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On a theorem of Erdös-Kac

by

A. Rényi (Budapest) and P. Turán (Budapest)

Introduction

Let $V(n)$ denote the number of all prime factors of n, *i. e.*, if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$$

where $p_1 < p_2 < \ldots < p_r$ are primes, $\alpha_1, \alpha_2, \ldots, \alpha_r$ natural numbers, then let us put $V(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r$.

It has been discovered by G. H. Hardy and S. Ramanujan [6] that the number of those integers $k \leqslant n$ for which

$$\left| \frac{V(k) - \log\log n}{\sqrt{\log\log n}} \right| > \omega(n), \tag{1}$$

where $\omega(n)$ is any function tending to $+\infty$ for $n \to \infty$, is $o(n)$. A very simple and elementary proof of this theorem has been given by P. Turán in his dissertation ([15] and [16]; for generalizations see [17]).

This proof consists in the application of Chebyshev's lemma, well known in the theory of probability. This was the first application of probabilistic methods to the investigation of additive number theoretic functions. Since that time a great number of important results have been achieved in this field of research. (As regards the bibliography of the subject see [8] and [9].)

The dissertation [15] contains also a second proof of the theorem of Hardy and Ramanujan. This second proof makes use of the standard tools of analytic number theory, Dirichlet series, contour integration, etc.

The aim of the present paper is to apply this analytic method to obtain the deeper statistical properties of the number theoretical function $V(n)$, or of other related functions.

We begin by giving in § 1 a new proof of the theorem of P. Erdös and M. Kac [3] concerning the function $V(n)$. This remarkable theorem states that if $N_n(V, x)$ denotes the number of those natural numbers $k \leqslant n$ for which

$$\frac{V(k) - \log\log n}{\sqrt{\log\log n}} < x,$$

then we have

$$\lim_{n\to\infty} \frac{N_n(V,x)}{n} = \Phi(x), \tag{2}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du. \tag{3}$$

In other words, the random variable ξ_n, which assumes the values $V(1), V(2), \ldots, V(n)$, each with the same probability $1/n$, is, for $n\to\infty$, asymptotically normally distributed with mean value $\log\log n$ and standard deviation $\sqrt{\log\log n}$.

The original proof of Erdös and Kac was not simple; besides the central limit theorem of the theory of probability, it used the sieve of Viggo Brun. Our proof given in § 1 is not elementary, but is much simpler than the original proof, or any other proof known to us, of the Erdös-Kac theorem. (It could be made still shorter, but in order to avoid duplications it contains also preparations to § 2.)

W. J. LeVeque [10] introduced certain mofidications of the proof of Erdös and Kac and obtained the following improvement of their result:

$$\frac{N_n(V,x)}{n} = \Phi(x) + O\left(\frac{\log\log\log n}{\sqrt[4]{\log\log n}}\right).$$

LeVeque conjectured that the error term is actually of order $1/\sqrt{\log\log n}$. Recently I. P. Kubilius [9] came very near to this conjecture, namely he proved

$$\frac{N_n(V,x)}{n} = \Phi(x) + O\left(\frac{\log\log\log n}{\sqrt{\log\log n}}\right).$$

In § 2 of the present paper we shall prove the conjecture of LeVeque; it can be shown by using a theorem of P. Erdös [2] and L. G. Sathe [12] that the result thus obtained, *i. e.*,

$$\frac{N_n(V,x)}{n} = \Phi(x) + O\left(\frac{1}{\sqrt{\log\log n}}\right), \tag{4}$$

is the best possible (¹).

(¹) The estimate in (4) is the best possible in the sense that $O(1/\sqrt{\log\log n})$ cannot be replaced by $o(1/\sqrt{\log\log n})$ uniformly in x. Nevertheless the remainder term can be improved in the sense that its dependence on x can be investigated (see [9]).

In § 3 we consider other number theoretical functions too. We show for example that (4) holds also for $U(n)$ instead of $V(n)$, where $U(n)$ denotes the number of different prime factors of n.

