Keeping on the left of (8.2.5) only the terms whose ϱ belongs to the square (8.1.2) and using (8.2.5) with

$$u = a_3 + \frac{\sqrt{2}-1}{2}\delta, \qquad v = \tau$$

we get

$$\sum_{\substack{|t_\varrho-\tau|\leqslant[\log(\tau/2)]\\ \sigma_\varrho\leqslant a_3}} \frac{u-\sigma_\varrho}{(u-\sigma_\varrho)^2+(v-t_\varrho)^2} \geqslant M_3(\tau, a_3, \delta) \min_{a_3-\delta\leqslant x\leqslant a_3} \frac{a_3+\frac{\sqrt{2}-1}{2}\delta - x}{\left(a_3+\frac{\sqrt{2}-1}{2}\delta - x\right)^2+\delta^2/4}$$

$$= M_3(\tau, a_3, \delta) \min_{\frac{\sqrt{2}-1}{2}\delta\leqslant y\leqslant\frac{\sqrt{2}+1}{2}\delta} \frac{y}{y^2+\delta^2/4} = M_3(\tau, a_3, \delta)\frac{1}{\sqrt{2}}\cdot\frac{1}{\delta}.$$

This and (8.2.5) give together

$$M_3(\tau, a_3, \delta) \leqslant \frac{\sqrt{2}}{2}\delta\log\tau + 4\sqrt{2}\,\delta\frac{\log\tau}{(\log\log\tau)^2} < 0{,}71\,\delta\log\frac{\tau}{2}$$

if $\tau > c_{64}(\delta)$, q. e. d.

References

[1] E. Landau, *Vorlesungen über Zahlentheorie*, Bd. II,

[2] V. T. Sós and P. Turán, *On some new theorems in the theory of diophantine approximation*, Acta Math. Hung. 6 (1955), p. 241-257.

[3] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford 1951.

[4] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953. The Chinese edition was printed in 1956.

[5] — *On Lindelöf's conjecture*, Acta Math. Hung. 5 (1954), p. 145-163.

[6] — *On the zeros of the zeta-function of Riemann*, Lecture at the Colloquium for zeta-functions, Bombay 1956.

Reçu par la Rédaction le 12. 12. 1956

On prime numbers in an arithmetical progression

by

S. Knapowski (Poznań)

1. Let $k \geqslant 3$, $0 < l < k$, $(l, k) = 1$ be integers ((l, k) denotes the greatest common divisor of l, k).

Throughout this paper p denotes prime numbers, $\pi(x, k, l)$ denotes the number of primes not exceeding x, belonging to the arithmetical progression

$$l, l+k, l+2k, \ldots,$$

$c, c_0, c_1, \ldots$ denote positive numerical constants, $\Lambda(n)$ denotes the Dirichlet symbol:

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^\alpha,\ \alpha = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

$\varphi(k) = h =$ Euler's function, $L(s, X)$ denotes Dirichlet L-functions.

It is well-known that

$$\pi(x, k, l) = \frac{1}{h}\int_2^x \frac{du}{\log u} + O\left(x\exp(-c\sqrt{\log x})\right)$$

for any fixed k.

Write

$$\Delta(x, k, l) = \pi(x, k, l) - \frac{1}{h}\int_2^x \frac{du}{\log u}.$$

We can show by classical methods that if k is fixed and

(1.1) $\quad \Delta(x, k, 1) = O(x^{\vartheta+\varepsilon}) \quad (\frac{1}{2} \leqslant \vartheta < 1,\ \varepsilon > 0$ freely fixed, $x \to \infty)$

then

(1.2) $\quad \Delta(x, k, l) = O(x^{\vartheta+\varepsilon}) \quad (x \to \infty)$

for all l $(0 < l < k,\ (l, k) = 1)$ and each fixed $\varepsilon > 0$.

These methods cannot, however, reduce the relation (1.1) $\to$ (1.2) to an explicit inequality.

In this paper I obtain such "explicit estimation". In the proof Turán's methods are used.

The result is as follows:

If

$$T \geqslant \max\left(c_0, \exp\exp\left(150(k\log k)^2\right)\right)$$

then

$$(1.3)\qquad \max_{1\leqslant x\leqslant T}|\Delta(x,k,l)| \leqslant T^{\delta(T)}\exp\left(\left(1+\frac{1}{k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\left(\max_{1\leqslant x\leqslant T}|\Delta(x,k,1)|+\sqrt{T}\right).$$

$\delta(T)$ is a function tending to zero for $T\to\infty$. The construction of $\delta(T)$ depends on the distribution of the non-trivial zeros of the Dirichlet L-functions $(\mathrm{mod}\,k)$. Under the assumption of the generalized Riemann hypothesis we can put in (1.3) $\delta(T)\equiv 0$. It is worth noting that the estimate (1.3) is true without any conjectures. It seems me to be a very interesting question whether one could replace $\max\limits_{1\leqslant x\leqslant T}|\Delta(x,k,1)|$ on the right side of the inequality (1.3) by $\max\limits_{1\leqslant x\leqslant T}|\Delta(x,k,l_1)|$ for any l_1, $0<l_1<k$, $(l_1,k)=1$. Even the relation (1.1)$\to$(1.2) is problematic in this case.

