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STRONG $\mathcal{S}$-GROUPS

BY

ULRICH ALBRECHT AND H. PAT GOETERS (AUBURN, ALABAMA)

1. Introduction. Virtually all classes of groups considered in the theory of torsion-free abelian groups of finite rank arise in an attempt to recover some of the properties of rank 1 torsion-free groups for groups of arbitrary (finite) rank. The motivation for this paper is the observation that quasi-isomorphic rank 1 groups are actually isomorphic. The failure of this property for torsion-free abelian groups of larger rank has led to the introduction of the classes of $\mathcal{J}$-groups and finitely faithful $\mathcal{S}$-groups by Arnold [3] and others: A *$\mathcal{J}$-group* is a torsion-free abelian group A of finite rank such that any torsion-free group which is quasi-isomorphic to A is actually isomorphic to A, while A is an *$\mathcal{S}$-group* if every subgroup B of finite index in A is of the form IA for some right ideal I of $E(A)$. The group A is *finitely faithful* if $IA \neq A$ for all maximal right ideals I of $E(A)$ which have finite index in $E(A)$. Every $\mathcal{J}$-group is an $\mathcal{S}$-group.

Arnold showed in [3] that the finitely faithful $\mathcal{S}$-groups are precisely the torsion-free abelian groups A of finite rank for which $r_p(E(A)) = [r_p(A)]^2$ for all primes where $r_p(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} A/pA$ denotes the p-rank of A. Furthermore, using a result of Warfield, Arnold showed that the finitely faithful $\mathcal{S}$-groups are the torsion-free groups of finite rank for which $\mathrm{Ext}(A, A)$ is torsion-free. In [6], it is shown that a finitely faithful $\mathcal{S}$-group A is a $\mathcal{J}$-group when A is reduced and satisfies $r_p(A) \neq 2$ for any p, or A is quasi-isomorphic to $A_1 \oplus \ldots \oplus A_n$ such that $E(A_j)$ is commutative for $j = 1, \ldots, n$, or $A = B \oplus B$ for some group B. The present authors show in [2] that a finitely faithful group A is an $\mathcal{S}$-group if and only if $S_A(G)$ is a pure subgroup for all torsion-free groups G, where $S_A(G) = \sum\{\phi(A) \mid \phi \in \mathrm{Hom}(A, G)\}$ is the A-socle of G. Equivalently, the A-socle of G is the largest subgroup of G which is an epimorphic image of a direct sum of copies of A.

It has become customary to study $\mathcal{S}$-groups only in conjunction with finite faithfulness partially due to the difficulties in handling the $\mathcal{S}$-group property alone, and in part because of the compatibility of the finitely faithful and $\mathcal{S}$-group conditions. We show in Section 3 of this paper that finite

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faithfulness is not necessary for the purity of A-socles. The purity condition gives rise to a new class of groups which properly contains the class of finitely faithful $\mathcal{S}$-groups: A torsion-free abelian group A of finite rank is a *strong* $\mathcal{S}$*-group* if A^n is an $\mathcal{S}$-group for all $n < \omega$. In Section 2 we give a characterization of the almost completely decomposable strong $\mathcal{S}$-groups, which allows us to construct an example of a strong $\mathcal{S}$-group which is flat as an $E(A)$-module, but not finitely faithful.

Section 3 gives further characterizations of strong $\mathcal{S}$-groups and strong $\mathcal{S}$-groups which are flat as modules over their endomorphism ring. In particular, we show that a strong $\mathcal{S}$-group A has the property that every reduced p-group is A-solvable whenever p is a prime with $A \neq pA$. If A is flat as an $E(A)$-module, then the converse holds as well. Finally, we show that a strong $\mathcal{S}$-group A is quotient divisible if and only if every reduced torsion group G with $G[p] = 0$ if $A = pA$ is A-solvable. Here, A is *quotient divisible* if $A/F \cong D \oplus T$ for some divisible group D and some finite group T whenever F is a full free subgroup of A.

