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A NOTE ON THE LIMIT POINTS ASSOCIATED WITH THE GENERALIZED *abc*-CONJECTURE FOR $\mathbb{Z}[t]$

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1. Introduction. For any non-zero polynomial $A \in \mathbb{Z}[t]$, let $\operatorname{rad}(A)$ denote the *radical* of A, i.e. the product of all the different irreducible factors of A. Also, let $A_1(t), \ldots, A_n(t) \in \mathbb{Z}$, where $n \geq 3$, satisfy the following conditions:

(1)
(i) $\max_{1\le j\le n} \deg A_j(t) = \deg A_n(t) > 0$,
(ii) $A_1(t) + \ldots + A_{n-1}(t) = A_n(t)$,
(iii) no subsum of the l.h.s. of (1) is equal to 0,
(iv) $\gcd(A_1(t), \ldots, A_n(t)) = 1$.

For concision, we shall henceforth denote the set of all such n-tuples $A = (A_1(t), \ldots, A_n(t))$ by T_n $(n \geq 3)$.

We also define the function $L_n : T_n \to \mathbb{R}^+$ as

$$L_n = L_n(A) = \frac{\deg A_n(t)}{\deg(\operatorname{rad}(A_1(t) \cdot \ldots \cdot A_n(t)))}.$$

Again for concision, we denote the set of all limit points of the set $\{L_n(A) : A \in T_n\}$ by P_n.

From the definition of L_n it follows that $L_n \geq 1/n$. The *n-conjecture* for $\mathbb{Z}[t]$ claims that $P_n \subseteq [1/n, 2n-5]$ for $n \geq 3$ (see [B-B]). It has been proved for $n = 3$ and $n = 4$.

We may now state our main result:

THEOREM 1.0. $[1/n, 2n-5] \subseteq P_n$ *for* $n \geq 3$.

Remark 1. It is clear from the method of proof that the same theorem holds for $K[t]$, where K is any integral domain of characteristic 0.

It is convenient to break the proof up into several steps.

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2. Preliminary lemmata

LEMMA 2.1 (J. Browkin). *For $m > k > 0$, the polynomial*

$$f(x) = x^m - x^k + 1$$

has no multiple roots over a field of characteristic zero.

Proof. (This was given in [Br].) From $f(x) = f'(x) = 0$ we get

$$\left(\frac{k}{m-k}\right)^k = \left(\frac{m}{m-k}\right)^m,$$

and hence $m^m = k^k \cdot (m-k)^{m-k}$. The last equality cannot hold since $k < m$, and $m - k < m$. ∎

LEMMA 2.2. $\left[\frac{1}{2}(2n-5), 2n-5\right] \subseteq P_n$ *for $n \geq 3$.*

Proof. Let $n \geq 3$ be chosen. By [B-B], Lemma 1, the polynomial of degree $k \geq 0$ defined as

$$f_k(z) = \sum_{j=0}^{k} \frac{2k+1}{k+j+1} \binom{k+j+1}{2j+1} z^j$$

has integral coefficients and satisfies the identity

$$\frac{x^{2k+1} - 1}{x - 1} = x^k f_k\left(\frac{(x-1)^2}{x}\right). \tag{2}$$

Making the substitution $k = n - 3$ in (2) and rearranging gives

$$x^{2n-5} = 1 + \sum_{j=0}^{n-3} s_j (x-1)^{2j+1} x^{n-j-3}, \tag{3}$$

where

$$s_j = \frac{2n-5}{n+j-2} \binom{n+j-2}{2j+1}.$$

Equation (3) is a sum of the form (1), with $A_n(x) = x^{2n-5}$, and $A_1(x) = 1$. Clearly conditions (i), (ii) and (iv) hold. We claim that (iii) also holds. Indeed, if it did not, then we could insert $x = 2^r$ ($r > 0$) in (3) and assert that the identity

$$2^{r(2n-5)} - 1 - \sum_{j=0}^{n-3} s_j (2^r - 1)^{2j+1} 2^{r(n-j-3)} = 0 \tag{4}$$

has a proper subsum equal to zero. However, this is impossible because only the first summand in (4) is positive (see [B-B], eqn. (7)). Thus we have a contradiction, and so (i)–(iv) all hold. Thus (3) $\in T_n$.