The same method as the one used in § 2 yields also the following result: if $d(n)$ denotes the number of divisors of n and $N_n(d,x)$ the number of those positive integers $k \leqslant n$ for which $d(k) \leqslant 2^{\log\log n + x\sqrt{\log\log n}}$, then

$$\frac{N_n(d,x)}{n} = \Phi(x) + O\left(\frac{1}{\sqrt{\log\log n}}\right).$$

As regards $d(n)$ the relation

$$\lim_{n\to\infty} \frac{N_n(d,x)}{n} = \Phi(x)$$

has been proved by M. Kac [7]. This has been improved by LeVeque [10] to

$$\frac{N_n(d,x)}{n} = \Phi(x) + O\left(\frac{\log\log\log n}{\sqrt[4]{\log\log n}}\right)$$

and by Kubilius [9] to

$$\frac{N_n(d,x)}{n} = \Phi(x) + O\left(\frac{\log\log\log n}{\sqrt{\log\log n}}\right).$$

Finally we give a new and simple proof of the formula proved recently by A. Rényi [11], according to which if the density of the sequence of those numbers n for which $V(n) - U(n) = k$ is denoted by d_k, then

$$\sum_{k=0}^{\infty} d_k z^k = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p-z}\right) \tag{5}$$

where p in the product on the right of (5) runs over all primes, and $|z| < 2$.

We shall return on an other occasion to the case of general additive functions to the dependence of the remainder-term upon x and the replacement of $\Phi(x)$ in (2) by an asymptotical expansion.

§ 1. Proof of the theorem of Erdös and Kac on the asymptotic distribution of the number of all prime factors of n

In this section we shall investigate the number-theoretical function $V(n)$. We put by definition $V(1) = 0$. We prove the following

THEOREM 1 (Erdös-Kac). *Let us denote by $N_n(V,x)$ the number of those positive integers $k \leqslant n$ for which*

$$\frac{V(k) - \log\log n}{\sqrt{\log\log n}} < x.$$

Then putting

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du,$$

we have

$$\lim_{n\to\infty} \frac{N_n(V, x)}{n} = \Phi(x) \qquad (-\infty < x < +\infty). \tag{1.1}$$

Proof of Theorem 1. Consider the Dirichlet series

$$\lambda(s, u) = \sum_{n=1}^{\infty} \frac{e^{iuV(n)}}{n^s} \tag{1.2}$$

where u is real and $s = \sigma + it$ a complex variable. The series on the right of (1.2) evidently converges for $\sigma > 1$. As $e^{iuV(n)}$ is (completely) multiplicative, *i. e.*,

$$e^{iuV(nm)} = e^{iuV(n)} e^{iuV(m)}$$

for any pair n, m of natural numbers, it follows that

$$\lambda(s, u) = \prod \frac{1}{(1-e^{iu}/p^s)}, \tag{1.3}$$

where p runs over all primes. Now let us put

$$\mu(s, u) = \frac{\lambda(s, u)}{(\zeta(s))^{e^{iu}}} \tag{1.4}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=2}^{\infty} \frac{1}{(1-1/p^s)} \tag{1.5}$$

is the zeta-function of Riemann, and the product on the right of (1.5) is extended over all primes p and $\log\zeta(s)$ is real for $\sigma > 1$.

Evidently for $\sigma > 1$

$$\log\mu(s, u) = \sum_{p=2}^{\infty} \sum_{k=2}^{\infty} \frac{e^{iu}(e^{iu(k-1)}-1)}{k p^{ks}}. \tag{1.6}$$

As the series on the right of (1.6) converges uniformly for $\sigma \geqslant \frac{1}{2}+\varepsilon$ where $\varepsilon > 0$ is arbitrary, it follows that, for any fixed real value of u, $\mu(s, u)$ is a regular function of s in the open half-plane $\sigma > \frac{1}{2}$. Later on we shall need the following estimation, which is a straightforward consequence of (1.6):

$$|\log\mu(s, u)| \leqslant |u| \tag{1.7}$$

for $s = \sigma + it$, $\sigma \geqslant 1$.

Now, by a well known formula for Dirichlet series, putting

$$S(n, u) = \sum_{k=1}^{n} e^{iuV(k)} \log\frac{n}{k} \tag{1.8}$$

we have

$$S(n, u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s \lambda(s, u)\,ds}{s^2} \tag{1.9}$$

where $c > 1$.

