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THE POLYNOMIAL HULL OF UNIONS OF CONVEX SETS IN $\mathbb{C}^n$

BY

ULF BACKLUND AND ANDERS FÄLLSTRÖM (UMEÅ)

We prove that three pairwise disjoint, convex sets can be found, all congruent to a set of the form $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^{2m} \leq 1\}$, such that their union has a non-trivial polynomial convex hull. This shows that not all holomorphic functions on the interior of the union can be approximated by polynomials in the open-closed topology.

I. In this paper we study polynomial convexity of unions of compact convex sets in $\mathbb{C}^n$. The *polynomial convex hull* $\widehat{K}$ of a compact set K in $\mathbb{C}^n$ is defined by

$$\widehat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_{\zeta \in K} |p(\zeta)| \text{ for all polynomials } p\}.$$

Furthermore, if $K = \widehat{K}$, then K is said to be *polynomially convex*.

The notion of polynomial convexity arises naturally in the theory of Banach algebras and is of importance in the area of polynomial approximation in $\mathbb{C}^n$. One reason to study polynomial convexity is that if $K \subset \mathbb{C}^n$ is a compact set, then the closure $\mathcal{P}(K)$ of the polynomials on K in the uniform norm is a Banach algebra and its maximal ideal space is homeomorphic to the polynomial convex hull of K. In fact, any finitely generated semisimple commutative Banach algebra $\mathcal{B}$ with unit is, via the Gelfand representation, isomorphic to $\mathcal{P}(K)$ for some polynomially convex compact K in $\mathbb{C}^N$, where N is the number of generators in $\mathcal{B}$. Moreover, the problem of determining whether every holomorphic function on an open set in $\mathbb{C}^n$ can be approximated by polynomials in the open-closed topology is linked to the problem of finding the polynomial convex hull of the closure of the given set.

In the complex plane polynomial convexity turns out to be a purely topological notion. Using the maximum modulus principle and the Runge approximation theorem, one proves that a compact set K is polynomially convex if and only if $\mathbb{C} \setminus K$ is connected. In higher dimensions the situation is in many ways different. That the complement of a polynomially convex set in

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$\mathbb{C}^n$ is connected is still a necessary condition but there are other obstructions to polynomial convexity making the theory considerably richer. For instance, there exist [Wer1] arcs with non-trivial polynomial convex hulls. Furthermore, the notion of polynomial convexity is not invariant under biholomorphic mappings. This phenomenon was first observed by J. Wermer in [Wer2].

Evidently, compact convex sets are polynomially convex. Using the following lemma (see e.g. [Kal]) one deduces that the union of two disjoint compact convex sets is also polynomially convex.

LEMMA 1.1. *If* X_1 *and* X_2 *are compact sets in* $\mathbb{C}^n$ *and* p *a polynomial such that* $p(X_1)^\wedge \cap p(X_2)^\wedge = \emptyset$, *then* $(X_1 \cup X_2)^\wedge = \widehat{X}_1 \cup \widehat{X}_2$.

This leads one to consider the following general problem: Let $K_1, \ldots, K_q$ be pairwise disjoint compact convex sets in $\mathbb{C}^n$. Is the union $\bigcup_{i=1}^q K_i$ polynomially convex?

Remark 1. If the sets are far enough apart, for instance if they have disjoint projections on some complex line, then the union is polynomially convex.

Remark 2. It is obvious that if $n = 1$, then the union is always polynomially convex.

Recall that an open set Ω in $\mathbb{C}^n$ is said to be *Runge* if every holomorphic function on Ω can be approximated by polynomials in the open-closed topology. This is equivalent to saying that for every compact subset K of Ω the intersection of the polynomial convex hull $\widehat{K}$ with Ω is relatively compact in Ω. As a consequence, the interior of the set $\bigcup_{i=1}^q K_i$ is Runge if and only if it is polynomially convex.

The first results when $q > 2$ in higher dimension were obtained by E. Kallin in 1964 and show that the answer is no longer independent of the geometry of the sets.