Finally, inserting $x = t^{r_i} - t^{m_i} + 1$, where $r_i > m_i > 0$, $r_i, m_i \in \mathbb{N}$, and $i \in \{1, 2, 3, \ldots\}$, we obtain (for each $i \in \mathbb{N}$), the identities

$$(5_i) \quad (t^{r_i} - t^{m_i} + 1)^{2n-5} = 1 + \sum_{j=0}^{n-3} s_j \cdot t^{(2j+1)m_i} \cdot (t^{r_i - m_i} - 1)^{2j+1} \cdot (t^{r_i} - t^{m_i} + 1)^{n-j-3}.$$

Since $(3) \in T_n$, so are the sums (5_i). The polynomials

$$t^{r_i} - t^{m_i} + 1, \quad t^{r_i - m_i} - 1, \quad \text{and} \quad t$$

are obviously pairwise coprime. The latter two polynomials clearly have no repeated roots over any field of characteristic zero; neither does $t^{r_i} - t^{m_i} + 1$, by Lemma 2.1. Hence the radical of (5_i) is

$$(t^{r_i} - t^{m_i} + 1) \cdot (t^{r_i - m_i} - 1) \cdot t^1,$$

which is of degree $2r_i - m_i + 1$. Therefore applying L_n to (5_i) gives

$$(6) \qquad L_n = \frac{(2n-5)r_i}{2r_i - m_i + 1} = \frac{2n-5}{2 - m_i/r_i + 1/r_i}.$$

Since $r_i > m_i > 0$, we may choose a sequence m_i/r_i converging to any $\alpha \in [0, 1]$ as $i \to \infty$. Applying this to (6) gives the stated result. ■

LEMMA 2.3. *$[1/n, 1/(n-1)] \subseteq P_n$ for $n \geq 3$.*

Proof. Let $n \geq 3$ be given. Choose any prime $q \geq n$. Then, for each $(i, r_i, m_i) \in \mathbb{N}^3$ (with $r_i > m_i$), we may form the n-tuples (in T_n)

$$(7) \quad (n-2)(t^{r_i} + qt^{m_i} + (n-1)q(q+1)) = \sum_{j=1}^{n-2} (t^{r_i} + 2qj) + q(n-2)(t^{m_i} + q(n-1)).$$

We note that the polynomials $t^{r_i} + qt^{m_i} + (n-1)q(q+1)$, $t^{r_i} + 2qj$, and $t^{m_i} + q(n-1)$ are all irreducible, by Eisenstein's irreducibility criterion, and hence do not have multiple roots. Furthermore, since the polynomials are distinct from each other, all their roots are distinct (in every integral domain containing $\mathbb{Z}$). Therefore the radical of (7) is

$$(t^{r_i} + qt^{m_i} + (n-1)q(q+1))(t^{m_i} + q(n-1)) \prod_{j=1}^{n-2} (t^{r_i} + 2qj),$$

which is of degree $(n-1)r_i + m_i$. Applying L_n therefore gives

$$\frac{r_i}{(n-1)r_i + m_i} = \frac{1}{(n-1) + m_i/r_i}.$$

Choosing a suitable sequence $m_i/r_i \to \alpha \in [0, 1]$ as before, we obtain the stated result. ■

LEMMA 2.4. *$[1/n, 2n-5] \subseteq P_n$ for $n = 3, 4$.*

Proof. The case $n = 3$ was solved in [Br] and will not be given here. In view of Lemmata 2.2 and 2.3, it is clear that we need now only show that $[1/3, 3/2] \subseteq P_4$. As before, $\langle r_i, m_i \rangle$ represents a sequence of integers in $\mathbb{N}^2$ with $r_i > m_i$ such that $m_i/r_i \to \alpha \in [0, 1]$. We consider families

$$A_{1,i}(t) + A_{2,i}(t) + A_{3,i}(t) = A_{4,i}(t)$$

as in (1).

That $[1/2, 1] \subseteq P_4$ follows from the substitution: $A_{1,i}(t) = A_{2,i}(t) = t^{r_i}$, $A_{3,i}(t) = 6(t^{m_i} + 1)$, $A_{4,i}(t) = 2(t^{r_i} + 3t^{m_i} + 3)$.