In what follows we shall always suppose $|u| \leqslant \pi/6$, which implies $\cos u \geqslant \frac{1}{2}$.

Let us effect the decomposition

$$\lambda(s, u) = \frac{\mu(s, u)}{(s-1)^{e^{iu}}} + \mu(s, u)\left((\zeta(s))^{e^{iu}} - \frac{1}{(s-1)^{e^{iu}}}\right) \tag{1.10}$$

with $\log(s-1)$ real for $s > 1$ and put

$$I_1 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s \mu(s, u)\,ds}{s^2 (s-1)^{e^{iu}}} \tag{1.11}$$

and

$$I_2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s \mu(s, u)}{s^2} \left((\zeta(s))^{e^{iu}} - \frac{1}{(s-1)^{e^{iu}}}\right) ds. \tag{1.12}$$

Then we have

$$S(n, u) = I_1 + I_2. \tag{1.13}$$

Let us consider first I_2. The integrand is regular for $s = \sigma + it$, $\sigma \geqslant 1$, except for $s = 1$, but it is continuous at this point also, because $(s-1)\zeta(s)$ is regular and equal to 1 at $s = 1$, and thus

$$\frac{[\zeta(s)(s-1)]^{e^{iu}} - 1}{(s-1)^{e^{iu}}}$$

is also continuous at $s = 1$, though of course it has a branching point there. Now let us push the path of integration to the line $s = 1 + it$ $(-\infty < t < +\infty)$ and apply partial integration in such a manner that n^s is chosen as the factor to be integrated, which results in the appearance of a factor $1/\log n$. By applying the well known estimates (see [14], Theorem 3.5 and 5.17)

$$|\zeta(1+it)| = O(\log t),$$

$$\left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right| = O(\log t),$$

$$|\log\zeta(1+it)| = O(\log t),$$

we obtain by routine calculations (the O-sign uniformly in $-\frac{1}{6}\pi \leqslant u \leqslant \frac{1}{6}\pi$)

$$I_2 = O\left(\frac{n}{\log n}\right). \tag{1.14}$$

Let us now turn to the investigation of I_1. Clearly we have

$$I_1 = I_{11} + I_{12} \tag{1.15}$$

where

$$I_{11} = \frac{\mu(1, u)}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s ds}{s^2(s-1)^{e^{iu}}} \tag{1.16}$$

and

$$I_{12} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s (s-1)^{1-e^{iu}}}{s^2} \left(\frac{\mu(s, u) - \mu(1, u)}{s-1}\right) ds. \tag{1.17}$$

As

$$\frac{\mu(s, u) - \mu(1, u)}{s-1}$$

is regular, and bounded for the half-plane $\mathrm{R}s \geqslant 1$ and further $\mathrm{R}(1-e^{iu}) \geqslant 0$, by transforming the path of integration of I_{12} to the line $\mathrm{R}s = 1$ and applying again partial integration we obtain again uniformly in u

$$I_{12} = O\left(\frac{n}{\log n}\right). \tag{1.18}$$

As regards I_{11}, we have

$$I_{11} = \mu(1, u)(I_{111} - I_{112}) \tag{1.19}$$

where

$$I_{111} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s ds}{(s-1)^{e^{iu}}} \tag{1.20}$$

and

$$I_{112} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{n^s (s-1)^{1-e^{iu}}(s+1)}{s^2} ds. \tag{1.21}$$

The integral I_{112} can clearly be transformed again to the line $\mathrm{R}s = 1$ and by integrating partially we obtain as before uniformly in u

$$I_{112} = O\left(\frac{n}{\log n}\right). \tag{1.22}$$

On the other hand, by transforming the integral I_{111} and using the well known integral representation of the Γ-function,

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du \qquad (\mathrm{R}z > 0),$$

further the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we obtain

$$I_{111} = \frac{n(\log n)^{e^{iu}-1}}{\Gamma(e^{iu})}. \tag{1.23}$$

Collecting our results, we obtain by virtue of (1.8), (1.13), (1.15), (1.16), (1.18), (1.19), (1.22) and (1.23) uniformly for $-\frac{1}{6}\pi \leqslant u \leqslant \frac{1}{6}\pi$

$$S(n, u) = n \frac{\mu(1, u)}{\Gamma(e^{iu})} (\log n)^{e^{iu}-1} + O\left(\frac{n}{\log n}\right). \tag{1.24}$$