I consider it my pleasant duty to express my deep gratitude to Professor P. Turán, who was kind enough to turn my attention to the aforesaid subject.

2. First, I shall prove some lemmas.

LEMMA 1. *Let X be a non-principal character* $(\mathrm{mod}\,k)$ *and let*

$$s=\sigma+it,\qquad \sigma\geqslant\tfrac{1}{4},\qquad -\infty<t<+\infty.$$

Then

$$|L(s,X)|\leqslant c_1k(|t|+1).$$

Proof. We apply partial summation.

LEMMA 2. *Let X be an non-principal character* $(\mathrm{mod}\,k)$. *Let N_T denote the number of zeros of $L(s,X)$ in the rectangle*

$$\tfrac{1}{4}\leqslant\sigma\leqslant 1,\qquad T\leqslant t\leqslant T+1,\qquad s=\sigma+it\qquad (T\geqslant 0).$$

Then

$$N_T\leqslant c_2\log k(T+1).$$

Proof. We apply Jensen's inequality for the circles

$$|s-s_0|\leqslant\tfrac{31}{16},\qquad |s-s_0|\leqslant\tfrac{15}{8}\qquad \left(s_0=2+(T+\tfrac{1}{2})i\right).$$

Then

$$N_T\leqslant c_3\max_{|s-s_0|=31/16}\left|\frac{L(s,X)}{L(s_0,X)}\right|\leqslant c_2\log k(T+1).$$

LEMMA 3 (compare [2], Lemma 13, p. 426). *Let X be a non-principal character* $(\mathrm{mod}\,k)$ *and* $n=0,1,2,\dots$ *If*

$$s=\sigma+it,\qquad \tfrac{1}{4}+\tfrac{1}{12}\leqslant\sigma\leqslant 2,\qquad n-\tfrac{1}{100}\leqslant t\leqslant n+1+\tfrac{1}{100},$$

then

$$\left|\frac{L'}{L}(s,X)-\sum_{\varrho}\frac{1}{s-\varrho}\right|\leqslant c_4\log k(n+1),$$

where ϱ denote some (not necessarily all) zeros of $L(s,X)$, lying in the rectangle

$$\tfrac{1}{4}+\tfrac{1}{24}\leqslant\sigma\leqslant 2,\qquad n-\tfrac{2}{100}\leqslant t\leqslant n+1+\tfrac{2}{100}.$$

Proof. We apply Turán's estimate (III.2.2) (see [3], p. 184), putting $R=\frac{7}{4}$, $\varepsilon>0$ numerical and so small that the rectangle $\frac{1}{4}+\frac{1}{12}\leqslant\sigma\leqslant 2$, $n-\frac{1}{100}\leqslant t\leqslant n+1+\frac{1}{100}$ lies in the interior of the circle

$$\left|s-\left(2+(n+\tfrac{1}{2})i\right)\right|\leqslant(1-2\varepsilon)R$$

and $R\cdot\varepsilon/2<\frac{1}{100}$ holds.

We can then take $M=c_5\log k(n+1)$ and the result follows.

LEMMA 4. *Let X be any character* $(\mathrm{mod}\,k)$.

1. *For* $n=0,1,2,\dots$ *there exist σ_n such that*

$$\tfrac{1}{4}+\tfrac{1}{12}<\sigma_n<\tfrac{1}{4}+\tfrac{1}{6}$$

and for $s=\sigma+it$, $\sigma=\sigma_n$, $n-\frac{1}{100}\leqslant t\leqslant n+1+\frac{1}{100}$,

$$\left|\frac{L'}{L}(s,X)\right|\leqslant c_6k\log^2\left(k(n+1)\right)$$

holds.

2. *For* $n=1,2,3,\dots$ *there exist t_n such that*

$$n-\tfrac{1}{100}<t_n<n+\tfrac{1}{100}$$

and for $s=\sigma+it$, $\frac{1}{4}+\frac{1}{12}\leqslant\sigma\leqslant 2$, $t=t_n$,

$$\left|\frac{L'}{L}(s,X)\right|\leqslant c_6k\log^2\left(k(n+1)\right)$$

holds.