2. The structure of almost completely decomposable $\mathcal{S}$-groups. Our first result shows that the requirement that A is a strong $\mathcal{S}$-group does not impose severe restrictions on the structure of A, in contrast to those observed in [3] for finitely faithful $\mathcal{S}$-groups.

LEMMA 2.1. *Let G be a torsion-free group of finite rank, and X a rank* 1 *group such that* $\mathrm{type}(X) \leq IT(G)$. *Then $A = X \oplus G$ is a strong $\mathcal{S}$-group which is flat as an $E(A)$-module.*

Proof. Ulmer's Theorem from [8] asserts that A is flat over its endomorphism ring if and only if A generates the kernel of any homomorphism between powers of A. Since $S_A(U) = U$ for all pure subgroups $U \subseteq A^n$, A is flat. If B is any group quasi-isomorphic to A, then $IT(B) = \mathrm{type}(X)$, and so $B = S_X(B) \subseteq S_A(B) \subseteq B$, and A is an $\mathcal{S}$-group. Finally, $A^n = X \oplus [X^{n-1} \oplus G^n]$ and $IT(X^{n-1} \oplus G^n) = \mathrm{type}(X)$ for all $1 < n < \infty$. By the first part of the proof, A^n is an $\mathcal{S}$-group. ■

Note that in the case above, $B = X \oplus H$ for some group H quasi-isomorphic to G. In particular, B is a $\mathcal{J}$-group if G is. As the next step in our characterization of almost completely decomposable strong $\mathcal{S}$-groups, we describe completely decomposable $\mathcal{J}$-groups.

LEMMA 2.2. *Let A and B be $\mathcal{J}$-groups such that* $\mathrm{Ext}(A, B)$ *is torsion-free. Then $A \oplus B$ is a $\mathcal{J}$-group.*

Proof. Suppose that $\mathrm{Ext}(A, B)$ is torsion-free. If G is quasi-isomorphic to $A \oplus B$, then there is a quasi-split sequence $0 \to A_1 \to G \to B_1 \to 0$ where A_1 is quasi-isomorphic to A and B_1 is quasi-isomorphic to B. Since

A and B are $\mathcal{J}$-groups, we have $A_1 \cong A$ and $B_1 \cong B$. Since $\mathrm{Ext}(A,B)$ is torsion-free, the sequence splits. ∎

Using the last result, we obtain the following one, which was originally shown in [7], but is restated here for the convenience of the reader since it will be used in Example 2.5.

PROPOSITION 2.3. *Let $A = X_1 \oplus \ldots \oplus X_n$ where each X_j is a subgroup of $\mathbb{Q}$ of type τ_j. Then A is a $\mathcal{J}$-group if and only if, for each $i \neq j$, either $\tau_i \leq \tau_j$, or $\tau_j \leq \tau_i$, or $\pi(\tau_i) \cap \pi(\tau_j) = \emptyset$ where $\pi(\tau) = \{p \mid \tau$ is finite at $p\}$.*

Proof. Suppose that A is a $\mathcal{J}$-group. If τ_i and τ_j are incomparable and $p \in \pi(\tau_i) \cap \pi(\tau_j)$, consider the group $G = X_i \oplus X_j + \frac{1}{p}\mathbb{Z}(a_i, a_j)$ where $a_i \in X_i$ and $a_j \in X_j$ have p-height 0. It is well known [5] that G is an indecomposable group, quasi-isomorphic to $X_i \oplus X_j$. It follows that A is quasi-isomorphic to $B = G \oplus \bigoplus_{k \neq i,j} X_k$. But A and B are not isomorphic since the class of completely decomposable groups is closed with respect to direct summands.