Let us now put

$$s(n, u) = \sum_{k \leqslant n} e^{iuV(k)}; \tag{1.25}$$

then trivially

$$|s(y, u) - s(x, u)| \leqslant |y - x|. \tag{1.26}$$

Since

$$S(n, u) = \int_1^n \frac{s(x, u)}{x} dx, \tag{1.27}$$

we have for any $\lambda > 0$

$$s(n, u) = \frac{S(n+\lambda n, u) - S(n, u) + \int_n^{n+n\lambda} \frac{s(n, u) - s(x, u)}{x} dx}{\log(1+\lambda)}. \tag{1.28}$$

Thus from (1.26) uniformly in u

$$s(n, u) = \frac{S\big(n(1+\lambda), u\big) - S(n, u)}{\log(1+\lambda)} + O(\lambda n). \tag{1.29}$$

But if $0 < \lambda \leqslant \frac{1}{2}$

$$\frac{(1+\lambda)\big(\log n(1+\lambda)\big)^{e^{iu}-1} - (\log n)^{e^{iu}-1}}{\log(1+\lambda)} = (\log n)^{e^{iu}-1}\left(1 + O(\lambda) + O\left(\frac{|u|}{\log n}\right)\right), \tag{1.30}$$

the O-estimates being uniform in n, u and λ. Therefore, choosing $\lambda = (|u|/\log n)^{1/2}$, we obtain uniformly in u

$$\frac{s(n,u)}{n} = \frac{\mu(1,u)}{\Gamma(e^{iu})}(\log n)^{e^{iu}-1}\left(1+O\left(\left(\frac{|u|}{\log n}\right)^{1/2}\right)\right)+O\left(\frac{1}{\sqrt{|u|\log n}}\right). \tag{1.31}$$

Now $s(n,u)/n$ is nothing else than the characteristic function of the probability distribution of a random variable ξ_n, which takes on the values $V(1), V(2), \ldots, V(n)$ with probability $1/n$.

Thus, in order to prove Theorem 1 according to a well known theorem (*e. g.* see [1], p. 96-98) it suffices to show that putting

$$\varphi_n(u) = \frac{s(n, u/\sqrt{\log\log n})\,e^{-iu\sqrt{\log\log n}}}{n} \tag{1.32}$$

we have

$$\lim_{n\to\infty}\varphi_n(u) = e^{-u^2/2}. \tag{1.33}$$

But (1.33) is evident for $u=0$ and for $u\neq 0$ it follows by easy calculation from (1.31). Thus Theorem 1 is proved.

§ 2. Proof of the conjecture of LeVeque

In this section we shall prove

THEOREM 2 (Conjecture of LeVeque). *Let $N_n(V,x)$ denote the number of those natural numbers $k\leqslant n$ for which*

$$\frac{V(k)-\log\log n}{\sqrt{\log\log n}} < x.$$

Then we have uniformly in x

$$\frac{N_n(V,x)}{n} = \Phi(x)+O\left(\frac{1}{\sqrt{\log\log n}}\right).$$

Proof of Theorem 2. In order to prove Theorem 2 we follow the same method as that used in proving Theorem 1. The only difference consists in the fact that now, as we want to estimate the rate of convergence of $(1/n)N_n(V,x)$ to $\Phi(x)$, we have to consider the rate of convergence of $\varphi_n(u)$, defined by (1.32), to $e^{-u^2/2}$ and apply the following theorem of C. G. Esseen [5]:

If $F(x)$ and $G(x)$ are two distribution functions, $G'(x)$ exists for all x and $|G'(x)|\leqslant A$, $f(u) = \int_{-\infty}^{+\infty} e^{iux}\,dF(x)$ and $g(u) = \int_{-\infty}^{+\infty} e^{iux}\,dG(x)$ denote the

characteristic functions of the two distribution functions respectively, and the following condition is satisfied:

$$\int_{-T}^{+T}\left|\frac{f(u)-g(u)}{u}\right|du < \varepsilon, \tag{$*$}$$

then for $-\infty < x < +\infty$

$$|F(x)-G(x)| < K\left(\varepsilon+\frac{A}{T}\right)$$

where K is an absolute constant.