THEOREM 1.1 (E. Kallin [Kal]). *If* B_1, B_2 *and* B_3 *are pairwise disjoint closed balls in* $\mathbb{C}^n$, *then* $B_1 \cup B_2 \cup B_3$ *is polynomially convex.*

THEOREM 1.2 (E. Kallin [Kal]). *There exist three congruent, pairwise disjoint, closed polydisks* P_1, P_2 *and* P_3 *in* $\mathbb{C}^3$ *such that* $P_1 \cup P_2 \cup P_3$ *is not polynomially convex.*

Remark 3. It is an open problem whether Theorem 1.1 still holds if the number of balls is larger than three. However, by a result of G. Khudaĭberganov [Khud], Theorem 1.1 holds for any finite number of balls if the centers of the balls are situated on $\mathbb{R}^n \subset \mathbb{C}^n$.

Remark 4. In the proof of Theorem 1.2 Kallin actually constructed polydisks parallel to the coordinate axes. This is, however, not possible in $\mathbb{C}^2$ (see Rosay [Ros]).

The following theorem was proved by A. M. Kytmanov and G. Khudaĭberganov:

THEOREM 1.3 (A. M. Kytmanov and G. Khudaĭberganov [KyKh]). *There exist three congruent, pairwise disjoint, closed complex ellipsoids* E_1, E_2 *and* E_3 *in* $\mathbb{C}^3$ *such that* $E_1 \cup E_2 \cup E_3$ *is not polynomially convex.*

The first example of three pairwise disjoint compact convex sets in $\mathbb{C}^2$ whose union has a non-trivial polynomial convex hull was published by J.-P. Rosay in 1989.

THEOREM 1.4 (J.-P. Rosay [Ros]). *There exist three congruent, pairwise disjoint, convex closed limited tubes* T_1, T_2 *and* T_3 *in* $\mathbb{C}^2$ *such that* $T_1 \cup T_2 \cup T_3$ *is not polynomially convex.*

Here the *limited tube* in $\mathbb{C}^2$ with base domain $B \subset \mathbb{R}^2$ and height M is the domain $\{(z_1, z_2) \in \mathbb{C}^2 : (\operatorname{Re} z_1, \operatorname{Re} z_2) \in B,\ |\operatorname{Im} z_1| < M,\ |\operatorname{Im} z_2| < M\}$.

II. We prove the existence of three pairwise disjoint convex sets all congruent to a set of the form $\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^{2m} \le 1\}$, m a positive integer, such that their union has a non-trivial polynomial convex hull.

Such domains have been studied by E. Bedford and S. Pinchuk [BePi]. One of their results is that any bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$ of finite type whose boundary is smooth such that the Levi form has rank at least $n-2$ at each point of the boundary is biholomorphically equivalent to the domain $\{z \in \mathbb{C}^n : |z_1|^2 + \ldots + |z_{n-1}|^2 + |z_n|^{2m} < 1\}$ for some integer $m \ge 1$ if the automorphism group $\operatorname{Aut}(\Omega)$ is non-compact.

THEOREM 2.1. *There exist a positive integer* m *and three pairwise disjoint, closed sets* S_1, S_2 *and* S_3 *in* $\mathbb{C}^3$ *all congruent to*

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^{2m} \le 1\}$$

such that $S_1 \cup S_2 \cup S_3$ *is not polynomially convex.*

Proof. Let $M > 2$. Furthermore, let

$$D_1 = \{z \in \mathbb{C} : |z| < M^{-1}\}, \qquad D_2 = \{z \in \mathbb{C} : |z-1| < M^{-1}\}$$

and

$$D_3 = \{z \in \mathbb{C} : |z| < M\}$$

and define $D \subset \mathbb{C}$ to be the domain $D = D_3 \setminus (\overline{D}_1 \cup \overline{D}_2)$. Define the mapping $\psi : D \to \mathbb{C}^3$ by

$$\psi(\xi) = \left(\xi, \frac{1}{\xi}, \frac{1}{1-\xi}\right)$$

and denote by γ_1, γ_2, γ_3 the components of the boundary of D, i.e. $\gamma_1 = \partial D_1$, $\gamma_2 = \partial D_2$, $\gamma_3 = \partial D_3$.