Proof. Suppose first that X is a non-principal character. In virtue of Lemma 2 we note that the number Q of all the zeros of $L(s,X)$ lying in the rectangle

$$\tfrac{1}{4}+\tfrac{1}{24}\leqslant\sigma\leqslant\tfrac{1}{4}+\tfrac{5}{24},\qquad n-\tfrac{2}{100}\leqslant t\leqslant n+1+\tfrac{2}{100},\qquad s=\sigma+it$$

is $O\left(k\log k(n+1)\right)$. We now divide the interval $\langle\frac{1}{4}+\frac{1}{12},\frac{1}{4}+\frac{1}{6}\rangle$ into $Q+1$ equal parts. One of the $Q+1$ rectangles constructed thus is obviously free of zeros. We choose as σ_n the middle line of this rectangle. Then we clearly have

$$\left|\frac{L'}{L}(s, X)\right| \leqslant c_4\log\big(k(n+1)\big)+c_7 k\log^2\big(k(n+1)\big)$$

and the result follows.

In the case $X = X_0 =$ principal character we have

$$\frac{L'}{L}(s, X_0) = \frac{\zeta'}{\zeta}(s)+\sum_{p|k}\frac{\log p}{p^s-1} = \frac{\zeta'}{\zeta}(s)+O(k) \quad \text{for} \quad \sigma \geqslant \frac{1}{4}.$$

Further (see [1], p. 339)

$$\frac{\zeta'}{\zeta}(s) = b-\frac{1}{s-1}-\frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}s+1\right)+\sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)$$

$$= \sum_{|\varrho-n|\leqslant 1}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right)+O\big(\log(n+2)\big) = O\big(k\log^2 k(n+1)\big).$$

A similar argument clearly applies to the second part of Lemma 4.

THEOREM. *Let A_k denote the set of non-trivial zeros of all the Dirichlet L-functions* $(\mathrm{mod}\, k)$. *Further, write* $\varepsilon(T) = \max\limits_{\substack{\varrho=\beta+i\gamma\in A_k\\ |\gamma|\leqslant T}} \beta$.

If

(2.1) $$T \geqslant \max\big(c_0,\ \exp\exp\big(150(k\log k)^2\big)\big),$$

where c_0 is an explicitly calculable numerical constant, then

$$\max_{1\leqslant x\leqslant T}|\Delta(x, k, l)| \leqslant T^{\delta(T)}\exp\left(\left(1+\frac{1}{k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\left(\max_{1\leqslant x\leqslant T}|\Delta(x, k, 1)|+\sqrt{T}\right)$$

where

$$\delta(T) = \varepsilon(\sqrt{T})-\varepsilon(\exp\sqrt{\log\log T}).$$

It is obvious that $\delta(T)\to 0$ for $T\to\infty$. Under the assumption of the generalized Riemann hypothesis, $\delta(T)\equiv 0$.

First I shall prove a similar inequality for the function

$$\tilde{\Delta}(x, k, l) = \sum_{n\leqslant x} a_n^{(l)}\Lambda(n)-\frac{1}{h}x$$

where

$$a_n^{(l)} = \begin{cases} 1 & \text{if} \quad n\equiv l\ (\mathrm{mod}\, k),\\ 0 & \text{otherwise.}\end{cases}$$

3. Lower estimation of $\max\limits_{1\leqslant x\leqslant T}|\tilde{\Delta}(x, k, 1)|$. Put

$$K_0 = a\frac{\log T}{\log\log T}, \quad N_0 = \log^{1/a}T(\log\log T)^2, \quad a = 1+1/4k.$$

For T satisfying (2.1) we have

(3.1) $$K_0 > N_0 \geqslant 3.$$

Indeed, it suffices to show that $(\log T)^{1-1/a} > (\log\log T)^3$, *i. e.*, that

$$\frac{1}{12k+3} > \frac{\log\log\log T}{\log\log T}.$$

The function

$$\frac{\log\log\log T}{\log\log T}$$

decreases for $T\geqslant \exp\exp e$, whence (3.1) is valid in virtue of (2.1).

For $T > c_8$ there exists $L > 2$ such that

$$(L^{K_0} <)\ L^{K_0+N_0} \leqslant T < L^{K_0+N_0+1}\ (< L^{3K_0}).$$

Hence

$$(\log T)^{1/3a} \leqslant L \leqslant (\log T)^{1/a}.$$

Denote by T_L the number t_n of Lemma 4 for $n = [L+1+\frac{1}{100}]$. Then we have

$$L \leqslant T_L \leqslant L+1+\tfrac{2}{100}.$$

We now introduce an integer r satisfying the inequalities

$$(3 <)\ K_0 \leqslant r+1 \leqslant K_0+N_0\ (< 2K_0)$$

and the numbers

$$L^{r+1} = \xi, \quad \frac{1}{\log\xi} = \eta.$$

Consider the integral

$$J(T) = \frac{1}{2\pi i}\int_{1+\eta-iT_L}^{1+\eta+iT_L}\frac{\xi^s}{s^{r+1}}F(s)\,ds,$$

where

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n^{(1)}\Lambda(n)-1/h}{n^s} = -\frac{1}{h}\sum_{(X)}\frac{L'}{L}(s, X)-\frac{1}{h}\zeta(s)$$

(the symbol $\sum_{(X)}$ means the summation over all the characters $(\operatorname{mod} k)$).