Let us verify the fulfilment of the condition $(*)$ with $G(x) = \Phi(x)$ (which implies $A = 1/\sqrt{2\pi}$),

$$F(x) = \frac{N_n(V,x)}{n}, \quad T = \frac{\pi}{6}\sqrt{\log\log n}, \quad \varepsilon = \frac{c}{\sqrt{\log\log n}}$$

where $c>0$ is a constant. We have only to prove that

$$\int_{-\pi\sqrt{\log\log n}/6}^{+\pi\sqrt{\log\log n}/6}\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right|du = O\left(\frac{1}{\sqrt{\log\log n}}\right). \tag{2.1}$$

We put

$$\int_{-\pi\sqrt{\log\log n}/6}^{+\pi\sqrt{\log\log n}/6}\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right|du = \Delta_1+\Delta_2 \tag{2.2}$$

where

$$\Delta_1 = \int_{|u|\leqslant 1/\sqrt{\log\log n}}\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right|du \tag{2.3}$$

and

$$\Delta_2 = \int_{1/\sqrt{\log\log n}\leqslant|u|\leqslant\pi\sqrt{\log\log n}/6}\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right|du. \tag{2.4}$$

Let us consider first Δ_1. Evidently, putting $a = 1/\sqrt{\log\log n}$, we have

$$\int_{-a}^{+a}\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right|du \leqslant \int_{-a}^{+a}\left|\frac{1-\varphi_n(u)}{u}\right|du+\int_{-a}^{+a}\left|\frac{1-e^{-u^2/2}}{u}\right|du. \tag{2.5}$$

Generally if $f(u)=\int\limits_{-\infty}^{+\infty} e^{ixu}dF(x)$, then

$$\int\limits_{-a}^{+a}\left|\frac{1-f(u)}{u}\right|du \leqslant 2a\sqrt{\int\limits_{-\infty}^{+\infty} x^2 dF(x)}. \tag{2.6}$$

Thus

$$\int\limits_{-a}^{+a}\left|\frac{1-\varphi_n(u)}{u}\right|du = O\left(\frac{1}{\sqrt{\log\log n}}\right) \tag{2.7}$$

because

$$\frac{\sum\limits_{k=1}^{n}(V(k)-\log\log n)^2}{n\log\log n}$$

is bounded (see [15]). As

$$\int\limits_{-a}^{+a}\left|\frac{1-e^{-u^2/2}}{u}\right|du = o\left(\frac{1}{\sqrt{\log\log n}}\right), \tag{2.8}$$

it follows that

$$\Delta_1 = O\left(\frac{1}{\sqrt{\log\log n}}\right). \tag{2.9}$$

Let us now turn to the estimation of Δ_2. Owing to the inequality $|e^{iz}-1-iz+z^2/2|\leqslant |z|^3/6$, valid for real z, we have from (1.31)

$$\left|\frac{\varphi_n(u)-e^{-u^2/2}}{u}\right| \leqslant \frac{1}{|u|^{3/2}}\,O\left(\frac{1}{\sqrt{\log n}}\right)+A(u) \tag{2.10}$$

where

$$A(u)=\frac{e^{-u^2/2}}{|u|}\left[\left(1+O\left(\frac{|u|}{\sqrt{\log\log n}}\right)\right)e^{\vartheta|u|^3/6\sqrt{\log\log n}}-1\right] \quad \text{and} \quad |\vartheta|\leqslant 1.$$

Thus

$$\Delta_2 \leqslant O\left(\frac{1}{\log^{1/3} n}\right)+\int\limits_{1/\sqrt{\log\log n}}^{\pi\sqrt{\log\log n}/6} A(u)\,du. \tag{2.11}$$