Conversely, we induct on n, and assume without loss of generality that τ_1 is minimal among $\tau_1, \ldots, \tau_n$. Recall Warfield has shown that, for rank 1 groups X and Y, the group $\mathrm{Ext}(X,Y)$ is torsion-free if and only if $\mathrm{type}(X) \leq \mathrm{type}(Y)$ or $\pi(X) \cap \pi(Y) = \emptyset$ (cf. [9]). Then $\mathrm{Ext}(X_1, \bigoplus_{j=2}^{n} X_j)$ is torsion-free, and A is a $\mathcal{J}$-group by Lemma 2.2. ∎

PROPOSITION 2.4. *Let $A = X_1 \oplus \ldots \oplus X_n$ where each X_j is a subgroup of $\mathbb{Q}$ of type τ_j. Then A is an $\mathcal{S}$-group if and only if, for all $i \neq j$ such that τ_i and τ_j are incomparable but $\pi(\tau_i) \cap \pi(\tau_j) \neq \emptyset$, there is k such that $\tau_k \leq \tau_i \wedge \tau_j$.*

Proof. The stated condition is equivalent to the following: For any two distinct minimal types τ_i and τ_j among $\{\tau_1, \ldots, \tau_n\}$, the set $\pi(\tau_i) \cap \pi(\tau_j)$ is empty. Suppose the collection of τ_i's satisfies the stated condition. If B is quasi-isomorphic to A, then $B = \bigoplus_{j=1}^{m} B(\mu_j)$ where $\mu_1, \ldots, \mu_m$ are the minimal types among $\{\tau_1, \ldots, \tau_n\}$. This holds because $B \doteq B(\mu_1) + \ldots + B(\mu_m)$, while the condition $\pi(\mu_i) \cap \pi(\mu_j) = \emptyset$ guarantees equality and directness of the decomposition. Observe that $B(\mu_j)$ and $A(\mu_j)$ are quasi-isomorphic, and that $A(\mu_j)$ has a direct summand of type $\mu_j = IT(A(\mu_j))$. By Lemma 2.1, $A(\mu_j)$ is an $\mathcal{S}$-group, and so $S_A(B(\mu_j)) = B(\mu_j)$, i.e. $S_A(B) = B$.

Conversely, suppose that A is an $\mathcal{S}$-group. We may rewrite the given decomposition of A as $A = A_1 \oplus \ldots \oplus A_k$ where each A_j is a homogeneous completely decomposable group of type τ_j, and $\tau_i \neq \tau_j$ for $i \neq j$. Suppose that τ_i and τ_j are minimal types for which we can find $p \in \pi(\tau_i) \cap \pi(\tau_j)$. Choose rank 1 summands Y_i of A_i and Y_j of A_j containing elements x_i and x_j of p-height 0, and set $B = A + \frac{1}{p}\mathbb{Z}(x_i, x_j, 0, \ldots)$. The element $x = (x_i, x_j, 0, \ldots)$ of B has type $\tau_i \wedge \tau_j$. Therefore, $B(\tau_l) = A(\tau_l)$ for $l = 1, \ldots, k$. Since A is an $\mathcal{S}$-group, $\frac{1}{p}x \in S_A(B)$, and we can find maps $\phi_1,\dots,\phi_m \in H_A(B)$ and elements $a_1,\dots,a_m \in A$ such that $\frac{1}{p}x = \sum_{t=1}^m \phi_t(a_t)$. No generality is lost if we assume that each ϕ_t maps A into $A_i \oplus A_j$. Any map $\phi : A \to B$ can be expressed as $\phi = \phi\eta_1 + \ldots + \phi\eta_k$ where $\eta_1,\dots,\eta_k$ are the idempotents of $E(A)$ induced by the decomposition $A = A_1 \oplus \ldots \oplus A_k$. Hence, we may assume that each of the ϕ_t has support either in A_i or in A_j. If $\phi_t(A_j) = 0$, then $\phi_t(A_i) \subseteq A_i$, while $\phi_t(A_i) = 0$ yields $\phi_t(A_j) \subseteq A_j$. Therefore, each $\phi_t : A \to A$, and $\frac{1}{p}x \in A$, a contradiction. ■