For a positive integer m we define the sets $\widetilde{S}_1$, $\widetilde{S}_2$ and $\widetilde{S}_3$ as

$$\widetilde{S}_1 = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \\ \left|\frac{z_1 - \big(-M + \frac{1}{M}\big)}{M+\delta}\right|^2 + \left|\frac{z_2}{M+\delta}\right|^2 + \left|\frac{z_3 - \big(M + \frac{M}{M+1}\big)}{M+\delta}\right|^{2m} \leq 1\Big\},$$

$$\widetilde{S}_2 = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \\ \left|\frac{z_1 - \big(M + 1 - \frac{1}{M}\big)}{M+\delta}\right|^2 + \left|\frac{z_2 - \big(M + \frac{M}{M+1}\big)}{M+\delta}\right|^2 + \left|\frac{z_3}{M+\delta}\right|^{2m} \leq 1\Big\},$$

$$\widetilde{S}_3 = \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : \\ \left|\frac{z_1}{M+\delta}\right|^2 + \left|\frac{z_2 - \big(-M + \frac{1}{M}\big)}{M+\delta}\right|^2 + \left|\frac{z_3 - \big(-M + \frac{1}{M+1}\big)}{M+\delta}\right|^{2m} \leq 1\Big\}.$$

We make the following estimates:

$$\left|\frac{M^{-1}e^{i\theta} - M^{-1} + M}{M+\delta}\right|^2 + \left|\frac{Me^{-i\theta}}{M+\delta}\right|^2 \\ + \left|\frac{M(M - e^{i\theta})^{-1} - M - M(M+1)^{-1}}{M+\delta}\right|^{2m} \\ \leq 2\left|\frac{M}{M+\delta}\right|^2 + \left|\frac{M}{M+\delta}\right|^{2m},$$

$$\left|\frac{1 + M^{-1}e^{i\theta} - M - 1 + M^{-1})}{M+\delta}\right|^2 + \left|\frac{M(M + e^{i\theta})^{-1} - M - M(M+1)^{-1})}{M+\delta}\right|^2 \\ + \left|\frac{-Me^{-i\theta}}{M+\delta}\right|^{2m} \\ \leq 2\left|\frac{M}{M+\delta}\right|^2 + \left|\frac{M}{M+\delta}\right|^{2m},$$

$$\left|\frac{Me^{i\theta}}{M+\delta}\right|^2 + \left|\frac{M^{-1}e^{-i\theta} + M - M^{-1}}{M+\delta}\right|^2 + \left|\frac{(1 - Me^{i\theta})^{-1} + M - (1+M)^{-1}}{M+\delta}\right|^{2m} \\ \leq 2\left|\frac{M}{M+\delta}\right|^2 + \left|\frac{M}{M+\delta}\right|^{2m}.$$

We can choose the positive integer m and the constants M and δ so that

$$2\left|\frac{M}{M+\delta}\right|^2 + \left|\frac{M}{M+\delta}\right|^{2m} \leq 1$$

and so that the sets $\widetilde{S}_i$ are pairwise disjoint. This implies that the image of γ_i under the mapping ψ will be contained in $\widetilde{S}_i$.

It now follows from the maximum modulus theorem that the polynomial convex hull of $\widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3$ contains the analytic variety $\psi(D)$. Hence $\widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3$ is not polynomially convex. By applying the complex linear isomorphism $(z_1, z_2, z_3) \to (z_1(M+\delta)^{-1}, z_2(M+\delta)^{-1}, z_3(M+\delta)^{-1})$ to $\widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3$ we obtain the sets S_i in the statement of the theorem. ∎

Remark 5. This shows that not all holomorphic functions on the interior of $S_1 \cup S_2 \cup S_3$ can be approximated by polynomials in the open-closed topology.

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DEPARTMENT OF MATHEMATICS
UMEÅ UNIVERSITY
S-901 87 UMEÅ, SWEDEN
E-mail: ULF.BACKLUND@MATHDEPT.UMU.SE
ANDERS.FALLSTROM@MATHDEPT.UMU.SE

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