We have

$$\left|\frac{1}{2\pi i}\int_{1+\eta\pm iT_L}^{1+\eta\pm i\infty}\frac{(\xi/n)^s}{s^{r+1}}ds\right| \leqslant \frac{\xi e}{2\pi n^{1+\eta}}\int_{T_L}^{\infty}\frac{dt}{t^{r+1}} = \frac{e}{2\pi r}\cdot\frac{\xi}{n^{1+\eta}}T_L^{-r},$$

whence

$$J(T) = \sum_{n=1}^{\infty}\left(a_n^{(1)}\Lambda(n)-\frac{1}{h}\right)\frac{1}{2\pi i}\int_{1+\eta-iT_L}^{1+\eta+iT_L}\frac{(\xi/n)^s}{s^{r+1}}ds$$

$$= \sum_{n\leqslant\xi}\frac{a_n^{(1)}\Lambda(n)-1/h}{r!}\log^r\frac{\xi}{n}+O\left(\sum_{n=1}^{\infty}\frac{\xi\log n}{rn^{1+\eta}}\cdot\frac{1}{T_L^r}\right).$$

But

$$\sum_{n=1}^{\infty}\frac{\xi\log n}{rn^{1+\eta}}\cdot\frac{1}{T_L^r} < \frac{L}{r}\sum_{n=1}^{\infty}\frac{\log n}{n^{1+\eta}} < c_9\frac{L}{r}\left(1+\frac{1}{\eta^2}\right) < c_{10}\log^3 T,$$

whence

$$|J(T)| \leqslant \left|\sum_{n\leqslant\xi}\frac{a_n^{(1)}\Lambda(n)-1/h}{r!}\log^r\frac{\xi}{n}\right| + c_{10}\log^3 T.$$

Hence, by partial summation, we obtain

$$|J(T)| \leqslant c_{10}\log^3 T+\frac{\log^r\xi}{r!}\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,1)|. \tag{3.2}$$

To obtain the lower estimate of $|J(T)|$ consider a contour C_T, consisting of the segment (I) $= \langle 1+\eta-iT_L, 1+\eta+iT_L\rangle$, a part of the polygonal line U given by Lemma 4 (between the ordinates $-T_L$ and $+T_L$) and of two segments (II), (III), joining U with the points $1+\eta\pm iT_L$ and parallel to the real axis.

In virtue of Cauchy's theorem

$$\frac{1}{2\pi i}\int_{C_T}\frac{\xi^s}{s^{r+1}}F(s)ds = -\frac{1}{h}\sum_{\substack{\varrho\in A_k,\varrho>U\\ |I\varrho|\leqslant T_L}}\frac{\xi^\varrho}{\varrho^{r+1}}$$

($\varrho > U$ denotes that ϱ lies to the right of U).

We shall now estimate the integrals $\int_{(\mathrm{I})}, \int_{(\mathrm{II})}, \int_{(\mathrm{III})}$.

$$\left|\frac{1}{2\pi i}\int_{(\mathrm{I})}\frac{\xi^s}{s^{r+1}}F(s)ds\right|$$

$$\leqslant c_{11}\xi^{1/4+1/6}\left(\int_{-T_L}^{T_L}\left\{k\log^2\big((|t|+1)k\big)+(|t|+1)^{(1-1/4)/2}\log(|t|+2)\right\}\frac{dt}{(t^2+\frac{1}{16})^{(r+1)/2}}+\right.$$

$$\left.+\sum_{n=0}^{T_L}\left\{\big(k\log^2(n+1)k+(n+1)^{(1-1/4)/2}\log(n+2)\big)\frac{1}{(\frac{1}{16}+n^2)^{(r+1)/2}}\right\}\right)$$

$$\leqslant c_{12}T^{1/4+1/6}\exp\left(12\frac{\log T}{\log\log T}\right)k\log^2 k.$$

Further

$$\left|\frac{1}{2\pi i}\int_{(\mathrm{II})}\frac{\xi^s}{s^{r+1}}F(s)ds\right|$$

$$\leqslant c_{13}\int_{1/4}^{1+\eta}\frac{\xi^\sigma}{T_L^{r+1}}\big(k\log^2(T_L+1)k+(T_L+1)^{(1-\sigma)/2}\log(T_L+2)\big)d\sigma$$

$$\leqslant c_{14}k\log^2 k\log T,$$

and a similar argument applies to the integral

$$\frac{1}{2\pi i}\int_{(\mathrm{III})}\frac{\xi^s}{s^{r+1}}F(s)ds.$$

Together

$$\left|\frac{1}{h}\sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\varrho\in A_k}}\frac{\xi^\varrho}{\varrho^{r+1}}\right| \leqslant |J(T)|+c_{15}T^{1/4+1/6}\exp\left(12\frac{\log T}{\log\log T}\right)k\log^2 k.$$

This and (3.2) give

$$\left|\frac{1}{h}\sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\varrho\in A_k}}\frac{\xi^\varrho}{\varrho^{r+1}}\right| \leqslant \frac{\log^r\xi}{r!}\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,1)|+T^{1/4+1/5} \tag{3.3}$$

for $T \geqslant \max\big(c_{16}, (k\log^2 k)^{60}\big)$.