As a direct consequence of the last two propositions and Ulmer's Theorem we obtain:

EXAMPLE 2.5. (a) Let X_1 and X_2 be subgroups of $\mathbb{Q}$ of incomparable types such that $\pi(\tau_1)\cap\pi(\tau_2) \neq \emptyset$, and choose a subgroup X_0 of $\mathbb{Q}$ such that $\mathrm{type}(X_0) < \mathrm{type}(X_1), \mathrm{type}(X_2)$. Then $A = X_0 \oplus X_1 \oplus X_2$ is a flat strong $\mathcal{S}$-group which is not a $\mathcal{J}$-group.

(b) Although the strong $\mathcal{S}$-group A constructed in Lemma 2.1 has the additional property that every pure rank 1 subgroup of A is A-generated, there are completely decomposable strong $\mathcal{S}$-groups without this property. For instance, let Π_1 and Π_2 be non-empty, disjoint subsets of the set Π of all primes of $\mathbb{Z}$, such that $\Pi = \Pi_1 \cup \Pi_2$, and define two subgroups A_1 and A_2 of $\mathbb{Q}$ by $A_i = \mathbb{Z}\big[\frac{1}{p} \mid p \in \Pi_i\big]$ for $i = 1, 2$. Since $\pi(A_1) = \Pi_2$ and $\pi(A_2) = \Pi_1$, the group $A = A_1 \oplus A_2$ is a strong $\mathcal{S}$-group which contains a pure subgroup U with $A/U \cong \mathbb{Q}$. Because $\Pi_1 \cap \Pi_2 = \emptyset$, one has $U \cong \mathbb{Z}$. Hence, U is not generated by A.

THEOREM 2.6. *An almost completely decomposable group A of finite rank is an $\mathcal{S}$-group if and only if $A = A_1 \oplus \ldots \oplus A_n$, where each $A_i = X_i \oplus G_i$ for some rank* 1 *group X_i with* $\mathrm{type}(X_i) \le IT(G_i)$, *and if $i \neq j$, then* $\pi(X_i) \cap \pi(X_j) = \emptyset$.

Proof. Suppose that A has the described form, and consider a group B quasi-isomorphic to A. Then $B = B_1 \oplus \ldots \oplus B_n$ where each B_j is quasi-isomorphic to A_j since $\pi(X_i) \cap \pi(X_j) = \emptyset$ and $S_{A_j}(B) = S_{X_j}(B)$. By Lemma 2.1, $S_{A_j}(B_j) = B_j$, and A is an $\mathcal{S}$-group.

Conversely, choose a non-zero integer m such that $mA \subseteq C_1 \oplus \ldots \oplus C_l \subseteq A$ where each C_j is a pure, homogeneous, completely decomposable subgroup of A of type τ_j such that $\tau_i \neq \tau_j$ whenever $i \neq j$. We show that $\tau_1,\dots,\tau_l$ satisfy the conditions of Proposition 2.4. Suppose to the contrary that, without loss of generality, τ_1 and τ_2 are minimal among $\tau_1,\dots,\tau_l$, but there is $p \in \pi(\tau_1) \cap \pi(\tau_2)$.

