphi(k) = h =$ Euler's function, $L(s, X)$ denotes Dirichlet L-functions.

It is well-known that

$$\pi(x, k, l) = \frac{1}{h}\int_2^x \frac{du}{\log u} + O\left(x\exp(-c\sqrt{\log x})\right)$$

for any fixed k.

Write

$$\Delta(x, k, l) = \pi(x, k, l) - \frac{1}{h}\int_2^x \frac{du}{\log u}.$$

We can show by classical methods that if k is fixed and

(1.1) $\quad \Delta(x, k, 1) = O(x^{\vartheta+\varepsilon}) \quad (\frac{1}{2} \leqslant \vartheta < 1,\ \varepsilon > 0$ freely fixed, $x \to \infty)$

then

(1.2) $$\Delta(x, k, l) = O(x^{\vartheta+\varepsilon}) \qquad (x \to \infty)$$

for all l $(0 < l < k,\ (l, k) = 1)$ and each fixed $\varepsilon > 0$.

These methods cannot, however, reduce the relation (1.1) $\to$ (1.2) to an explicit inequality.