In order to estimate the integral on the right of (2.11) we remark that for $|u|\leqslant \sqrt[6]{\log\log n}$ we have

$$e^{\vartheta|u|^3/\sqrt{\log\log n}} = 1+O\left(\frac{|u|^3}{\sqrt{\log\log n}}\right),$$

which implies

$$\int\limits_{1/\sqrt{\log\log n}}^{\sqrt[6]{\log\log n}} A(u)\,du = O\left(\frac{1}{\sqrt{\log\log n}}\right). \tag{2.12}$$

On the other hand for $\sqrt[6]{\log\log n}<|u|\leqslant \pi\sqrt{\log\log n}/6$ we have

$$-\frac{u^2}{2}+\frac{\vartheta|u|^3}{6\sqrt{\log\log n}} \leqslant -\frac{u^2}{4} \quad \text{for} \quad |\vartheta|\leqslant 1,$$

and thus we obtain

$$\int\limits_{\sqrt[6]{\log\log n}}^{\pi\sqrt{\log\log n}/6} A(u)\,du = o\left(\frac{1}{\sqrt{\log\log n}}\right). \tag{2.13}$$

Thus we have proved (2.1) and therewith completed the proof of Theorem 2.

§ 3. A general theorem

The reasoning of §§ 1-2 can also be applied in other cases. We restrict ourselves here to the proof of a result, which contains Theorem 2 as a special case:

THEOREM 3. *Let $f(n)$ denote an additive arithmetic function, i. e., suppose that $f(nm)=f(n)+f(m)$ if n and m are relatively prime. Suppose that $f(p)=1$ for any prime p and that $|f(p^k)|\leqslant k^a$ $(k=1,2,\ldots)$ where $a>0$ is a constant, independent of p. Then, denoting by $N_n(f,x)$ the number of those positive integers $k\leqslant n$ for which*

$$\frac{f(k)-\log\log n}{\sqrt{\log\log n}}<x,$$

we have

$$\frac{N_n(f,x)}{n}=\Phi(x)+O\left(\frac{1}{\sqrt{\log\log n}}\right).$$

Proof. If

$$a(s,u)=\sum_{n=1}^{\infty}\frac{e^{iuf(n)}}{n^s}$$

and

$$\beta(s,u)=\frac{a(s,u)}{(\zeta(s))^{e^{iu}}},$$

then clearly $\beta(s, u)$ is regular in the half plane $s = \sigma + it$, $\sigma > \frac{1}{2}$ and

$$|\log \beta(s, u)| = O(|u|)$$

for $u \to 0$ and $s = \sigma + it$, $\sigma \geqslant 1$, and everything follows exactly as in the proof of Theorem 2.

Among the arithmetical functions $f(n)$ for which the hypotheses of Theorem 3 are satisfied let us mention besides $V(n)$ the functions $U(n)$ and $\log_2 d(n)$.

§ 4. The distribution of the function $\Delta(n) = V(n) - U(n)$

Let us consider the arithmetical function

$$\Delta(n) = V(n) - U(n) \quad (n = 1, 2, \ldots). \tag{4.1}$$

Clearly if $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$ where $p_1 < p_2 < \ldots < p_r$ are primes and $a_j \geqslant 1$ natural numbers, we have

$$\Delta(n) = \sum_{j=1}^{r} (a_j - 1). \tag{4.2}$$

The following result has been proved in [11](2):

The sequence of those integers n for which $\Delta(n) = k$ where k is any fixed nonnegative integer has a definite density d_k $(k = 0, 1, \ldots)$ and these densities d_k can be determined by means of their generating function

$$\sum_{k=0}^{\infty} d_k z^k = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p - z}\right) \tag{4.3}$$

where $|z| < 2$ and p runs over all primes.

(For $z = 0$ (4.3) reduces the relation $d_0 = 6/\pi^2$, which is well-known as d_0 is the density of squarefree integers.)

We shall now give a new proof of (4.3). Let us consider the Dirichlet series

$$\delta(s, u) = \sum_{n=1}^{\infty} \frac{e^{iu\Delta(n)}}{n^s}. \tag{4.4}$$

Evidently

$$\delta(s, u) = \zeta(s) A(s, u) \tag{4.5}$$

where

$$A(s, u) = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p^s}\right)\left(1 + \frac{1}{p^s - e^{iu}}\right). \tag{4.6}$$

(2) This occasion is used to acknowledge that formula (15) of [11] has been proved previously by I. J. Schoenberg [13].