Let e be the exponent of p in m, and consider

$$B = A + \frac{1}{p^{2e+1}}\mathbb{Z}(c_1, c_2, 0, \ldots)$$

where $c_i \in C_i$ has p-height 0 for $i = 1, 2$. Set $x = (c_1, c_2, 0, \ldots)$, and observe that $(1/p^{2e+1})x \in S_A(B)$ since A is an $\mathcal{S}$-group. Hence, we can find $\phi_1, \ldots, \phi_k \in H_A(B)$ and $a_1, \ldots, a_k \in A$ with $(1/p^{2e+1})x = \sum_{j=1}^{k} \phi_j(a_j)$. Let $j \in \{1, \ldots, k\}$. Since $B(\tau_i) = A(\tau_i)$ for $i = 1, 2$, we have $\phi_j(A_i) \subseteq \phi_j(A(\tau_i)) \subseteq B(\tau_i) = A(\tau_i)$. Furthermore, $\phi_j(A_t) = 0$ for $t > 2$ since τ_1 and τ_2 are minimal. So, $\phi_j(mA) \subseteq \phi_j(A_1 \oplus A_2) \subseteq A(\tau_1) \oplus A(\tau_2) \subseteq A$. This shows $\phi_j(A) \subseteq \left(\frac{1}{m}A\right) \cap B$ where $\frac{1}{m}A = \{u \in \mathbb{Q}A \mid mu \in A\}$. Therefore, $(m/p^{2e+1})x = \sum_{j=1}^{k} m\phi_j(a_j) \in A$. But $A/[C_1 \oplus \ldots \oplus C_l]$ has p-component bounded by p^e in view of the choice of e. So, x has p-height at most e in A, while $(m/p^{2e+1})x \in A$ implies that x has p-height at least $e + 1$, a contradiction. It follows that $A = A_1 \oplus \ldots \oplus A_n$ where $A_j = A(\tau_j)$ and $\tau_1, \ldots, \tau_n$ are minimal among $\text{type}(C_1), \ldots, \text{type}(C_l)$. If X_j is a pure rank 1 subgroup of A_j of type τ_j, then $A_j = X_j \oplus A'_j$, and the remainder follows from Lemma 2.1. ∎

While the question whether every $\mathcal{S}$-group is a strong $\mathcal{S}$-group remains open, we can give an affirmative answer for almost completely decomposable $\mathcal{S}$-groups.

Corollary 2.7. *Let A be an almost completely decomposable $\mathcal{S}$-group. Then A is a strong $\mathcal{S}$-group.*

3. Strong $\mathcal{S}$-groups and A-solvability. In this section we give several characterizations of strong $\mathcal{S}$-groups, and discuss their most important properties. For the convenience of the reader, we give a short summary of the notation used in discussion of endomorphism rings which goes back to [4]: Associated with every abelian group A is a pair (H_A, T_A) of adjoint functors between the category of abelian groups and the category of right $E(A)$-modules which are defined as $H_A(G) = \text{Hom}(A, G)$ for an abelian group G and $T_A(M) = M \otimes_{E(A)} A$ for a right $E(A)$-module M. The module structure on $H_A(G)$ is induced by composition of maps. The natural maps $\theta_G : T_A H_A(G) \to G$ for an abelian group G and $\Phi_M : M \to H_A T_A(M)$ for a right $E(A)$-module M are defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ and $[\Phi_M(m)](a) = m \otimes a$ for all $\alpha \in H_A(G)$, $m \in M$, and $a \in A$. The *A-generated abelian groups* are the groups G for which θ_G is onto, while the *A-solvable abelian groups* are those for which θ_G is an isomorphism.

An exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ is (*almost*) *A-balanced* if the induced exact sequence

$$0 \to H_A(B) \xrightarrow{H_A(\alpha)} H_A(C) \xrightarrow{H_A(\beta)} H_A(G)$$

has the property that $\text{coker}\, H_A(\beta) = 0$ ($\text{coker}\, H_A(\beta)$ is torsion).