To find the lower estimate of the sum

$$\left|\frac{1}{h}\sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\varrho\in A_k}}\frac{\xi^\varrho}{\varrho^{r+1}}\right|$$

apply Turán's method. In virtue of Lemma 2, the number of terms of this sum is

$$\leqslant c_{17}k(\log k)\log^{1/a}T(\log\log T) = c_{17}k(\log k)(\log T)^{4k/(4k+1)}\log\log T.$$

We now choose a zero

$$\varrho_0 = \beta_0 + i\gamma_0$$

satisfying the conditions

$$\beta_0 = \varepsilon(\exp\sqrt{\log\log T}), \quad |\gamma_0| \leqslant \exp\sqrt{\log\log T}, \quad \beta_0 \geqslant \tfrac{1}{2}, \quad \varrho_0 \epsilon A_k.$$

Then

$$\Bigg|\sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\, \varrho\epsilon A_k}} \frac{\xi^\varrho}{\varrho^{r+1}}\Bigg| = \frac{\xi^{\beta_0}}{|\varrho_0|^{r+1}}\Bigg|\sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\, \varrho\epsilon A_k}} \xi^{\varrho-\varrho_0}\left(\frac{\varrho_0}{\varrho}\right)^{r+1}\Bigg|.$$

Further

$$L^{\beta_0} \geqslant L^{1/2} \geqslant \log^{1/12}T > |\varrho_0|,$$

whence

$$\left(\frac{L^{\beta_0}}{|\varrho_0|}\right)^{r+1} > \frac{L^{\beta_0 K_0}}{|\varrho_0|^{K_0}}$$

$$> T^{\beta_0}\exp\left(-a\frac{\log T}{\sqrt{\log\log T}} - \frac{2}{a}(\log T)^{4k/(4k+1)}(\log\log T)^3\right).$$

We easily show under (2.1) that

$$\frac{2}{a}(\log T)^{4k/(4k+1)}(\log\log T)^3 < \frac{1}{16k}\cdot\frac{\log T}{\sqrt{\log\log T}}.$$

Indeed, it suffices to prove

$$\log^{1/(8k+2)}T > 32k \quad \text{and} \quad \log^{1/(8k+2)}T > (\log\log T)^{7/2}.$$

But both these inequalities are valid for

$$T \geqslant \max\big(c_{18},\ \exp\exp(30k\log k)\big).$$

Hence

$$\frac{\xi^{\beta_0}}{|\varrho_0|^{r+1}} > T^{\beta_0}\exp\left(-\left(1+\frac{5}{16k}\right)\frac{\log T}{\sqrt{\log\log T}}\right). \tag{3.4}$$

Now write

$$Z = \sum_{\substack{|I\varrho|<T_L\\ \varrho>U,\, \varrho\epsilon A_k}} \xi^{\varrho-\varrho_0}\left(\frac{\varrho_0}{\varrho}\right)^{r+1}$$

and apply Turán's theorem (see [3], Theorem X, p. 52), putting

$$N = k(\log k)\log^{1/a}T(\log\log T)^{3/2},$$

$$m = K_0 = a\frac{\log T}{\log\log T} > N.$$

We obviously have $N < N_0$ under (2.1).

Taking now a suitable r we obtain

$$\frac{1}{h}|Z| > \frac{1}{k}\left(\frac{1}{144e^2}\cdot\frac{N}{m}\right)^N > \exp\big(-k\log k\log^{4k/(4k+1)}T(\log\log T)^{5/2} - \log k\big)$$

$$> \exp\big(-2k\log k\log^{4k/(4k+1)}T(\log\log T)^{5/2}\big).$$

It is easy to prove

$$\exp\big(-2k\log k\log^{4k/(4k+1)}T(\log\log T)^{5/2}\big) > \exp\left(-\frac{1}{8k}\cdot\frac{\log T}{\sqrt{\log\log T}}\right)$$

if

$$T \geqslant \max\big(c_{19},\ \exp\exp(30k^2\log k)\big),$$

whence

$$\frac{1}{h}|Z| > \exp\left(-\frac{1}{8k}\cdot\frac{\log T}{\sqrt{\log\log T}}\right). \tag{3.5}$$