Thus it follows by standard methods (much simpler than those used in § 1) that

$$\frac{1}{n} \sum_{k=1}^{n} e^{iu\Delta(k)} = A(1, u) + o(1), \tag{4.7}$$

which implies that the distribution of the random variable, which takes on the values $\Delta(1), \Delta(2), \ldots, \Delta(n)$ each with the probability $1/n$, tends for $n \to \infty$ to a limiting distribution, having the characteristic function $A(1, u)$. But this is equivalent to the relation (4.3), which is thus proved.

Of course the existence of the asymptotic distribution of $\Delta(n)$ follows from a well known general theorem of P. Erdös and A. Wintner (see [4]). In [4] the characteristic function of the asymptotic distribution of additive arithmetical functions is also considered in connection with the theory of infinite convolutions, and our explicit formula (4.3) could also be deduced from the general theory. Nevertheless it is not without interest to note, that formula (4.3) (and similar formulae for other additive functions) can also be deduced by the method of the present paper. It should also be mentioned that $\Delta(n)$ has already been thoroughly investigated by A. Wintner [18], who showed as early as 1942 that $\Delta(n)$ is almost periodic (B^∞) and determined its Fourier-series in terms of Ramanujan sums. A. Wintner [18] proved also that all moments $D_m = \sum_{k=0}^{\infty} d_k \cdot k^m$ $(m = 1, 2, \ldots)$ of the asymptotic distribution of $\Delta(n)$ exist, which can also be seen from (4.3). As a matter of fact (4.3) implies that

$$d_k \sim \frac{1}{2^{k+1}} \prod_{p=3}^{\infty} \frac{(p-1)^2}{p(p-2)} \quad \text{for} \quad k \to \infty.$$

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BUDAPEST, MATHEMATICAL INSTITUTE
OF THE HUNGARIAN ACADEMY OF SCIENCES

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The inhomogeneous minimum of quadratic forms of signature zero

by

B. J. Birch (Cambridge)

1. Minkowski proved that, if L_1, L_2 are linear forms in x, y of determinant Δ, then, given any x^*, y^*, we can find $(x, y) \equiv (x^*, y^*) \pmod 1$ so that

$$|L_1 L_2| \leqslant \tfrac{1}{4}|\Delta|.$$

He conjectured that a similar result remained true for the product of n linear forms; but this has been proved only for $n = 3$ and $n = 4$.

The result proved by Minkowski may be restated in terms of quadratic forms: *If $Q_2(x, y)$ is an indefinite binary quadratic form of determinant D, then, given any x^*, y^*, we can find $x \equiv x^*$ and $y \equiv y^*$* (mod 1) *so that*

$$|Q_2(x, y)| \leqslant |\tfrac{1}{4}D|^{1/2}.$$

Put in this way, the result may be generalized in a different way, as follows.

Given a quadratic form Q_r in r variables $x_1, \ldots, x_r$, we define the inhomogeneous minimum $M_I(Q_r)$ by

$$M_I(Q_r) = \sup_{x_1^*, \ldots, x_n^*} \Big\{ \inf_{x_i \equiv x_i^* (\mathrm{mod}\, 1)} [Q_r(x_1, \ldots, x_r)] \Big\}.$$

Then the natural generalization for quadratic forms of Minkowski's result is: "*If Q_r is any indefinite quadratic form in r variables of determinant $D \neq 0$, then*

$$M_I(Q_r) \leqslant |\tfrac{1}{4}D|^{1/r}."$$

By giving an example of an indefinite ternary form with $M_I(Q_3) = |\tfrac{27}{100}D|^{1/3}$, Davenport [4] showed that such a wide generalization is false. However, if we restrict ourselves to forms of signature zero the conjecture is valid; I will prove

Theorem 1. *Let Q_{2n} be any indefinite quadratic form in $2n$ variables, with signature zero and determinant $D \neq 0$. Then*

$$M_I(Q_{2n}) \leqslant |\tfrac{1}{4}D|^{1/2n}.$$