THEOREM 3.1. *The first three of the following conditions are equivalent for a torsion-free abelian group A of finite rank. Moreover, they imply the fourth, and the converse holds if A is flat as an $E(A)$-module.*

(a) *A is a strong $\mathcal{S}$-group.*

(b) *If G is an A-generated torsion-free group, and $H \doteq G$, then H is A-generated.*

(c) *$S_A(G)$ is a pure subgroup of G whenever G is torsion-free.*

(d) *If p is a prime with $A \neq pA$, then all reduced p-groups are A-solvable.*

Proof. (a)⇒(b). Let H be a subgroup of the torsion-free group G such that $mG \subseteq H \subseteq G$ for some non-zero integer m. For every $h \in H$, we can find $\phi_1, \ldots, \phi_n \in H_A(H)$ such that $mh \in \langle \phi_1(A), \ldots, \phi_n(A) \rangle$. To simplify our notation, we denote the latter subgroup of G by U, and set $V = \langle U, h \rangle$. Without loss of generality, we may assume $V \subseteq \mathbb{Q}U$. Since $mV \subseteq U$, we have $V \subseteq \frac{1}{m}U \cong U$. The maps $\phi_1, \ldots, \phi_n$ induce an epimorphism $\delta : A^n \to U$ which extends to a map $\delta' : \mathbb{Q}A^n \to \mathbb{Q}U$ such that $\delta'\big(\frac{1}{m}A^n\big) = \frac{1}{m}U$. The subgroup $W = (\delta')^{-1}(V)$ of $\frac{1}{m}A^n$ contains A^n. Since A^n is an $\mathcal{S}$-group, we can find an ideal I of $E(A^n)$ such that $W \cong mW = IA^n$. In particular, W is A-generated, and the same holds for V as an epimorphic image of W.

(b)⇒(c). Let $S_A(G)_*$ denote the $\mathbb{Z}$-purification of $S_A(G)$ in the torsion-free group G. When $x \in S_A(G)_*$, the subgroup $\langle S_A(G), x \rangle$ is quasi-equal to $S_A(G)$ and hence A-generated by virtue of (b). Therefore, $x \in S_A(G)$, and (c) holds.

(c)⇒(a). If a subgroup U of A^n is quasi-equal to A^n, then $S_A(U) \doteq U$. Since $U/S_A(U)$ is also torsion-free by (c), we see that U is A-generated. Thus, $I = \mathrm{Hom}(A^n, U)$ is a right ideal of $E(A^n)$ with $U = IA^n$, and consequently, A^n is an $\mathcal{S}$-group.

(c)⇒(d). Let p be a prime such that $A \neq pA$. As a first step, we show that every bounded p-group G is A-solvable. If $p^m G = 0$, then G is an epimorphic image of a direct sum of cyclic groups of order p^m. Since $A/p^m A$ contains at least one element of order p^m, the group G is A-generated. So, there exists an A-balanced exact sequence $0 \to U \overset{\alpha}{\to} \bigoplus_I A \overset{\beta}{\to} G \to 0$ for some index-set I. Since $p^m G = 0$, we have $p^m \bigoplus_I A \subseteq \alpha(U)$. In particular, $S_A(U)$ is quasi-equal to U. On the other hand, $S_A(U)$ is pure in U by (c), so that U is A-generated. Consequently, the map θ_U in the commutative diagram

$$\begin{array}{ccccccc} & T_A H_A(U) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(\bigoplus_I A) & \xrightarrow{T_A H_A(\beta)} & T_A H_A(G) & \to 0 \\ & \downarrow \theta_U & & \wr\downarrow \theta_{\oplus_I A} & & \downarrow \theta_G & \\ 0 \to & U & \overset{\alpha}{\to} & \bigoplus_I A & \overset{\beta}{\to} & G & \to 0 \end{array}$$

is onto. By the Snake Lemma, θ_G is an isomorphism.

Now assume that G is a reduced p-group. For every p-basic subgroup F of A, the group A/F is p-divisible. Therefore, $\mathrm{Hom}(A/F, G) = 0$, and we have an embedding $0 \to \mathrm{Hom}(A, G) \to \mathrm{Hom}(F, G)$. Since F is finitely generated, $\mathrm{Hom}(A, G)$ is a p-group. If $\phi_1, \ldots, \phi_n \in H_A(G)$, then there is $k < \omega$ such that $p^k\phi_1 = \ldots = p^k\phi_n = 0$. Therefore, $\langle \phi_1(A), \ldots, \phi_n(A) \rangle$ is bounded by p^k, and hence A-solvable by the results of the first paragraph. Hence, all finitely A-generated subgroups of G are A-solvable; the same holds for G.