From (3.3), (3.4), (3.5) we obtain

$$\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,1)| \tag{3.6}$$

$$\geqslant \frac{r!}{\log^r\xi}\left(T^{\varepsilon(\exp\sqrt{\log\log T})}\exp\left(-\left(1+\frac{7}{16k}\right)\frac{\log T}{\sqrt{\log\log T}}\right) - T^{1/4+1/5}\right)$$

$$> \frac{r!}{\log^r\xi}T^{\varepsilon(\exp\sqrt{\log\log T})}\exp\left(-\left(1+\frac{7}{16k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\times$$

$$\times\left(1-\frac{\exp\left(\left(1+\frac{1}{6}\right)\frac{\log T}{\sqrt{\log\log T}}\right)}{T^{1/20}}\right)$$

$$> \frac{r!}{2\log^r\xi}T^{\varepsilon(\exp\sqrt{\log\log T})}\exp\left(-\left(1+\frac{7}{16k}\right)\frac{\log T}{\sqrt{\log\log T}}\right).$$

We now show that

$$2\frac{\log^r\xi}{r!} \leqslant \exp\left(\frac{1}{2k}\cdot\frac{\log T}{\sqrt{\log\log T}}\right). \tag{3.7}$$

Clearly

$$2\frac{\log^r\xi}{r!}\leqslant 2\frac{\log^r T}{r!}<2\left(\frac{e\log T}{r}\right)^r$$

$$<2\left(\frac{2e\log T}{r+1}\right)^r<2\left(\frac{2e\log T}{a\dfrac{\log T}{\log\log T}}\right)^{2a\log T/\log\log T}$$

$$=\exp\left(2a\frac{\log T}{\log\log T}\left(\log\log\log T+\log\frac{2e}{a}\right)+\log 2\right).$$

It suffices to prove

$$3\frac{\log T}{\log\log T}\log\log\log T<\frac{1}{2k}\cdot\frac{\log T}{\sqrt{\log\log T}},$$

i. e.,

$$6k<\frac{\sqrt{\log\log T}}{\log\log\log T},$$

and this inequality follows from (2.1).

From (3.6) and (3.7)

$$\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,1)|\geqslant T^{\varepsilon(\exp\sqrt{\log\log T})}\exp\left(-\left(1+\frac{15}{16k}\right)\frac{\log T}{\sqrt{\log\log T}}\right). \tag{3.8}$$

4. Upper estimation of $\tilde{\Delta}(x,k,l)$. Let $x\geqslant 3$. Put $T_x=t_n$ given by Lemma 4 for

$$n=[\sqrt{x}-\tfrac{1}{100}],\qquad a=1+\frac{1}{\log x},$$

$$f_l(s)=\sum_{n=1}^{\infty}\frac{a_n^{(l)}\Lambda(n)}{n^s}=-\frac{1}{h}\sum_{(X)}\frac{1}{X(l)}\cdot\frac{L'}{L}(s,X).$$

Then (*e. g.*, see [4], Theorem 24, p. 69)

$$\frac{1}{h}x+\tilde{\Delta}(x,k,l)=\frac{1}{2\pi i}\int\limits_{a-iT_x}^{a+iT_x}\frac{x^s}{s}f_l(s)\,ds+O(\sqrt{x}\log^2 x).$$

Clearly

$$\sqrt{x}-1-\tfrac{2}{100}\leqslant T_x\leqslant\sqrt{x}.$$

Consider an analogous contour to that in section 3 and apply Cauchy's theorem. The estimation of the integrals $\int\limits_{(\mathrm{I})},\int\limits_{(\mathrm{II})},\int\limits_{(\mathrm{III})}$ is quite similar to that in the preceding case.

$$\left|\frac{1}{2\pi i}\int\limits_{(\mathrm{I})}\frac{x^s}{s}f_l(s)\,ds\right|$$

$$\leqslant c_{20}x^{1/4+1/6}\left(\int\limits_{-\sqrt{x}}^{\sqrt{x}}k\log^2\big((|t|+1)k\big)\frac{dt}{(t^2+\frac{1}{16})^{1/2}}+\sum_{n=0}^{\sqrt{x}+1}k\log^2(n+1)k\frac{1}{(\frac{1}{16}+n^2)^{1/2}}\right)$$

$$\leqslant c_{21}x^{1/4+1/6}k\log^2 k\log^3 x.$$

Further

$$\left|\frac{1}{2\pi i}\int\limits_{(\mathrm{II})}\frac{x^s}{s}f_l(s)\,ds\right|\leqslant c_{22}\int\limits_{1/4}^{1+1/\log x}\frac{x^\sigma}{\sqrt{x}}\big(k\log^2(\sqrt{x}k)\big)\,d\sigma\leqslant c_{23}k\log^2 k\sqrt{x}\log^2 x.$$

We finally obtain

$$\tilde{\Delta}(x,k,l)=\sum_{\substack{|I\varrho|<T_x\\ \varrho>U,\,\varrho\epsilon\Delta_k}}\bar{X}(l)\frac{x^\varrho}{\varrho}+O(k\log^2 k\sqrt{x}\log^3 x).$$

Hence

$$|\tilde{\Delta}(x,k,l)|\leqslant x^{\varepsilon(\sqrt{x})}\sum_{\substack{|I\varrho|\leqslant\sqrt{x}\\ \varrho>U,\,\varrho\epsilon\Delta_k}}\frac{1}{|\varrho|}+O(\sqrt{x}\log^3 x\,k\log^2 k).$$

