A COUNTEREXAMPLE IN COMONOTONE
APPROXIMATION IN L^p SPACE

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Refining the idea used in [24] and employing very careful computation, the present paper shows that for $0 < p \leq \infty$ and $k \geq 1$, there exists a function $f \in C^k_{[-1,1]}$, with $f^{(k)}(x) \geq 0$ for $x \in [0,1]$ and $f^{(k)}(x) \leq 0$ for $x \in [-1,0]$, such that

$$\limsup_{n \to \infty} \frac{e_n^{(k)}(f)_p}{\omega_{k+2+[1/p]}(f, n^{-1})_p} = +\infty \,,$$

where $e_n^{(k)}(f)_p$ is the best approximation of degree n to f in L^p by polynomials which are comonotone with f, that is, polynomials P so that $P^{(k)}(x)f^{(k)}(x) \geq 0$ for all $x \in [-1,1]$. This theorem, which is a particular case of a more general one, gives a complete solution to the converse result in comonotone approximation in L^p space for $1 < p \leq \infty$.

1. Introduction. Denote by $C^N_{[-1,1]}$ the class of real functions which have N continuous derivatives on the interval $[-1,1]$, $C_{[-1,1]} = L^\infty_{[-1,1]} = C^0_{[-1,1]}$, $C^\infty_{[-1,1]}$ the class of infinitely differentiable real functions on $[-1,1]$. Let $L^p_{[a,b]}$ be the space of pth power integrable real functions on $[a,b]$, Π_n the class of algebraic polynomials of degree at most n, and

$$\Delta^k = \{f : \Delta^k_h f(x) \geq 0, x \in [-1, 1-kh], h > 0\} \,,$$

where

$$\Delta^k_h f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh) \,.$$

Define

$$\overline{\Delta}^k = \{f : \operatorname{sgn}(x)\Delta^k_h f(x) \geq 0, x \in [-1, 1] \setminus \{0\}, x + kh \in [-1, 1]\} \,.$$

For $f \in C_{[-1,1]}$, let

$$\|f\| = \|f\|_{L^\infty_{[-1,1]}} = \max_{-1 \le x \le 1} |f(x)|,$$

and for $f \in L^p_{[a,b]}$ and $0 < p < \infty$,

$$\|f\|_{L^p_{[a,b]}} = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}.$$

As usual, let $E_n(f)_p$ denote the best approximation to $f \in L^p_{[-1,1]}$ in L^p by algebraic polynomials of degree n; moreover, set $E_n(f) = E_n(f)_\infty$ and

$$\omega_m(f,t)_p = \sup\{\|\Delta_h^m f(x)\|_{L^p_{[-1,1-mh]}} : 0 < h \le t\},$$
$$\omega_m(f,t) = \omega_m(f,t)_\infty.$$

For $f \in L^p_{[-1,1]} \cap \Delta^k$, let

$$E_n^{(k)}(f)_p = \min\{\|f - P\|_{L^p_{[-1,1]}} : P \in \Pi_n \cap \Delta^k\},$$
$$E_n^{(k)}(f) = E_n^{(k)}(f)_\infty,$$

and for $f \in L^p_{[-1,1]} \cap \overline{\Delta}^k$,

$$e_n^{(k)}(f)_p = \min\{\|f - P\|_{L^p_{[-1,1]}} : P \in \Pi_n \cap \overline{\Delta}^k\},$$
$$e_n^{(k)}(f) = e_n^{(k)}(f)_\infty.$$

Presently, coapproximation of functions by algebraic polynomials is one of the most active and interesting fields in approximation theory, and many scholars focus especially on Jackson type estimates. In monotone approximation, G. G. Lorentz and K. Zeller [12] showed that for $f \in C_{[-1,1]} \cap \Delta^1$,

$$E_n^{(1)}(f) \le C\omega_1(f, n^{-1}),$$

and R. A. DeVore [3] proved that for $f \in C_{[-1,1]} \cap \Delta^1$,

$$E_n^{(1)}(f) \le C\omega_2(f, n^{-1}).$$

In the case that $f \in C_{[-1,1]}^k$, G. G. Lorentz [11] and R. A. DeVore [4] showed that for $k \ge 1$,

$$E_n^{(1)}(f) \le C(k)n^{-k}\omega_1(f^{(k)}, n^{-1}).$$

There are corresponding results due to R. K. Beatson [1] and A. S. Shvedov [21], [22] in convex approximation for $\omega_1(f,t)$ and $\omega_2(f,t)$. A. S. Shvedov also investigated monotone and convex approximations in L^p space; his result states that for $f \in L^p_{[-1,1]} \cap \Delta^k$, $k = 1, 2$ and $1 \le p \le \infty$,

$$E_n^{(k)}(f)_p \le C\omega_2(f, n^{-1})_p.$$

The above estimate also holds for general $k \geq 1$, which was established recently by X. M. Yu and Y. P. Ma [27].