(d)$\Rightarrow$(a). Suppose that A is flat as an $E(A)$-module. Since the class of A-solvable groups is closed with respect to finite direct sums, every bounded group G such that $A = pA$ implies $G[p] = 0$ is A-solvable by (d). To show that A is a strong $\mathcal{S}$-group, we consider a subgroup U of A^n such that $mA^n \subseteq U$ for some non-zero integer m. Without loss of generality, $A \neq pA$ for all primes $p \mid m$. Therefore, A^n/U is A-solvable by the initial remarks. In view of the flatness of A as an $E(A)$-module, U is A-solvable since kernels of maps between A-solvable groups are A-solvable. But then $I = \mathrm{Hom}(A^n, U)$ is a right ideal of $E(A^n)$ with $U = IA^n$. ∎

However, even if A is a strong $\mathcal{S}$-group which is flat as an $E(A)$-module, not every reduced torsion group G such that $G[p] = 0$ whenever $A = pA$ needs to be A-solvable, as the following result shows. It is easy to see that a torsion-free group A of finite rank is quotient divisible if and only if, for every full subgroup U of A, the group $(A/U)_p$ is divisible for all but finitely many primes.

Corollary 3.2. *Let A be a strong $\mathcal{S}$-group of finite rank. Every reduced torsion group G such that $A = pA$ implies $G[p] = 0$ is A-solvable if and only if A is quotient divisible.*

Proof. Suppose that A is a quotient divisible strong $\mathcal{S}$-group. We know by Theorem 3.1 that every reduced p-group is A-solvable. Consider a reduced torsion group G such that $A = pA$ implies $G[p] = 0$, and write $G = \bigoplus_p G_p$ where G_p denotes the p-primary component of G. By [1], we know that a direct sum of A-solvable groups $\{U_i \mid i \in I\}$ is A-solvable if and only if $\{U_i \mid i \in I\}$ is A-small, i.e., for every map $\alpha \in H_A(\bigoplus_{i \in I} U_i)$, there is a finite subset I' of I with $\alpha(A) \subseteq \bigoplus_{i \in I'} U_i$. Thus, it suffices to show that $\{G_p \mid p \text{ is a prime with } A \neq pA\}$ is an A-small family to ensure that G is A-solvable. For a morphism $\alpha : A \to G$, we choose a free subgroup F of $\ker \alpha$ such that A/F is torsion. Since A is quotient divisible, $(A/F)_p$ is divisible for all but finitely many primes p. We write $(A/F)_p = U_p/F$ for some subgroup U_p of A containing F, and choose a cofinite subgroup V_p of U_p containing F such that V_p/F is the divisible subgroup of $(A/F)_p$. Since A is quotient divisible, we have $V_p = U_p$ for almost all primes, and $A/\langle V_p \mid A \neq pA \rangle$ is finite. Since G is reduced, $V_p \subset \ker \alpha$ for all primes, and so $A/\ker\alpha$ is finite. Thus, there are finitely many primes $p_1,\dots,p_n$ such that $\alpha(A)\subseteq G_{p_1}\oplus\ldots\oplus G_{p_n}$, and $\{G_p \mid p$ is a prime with $A\neq pA\}$ is A-small.