In virtue of Lemma 2

$$\sum_{\substack{|I\varrho|\leqslant\sqrt{x}\\ \varrho\epsilon\Delta_k}}\frac{1}{|\varrho|}\leqslant c_{24}k\log k\log^2 x.$$

Thus, for $x\geqslant\max\big(c_{25},\exp(k\log^2 k)\big)$

$$|\tilde{\Delta}(x,k,l)|\leqslant x^{\varepsilon(\sqrt{x})}\log^5 x$$

and for $x<\max\big(c_{25},\exp(k\log^2 k)\big)$

$$|\tilde{\Delta}(x,k,l)|\leqslant c_{26}+\exp(k\log^2 k)k\log^2 k.$$

Hence

$$\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,l)|\leqslant T^{\varepsilon(\sqrt{T})}\log^6 T\quad\text{for}\quad T\geqslant\max\big(c_{27},\exp(2k\log^2 k)\big). \tag{4.1}$$

Together with (3.8)

$$\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,l)|\leqslant T^{\delta(T)}\exp\left(\left(1+\frac{31}{32k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\max_{1\leqslant x\leqslant T}|\tilde{\Delta}(x,k,1)| \tag{4.2}$$

for $T\geqslant\max\Big(c_{28},\exp\exp\big(150(k\log k)^2\big)\Big)$.

5. The inequality for $\varDelta(x,k,l)$. Write

$$S_l(x) = \sum_{n\leqslant x} a_n^{(l)}\varLambda(n).$$

Then

$$\varDelta(x,k,l) = \sum_{2\leqslant n\leqslant x}\frac{S_l(n)-S_l(n-1)}{\log n} - \frac{1}{h}\int_2^x\frac{du}{\log u} + O(\sqrt{x}\log x)$$

$$= \sum_{2\leqslant n\leqslant x}\frac{\tilde{\varDelta}(n,k,l)-\tilde{\varDelta}(n-1,k,l)}{\log n} + O(\sqrt{x}\log x).$$

Hence

$$|\varDelta(x,k,l)| \leqslant \max_{2\leqslant n\leqslant x}|\tilde{\varDelta}(n,k,l)|\cdot\frac{1}{\log 2} + O(\sqrt{x}\log x)$$

and

$$\text{(5.1)}\qquad \max_{1\leqslant x\leqslant T}|\varDelta(x,k,l)| \leqslant \frac{1}{\log 2}\max_{1\leqslant x\leqslant T}|\tilde{\varDelta}(x,k,l)| + O(\sqrt{T}\log T).$$

On the other hand

$$\tilde{\varDelta}(x,k,1) = \sum_{n\leqslant x}\big(\pi(n,k,1)-\pi(n-1,k,1)\big)\log n - \frac{x}{h} + O(\sqrt{x}\log x)$$

$$= \frac{1}{h}\sum_{3\leqslant n\leqslant x}\log n\left(\int_{n-1}^{n}\frac{du}{\log u} - \frac{1}{\log n}\right) +$$

$$+ \frac{1}{h}\sum_{3\leqslant n\leqslant x}\big(\varDelta(n,k,1)-\varDelta(n-1,k,1)\big)\log n + O(\sqrt{x}\log x)$$

$$= \frac{1}{h}\sum_{n=3}^{x}\left(\frac{\log n}{\log(n-\Theta)}-1\right) - \sum_{n=2}^{x}\big(\log(n+1)-\log n\big)\varDelta(n,k,1) +$$

$$+\log([x]+1)\varDelta([x],k,1) + O(\sqrt{x}\log x),\quad\text{where}\quad 0<\Theta<1.$$

Hence

$$|\tilde{\varDelta}(x,k,1)| \leqslant c_{29}\sum_{n=2}^{x}\frac{1}{n\log n} + \max_{1\leqslant n\leqslant x}|\varDelta(n,k,1)|\cdot 2\log(x+1) + O(\sqrt{x}\log x)$$

$$\leqslant 3\log x\max_{1\leqslant n\leqslant x}|\varDelta(n,k,1)| + O(\sqrt{x}\log x)$$

and

$$\text{(5.2)}\qquad \max_{1\leqslant x\leqslant T}|\tilde{\varDelta}(x,k,1)| \leqslant 3\log T\max_{1\leqslant x\leqslant T}|\varDelta(x,k,1)| + c_{30}\sqrt{T}\log T.$$

(5.1) and (5.2) give in virtue of (4.2)

$$\max_{1\leqslant x\leqslant T}|\varDelta(x,k,l)|$$

$$\leqslant \frac{1}{\log 2}\left(T^{\delta(T)}\exp\left(\left(1+\frac{31}{32k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\left(3\log T\max_{1\leqslant x\leqslant T}|\varDelta(x,k,1)|+\right.\right.$$

$$\left.\left.+c_{31}\sqrt{T}\log T\right)\right)$$

$$\leqslant T^{\delta(T)}\exp\left(\left(1+\frac{1}{k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\left(\max_{1\leqslant x\leqslant T}|\varDelta(x,k,1)|+\sqrt{T}\right)$$

for $T\geqslant\max\big(c_0,\ \exp\exp(150(k\log k)^2)\big)$.