Concerning the comonotone case, X. M. Yu [26] considers the problem in which one approximates a continuous function f with a finite number of changes of monotonicity on $[-1, 1]$ by a polynomial comonotone with it, and shows that such an approximation still has the Jackson type estimates for $\omega_2(f, n^{-1})$.

Other relevant materials can be found in [2], [5], [7], [9], [10], [15]–[20] and [25].

On the other hand, there are several converse results. G. G. Lorentz and K. L. Zeller [13] showed that there exists a function $f \in C_{[-1,1]} \cap \Delta^k$ such that for $k \geq 1$,

$$\limsup_{n \to \infty} E_n^{(k)}(f) / E_n(f) = +\infty \,.$$

A. S. Shvedov [22] proved that for any given A, k and $n \geq k + 1$, there exists a function $f_{n,k} \in C_{[-1,1]}^k \cap \Delta^k$ such that

$$E_n^{(k)}(f_{n,k})_p \geq A\omega_{k+2}(f_{n,k}, (k+2)^{-1})_p, \qquad 0 < p \leq \infty \,.$$

Although Shvedov's result shows that the Jackson type estimate

$$E_n^{(k)}(f) \leq C(k)\omega_{k+2}(f, n^{-1})$$

cannot hold for all continuous functions, is it possible that it holds for any particular $f \in C_{[-1,1]}$?

Using a result from [6] or [28], we can prove that for any fixed n and k, Shvedov's example $f_{n,k}$ satisfies

$$E_m(f_{n,k}) \geq Cm^{-k} \,,$$

while since $f_{n,k}^{(k-1)} \in \operatorname{Lip} 1$ we get

$$\limsup_{m \to \infty} E_m^{(1)}(f_{n,k}) / E_m(f_{n,k}) < +\infty \,.$$

This discussion leads to the following problem:

PROBLEM 1. Does there exist a function $f \in C_{[-1,1]} \cap \Delta^k$ for $k \geq 1$ such that

$$\limsup_{n \to \infty} E_n^{(k)}(f) / \omega_{k+2}(f, n^{-1}) = +\infty?$$

In comonotone case one can ask a weak form of that:

PROBLEM 2. Does there exist a function $f \in C_{[-1,1]} \cap \overline{\Delta}^k$ for $k \geq 1$ such that

$$\limsup_{n \to \infty} e_n^{(k)}(f) / \omega_{k+2}(f, n^{-1}) = +\infty?$$

They appear not to be easy questions. In X. Wu and S. P. Zhou [23], we showed a weaker result, which asserts that there exists a function $f \in C^k_{[-1,1]} \cap \Delta^k$ such that

$$\limsup_{n \to \infty} E_n^{(k)}(f)/\omega_{2k+1}(f, n^{-1}) = +\infty$$

for $k \geq 2$ and

$$\limsup_{n \to \infty} E_n^{(1)}(f)/\omega_4(f, n^{-1}) = +\infty$$

for $k = 1$, while in [24] by using a new idea we constructed a counterexample $f \in C^k_{[-1,1]} \cap \Delta^k$ such that for $k \geq 1$,

$$\limsup_{n \to \infty} E_n^{(k)}(f)/\omega_{k+3}(f, n^{-1}) = +\infty \, .$$

We are still unable to give a complete answer to Problem 1. However, by refining the basic idea used in [24] and employing very careful computation, the present paper will show a positive answer to Problem 2. Indeed, we will consider Problem 2 in general L^p spaces.

THEOREM 1. *Let $0 < p \leq \infty$ and $k \geq 1$. Then there exists a function* $f \in C^k_{[-1,1]} \cap \overline{\Delta}^k$ *such that*

$$\limsup_{n \to \infty} e_n^{(k)}(f)_p/\omega_{k+2+[1/p]}(f, n^{-1})_p = +\infty \, .$$

Theorem 1 follows as a particular case from the following more general result.

Let $f \in L^p_{[-1,1]}$ with $f^{(k)}(0) = 0$. Write

$$\widetilde{e}_n^{(k)}(f)_p = \min\{\|f - P\|_{L^p_{[-1,1]}} : P \in \Pi_n \text{ with } P^{(k)}(0) = 0\} \, ,$$

and

$$k_p = \begin{cases} k, & 0 < p < \infty, \\ k - 1, & p = \infty. \end{cases}$$

THEOREM 2. *Let $0 < p \leq \infty$, $k \geq 1$ and $0 \leq m \leq k_p$. Then there exists a function $f \in C^k_{[-1,1]} \cap \overline{\Delta}^k$ such that*

$$\limsup_{n \to \infty} \frac{\widetilde{e}_n^{(k)}(f)_p}{n^{-m}\omega_{k-m+2+[1/p]}(f^{(m)}, n^{-1})_p} = +\infty \, .$$

COROLLARY. *Let $0 < p \leq \infty$, $k \geq 1$ and $0 \leq m \leq k_p$. Then there exists a function $f \in C^k_{[-1,1]} \cap \overline{\Delta}^k$ such that*

$$\limsup_{n \to \infty} \frac{\widetilde{e}_n^{(k)}(f)_p}{n^{-m}E_n(f^{(m)})_p} = +\infty \, .$$

Throughout the paper, we will use $C(x)$ to denote a positive constant depending only upon x in case $1 \le p \le \infty$ or depending only upon x and p in case $0 < p < 1$, which is different, in general, in different relations.