Conversely, suppose that all the described torsion groups are A-solvable, and choose a full free subgroup F of A. Suppose that A/F is not divisible for infinitely many primes. Then there are subgroups V and W of A containing F such that V/F is divisible, W/F is reduced and infinite, and $A/F = V/F\oplus W/F$. Observe that $(W/F)_p$ is finite for all primes p. By our hypothesis, W/F is A-solvable since the fact that it is A-generated guarantees that $A = pA$ implies $W/F[p]=0$. However, since W/F is an epimorphic image of A, the family $\{(W/F)_p \mid p$ a prime$\}$ is not A-small, which is not possible. ■

THEOREM 3.3. *The following are equivalent for a self-small abelian group A which is flat as an $E(A)$-module, and a group B quasi-isomorphic to A.*

(a) $S_A(B)=B$ *and* $S_B(A)=A$.

(b) *The class of torsion-free A-solvable groups coincides with the class of torsion-free B-solvable groups.*

Proof. It remains to show that (a) implies (b). Choose maps $\sigma: A\to B$ and $\tau: B\to A$ such that $\sigma\tau = m1_B$ and $\tau\sigma = m1_A$ for some non-zero integer m. For a torsion-free B-solvable group G, we choose a B-balanced exact sequence $0\to U\overset{\alpha}{\to}\bigoplus_I B\overset{\beta}{\to} G\to 0$ such that $S_B(U)=U$. Since $S_A(B)=B$, every B-generated group is A-generated. Furthermore, since A is flat as an $E(A)$-module, the direct sum of a collection of A-generated subgroups of A is A-solvable. In particular, this holds for $\bigoplus_I B$; the group G is A-solvable once we have established that the above sequence is almost A-balanced.

Then, $M=\operatorname{im} H_A(\beta)$ is a submodule of $H_A(G)$ such that $H_A(G)/M$ is torsion as an abelian group. By a standard argument, we deduce that the evaluation map $\theta: T_A(M)\to G$ is an isomorphism. If $\iota: M\to H_A(G)$ is the inclusion map, then $\theta_G T_A(\iota)=\theta$. For $x\in\ker\theta_G$ we can find a non-zero integer k and $y\in T_A(M)$ such that $kx = T_A(\iota)(y)$. But then $\theta(y)=0$ yields $y=0$. Since $T_AH_A(G)$ is torsion-free because A is flat, we have $x=0$, and G is A-solvable.

If $\phi: A\to G$, then $\phi\tau: B\to G$, and there is $\lambda: B\to\bigoplus_I B$ with $\phi\tau=\beta\lambda$. Hence, $\beta\lambda\sigma = m\phi$ and the given sequence is almost A-balanced. Hence, every torsion-free B-solvable group is A-solvable.

The converse holds by symmetry once we have shown that B is $E(B)$-flat. To show this, we consider an exact sequence $0\to U\to B^n\to B$. The flatness of B follows directly from Ulmer's Theorem once we have shown that $S_B(U)=U$. Since B is A-solvable, and A is flat as an $E(A)$-module, we obtain $S_A(U)=U$. As before, U is B-generated since $S_B(A)=A$. ■

COROLLARY 3.4. *The following are equivalent for a torsion-free abelian group of finite rank which is flat as an $E(A)$-module.*

(a) *A is a strong $\mathcal{S}$-group.*

(b) *If B is quasi-isomorphic to A^n for some $0 < n < \omega$, then the class of torsion-free B-solvable groups coincides with the class of torsion-free A-solvable groups.*

Proof. (a)⇒(b). Since A^n is an $\mathcal{S}$-group, the same holds for B, and $S_A(B) = B$ and $S_B(A^n) = A^n$. By Theorem 3.3, the class of torsion-free B-solvable groups coincides with the class of torsion-free A^n-solvable groups, which is the class of torsion-free A-solvables.

(b)⇒(a). If $B \doteq A^n$ for some n, then B is A-solvable by (b), and $B = H_A(B)A = H_{A^n}(B)A^n$. This shows that A^n is an $\mathcal{S}$-group. ∎

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Department of Mathematics
Auburn University
Auburn, Alabama 36849-5310
U.S.A.
E-mail: albreuf@mail.auburn.edu

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