Remark 1. Note that the estimate (4.2) might be improved if we replace $\max_{1\leqslant x\leqslant T}|\tilde{\varDelta}(x,k,1)|$ on the right side of the inequality by

$$\max_{T^{1/2-\varepsilon}\leqslant x\leqslant T}|\tilde{\varDelta}(x,k,1)|$$

for any fixed $\varepsilon>0$. Of course, condition (2.1) must be replaced in this case by $T\geqslant c(k,\varepsilon)$.

Remark 2. By slight modifications in the proof we could obtain the following inequality:

$$\max_{1\leqslant x\leqslant T}|\varDelta(x,k,l)| \leqslant \exp\left(\left(1+\frac{1}{k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\left(\max_{1\leqslant x\leqslant T^{\zeta}}|\varDelta(x,k,1)+T^{\zeta/2}\right),$$

where

$$\zeta = \frac{\varepsilon(\sqrt{T})}{\varepsilon(\exp\sqrt{\log\log T})}\ (\to 1,\ T\to\infty),\qquad\text{if}\qquad T\geqslant c(k).$$

Remark 3. Write

$$\tilde{\tilde{\varDelta}}(x,k,l) = \sum_{\substack{n\equiv l\,(\mathrm{mod}\,k)\\ n\leqslant x}}\frac{\varLambda(n)}{\log n} - \frac{1}{h}\int_2^x\frac{du}{\log u}.$$

Let σ_0 be the lower bound of such σ_1 that in the half-plane $\sigma>\sigma_1$ none of the L-functions $(\mathrm{mod}\,k)$ has zeros. Suppose that there is a zero $\varrho\in A_k$ lying on the "border-line" $\sigma=\sigma_0$.

Then

$$\max_{1\leqslant x\leqslant T}|\tilde{\tilde{\varDelta}}(x,k,l)| \leqslant \exp\left(\left(1+\frac{1}{k}\right)\frac{\log T}{\sqrt{\log\log T}}\right)\max_{1\leqslant x\leqslant T}|\tilde{\tilde{\varDelta}}(x,k,1)|$$

for $(l,k)=1$, $0<l<k$, if

$$T\geqslant\max\big(c_0,\ \exp\exp(150(k\log k)^2),\ \exp\exp(\log^2|\varrho|)\big).$$

References

[1] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Bd I, Leipzig-Berlin 1909.

[2] E. C. Titchmarsh, *A divisor problem*, Rend. Circ. Mat. Palermo 54 (1930), p. 414-429.

[3] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Budapest 1953.

[4] Н. Чудаков, *Введение в теорию L-функций Дирихле*, Москва - Ленинград 1947.

Reçu par la Rédaction le 12. 1. 1957

On a theorem of Erdös-Kac

by

A. Rényi (Budapest) and P. Turán (Budapest)

Introduction

Let $V(n)$ denote the number of all prime factors of n, *i. e.*, if

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$$

where $p_1 < p_2 < \dots < p_r$ are primes, $a_1, a_2, \dots, a_r$ natural numbers, then let us put $V(n) = a_1 + a_2 + \dots + a_r$.

It has been discovered by G. H. Hardy and S. Ramanujan [6] that the number of those integers $k \leqslant n$ for which

$$(1) \qquad \left| \frac{V(k) - \log\log n}{\sqrt{\log\log n}} \right| > \omega(n),$$

where $\omega(n)$ is any function tending to $+\infty$ for $n \to \infty$, is $o(n)$. A very simple and elementary proof of this theorem has been given by P. Turán in his dissertation ([15] and [16]; for generalizations see [17]).

This proof consists in the application of Chebyshev's lemma, well known in the theory of probability. This was the first application of probabilistic methods to the investigation of additive number theoretic functions. Since that time a great number of important results have been achieved in this field of research. (As regards the bibliography of the subject see [8] and [9].)

The dissertation [15] contains also a second proof of the theorem of Hardy and Ramanujan. This second proof makes use of the standard tools of analytic number theory, Dirichlet series, contour integration, etc.

The aim of the present paper is to apply this analytic method to obtain the deeper statistical properties of the number theoretical function $V(n)$, or of other related functions.

We begin by giving in § 1 a new proof of the theorem of P. Erdös and M. Kac [3] concerning the function $V(n)$. This remarkable theorem states that if $N_n(V, x)$ denotes the number of those natural numbers $k \leqslant n$ for which

$$\frac{V(k) - \log\log n}{\sqrt{\log\log n}} < x,$$