That $[1, 2] \subseteq P_4$ follows from the substitution: $A_{1,i}(t) = t^{2r_i}$, $A_{2,i}(t) = 2t^{r_i}(t^{m_i} + 1)$, $A_{3,i}(t) = (t^{m_i} + 1)^2$, $A_{4,i}(t) = (t^{r_i} + t^{m_i} + 1)^2$.

Finally, that $[1/3, 1/2] \subseteq P_4$ follows from the substitution: $A_{1,i}(t) = A_{2,i}(t) = t^{r_i} + t^{m_i} + 1$, $A_{3,i}(t) = 2(t^{m_i} + 1)$, $A_{4,i}(t) = 2(t^{r_i} + 2t^{m_i} + 2)$. ∎

3. Proof (by induction) of Theorem 1.0. Our induction hypothesis (H) is that $[1/n, 2n-5] \subseteq P_n$.

Let

$$A_{1,i}(t) + \ldots + A_{n-1,i}(t) = A_{n,i}(t) \tag{8_i}$$

be any family of elements of T_n such that $L_n(A_{1,i}(t), \ldots, A_{n,i}(t))$ converges, under (H), to any $\alpha \in [1/n, 2n-5]$ as $i \to \infty$.

Let S_i be the set of all the finite subsums of the l.h.s. of (8_i), for every $i \in \mathbb{N}$. Since every S_i is finite, there must exist a corresponding least integer $m_{0,i} \in \mathbb{N}$ such that

$$(m_{0,i} + 1)A_{n,i}(t) \notin S_i, \qquad -m_{0,i}A_{n,i}(t) \notin S_i,$$

for every $i \in \mathbb{N}$. Choosing such a collection of $m_{0,i}$ we may construct a family of $(n+1)$-tuples

$$(m_{0,i} + 1)A_{n,i}(t) - m_{0,i}A_{n,i}(t) = A_{1,i}(t) + \ldots + A_{n-1,i}(t). \tag{9_i}$$

A straightforward check shows that we have constructed a family (9_i) of elements of T_{n+1}. Furthermore, applying L_{n+1} to the family (9_i) yields the same limit point as applying L_n to the family (8_i). This implies, by (H), that

$$[1/n, 2n-5] \subseteq P_{n+1}.$$

Combining this with Lemmata 2.2 and 2.3, we have shown that

$$\left[\frac{1}{n+1}, 2n-5\right] \cup \left[\frac{2(n+1)-5}{2}, 2(n+1)-5\right] \subseteq P_{n+1}.$$

Now, these two intervals overlap iff $2n - 5 \geq (2(n+1)-5)/2$, i.e. when $n \geq 4$.

Using Lemma 2.3, the theorem now follows by induction. ∎

4. Some general comments. The conjectures in [B-B] strongly suggest an n-conjecture for all number/function fields for *all* $n \geq 3$, thereby generalizing a conjecture of Vojta ([V], p. 84). The following conjecture is a logical consequence of the work in [B-B], but it has not been explicitly stated yet, as far as I am aware.

We state a

Generalized Vojta Conjecture. Let k be a global field, O_k its ring of integers and S a finite set of places of k containing all the archimedean places.

Suppose $\alpha_1, \ldots, \alpha_n \in O_k$ $(n \geq 3)$ satisfy

$$\alpha_1 + \ldots + \alpha_{n-1} = \alpha_n. \tag{10}$$

Then, for every $\varepsilon > 0$ and all $\{\alpha_1, \ldots, \alpha_n\}$ satisfying (10), we have

$$h([\alpha_1; \ldots; \alpha_n]) < (2n - 5 + \varepsilon) \sum_{\nu \in S,\, \nu(\alpha_1 \ldots \alpha_n) > 0} N_\nu + O(1).$$

The constant in $O(1)$ depends only on ε, k, n and S. Here $h([\ ])$ denotes the usual logarithmic height. N_ν is as in [V]. That is to say, $N_\nu = 1$ in the function field case, and $N_\nu = (f \log p)/[k : \mathbb{Q}]$ in the number field case, if the residue field of ν has p^f elements.

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