2. Proof of Theorem 2

LEMMA 1 (^{1}). *Suppose that* $a > 0$, $\alpha(x) = a^2/(x^2 - a^2)$, $g_k(x, a) = x^k e^{\alpha(x)+1}$, $x \in (-a, a)$. *Then*

$$|g_k^{(k)}(x, a) - k!| \le C(k)a^{-2}x^2, \qquad |x| < a.$$

P r o o f. This lemma is evidently true for $a/2 \le |x| < a$. Now suppose $|x| < a/2$. Write

$$g_k^{(k)}(x, a) = k!e^{\alpha(x)+1} + e \sum_{j=1}^{k} \frac{k!}{j!}\binom{k}{j}x^j \frac{d^j}{dx^j}e^{\alpha(x)}$$

$$= k!\left(1 + \frac{x^2}{x^2 - a^2} + O\left(\left(\frac{x^2}{x^2 - a^2}\right)^2\right)\right)$$

$$+ e \sum_{j=1}^{k} \frac{k!}{j!}\binom{k}{j}x^j \frac{d^j}{dx^j}e^{\alpha(x)}.$$

We verify that

$$\left|\frac{d}{dx}e^{\alpha(x)}\right| \le C|x|a^{-2},$$

and by substituting $y = x/a$, we have

$$\left|\frac{d^j}{dx^j}e^{\alpha(x)}\right| = a^{-j}\left|\frac{d^j}{dy^j}\exp\left(\frac{1}{y^2 - 1}\right)\right| \le C(k)a^{-j}, \qquad 2 \le j \le k.$$

The proof is completed by combining the above results. ∎

LEMMA 2. *Under the same notations as in Lemma* 1, *we have for* $0 \le m \le k$,

$$|g_k^{(m)}(x, a)| = O(a^{k-m}), \qquad x \in [-a, a].$$

P r o o f. The argument is quite straightforward. ∎

P r o o f o f T h e o r e m 2. We begin with the construction of a sequence of functions $\{f_n\}$ such that for sufficiently large $n \ge N_0$,

$$f_n \in C_{[-1,1]}^{\infty} \cap \overline{\Delta}^k,$$

(^{1}) This is Lemma 1 in [24]. We give the proof to make the paper self-contained.

and for $0 \leq m \leq k$,

$$(1) \quad \left\| f_n^{(m)}(x) - \frac{(k+1)!}{(k-m+1)!} x^{k-m+1} + \frac{k!}{(k-m)!} \varepsilon_n^{1+\sigma} x^{k-m} \right\|_{L_{[-1,1]}^p}$$

$$= O(\varepsilon_n^{k-m+1+\sigma+1/p}),$$

$$(2) \qquad \qquad \|f_n^{(m)}\| = O(1),$$

where $(^2)$

$$\varepsilon_n = n^{-1-\theta/2}, \quad \theta = 1 + [1/p] - 1/p, \quad \sigma = \frac{\theta}{2(2+\theta)}.$$

In fact, let

$$\overline{g}_k(x, \varepsilon_n, \sigma) = \varepsilon_n^{1+\sigma} g_k(x, \varepsilon_n) + x^{k+1} - \varepsilon_n^{1+\sigma} x^k, \quad x \in (-\varepsilon_n, \varepsilon_n).$$

Then

$$\overline{g}_k^{(k)}(x, \varepsilon_n, \sigma) = \varepsilon_n^{1+\sigma} (g_k^{(k)}(x, \varepsilon_n) - k!) + (k+1)! x.$$

By Lemma 1,

$$\overline{g}_k^{(k)}(x, \varepsilon_n, \sigma) \geq (k+1)! x - O(\varepsilon_n^\sigma x) \quad \text{for } x \in [0, \varepsilon_n),$$

$$\overline{g}_k^{(k)}(x, \varepsilon_n, \sigma) \leq (k+1)! x + O(\varepsilon_n^\sigma x) \quad \text{for } x \in (-\varepsilon_n, 0),$$

that is, for sufficiently large $n \geq N_0$,

$$\overline{g}_k^{(k)}(x, \varepsilon_n, \sigma) \geq 0 \quad \text{for } x \in [0, \varepsilon_n),$$

$$\overline{g}_k^{(k)}(x, \varepsilon_n, \sigma) \leq 0 \quad \text{for } x \in (-\varepsilon_n, 0).$$

Put

$$f_n(x) = \begin{cases} x^{k+1} - \varepsilon_n^{1+\sigma} x^k, & |x| \geq \varepsilon_n, \\ \overline{g}_k(x, \varepsilon_n, \sigma), & |x| < \varepsilon_n. \end{cases}$$

Then it is not difficult to verify (2) and that $f_n \in C_{[-1,1]}^\infty \cap \overline{\Delta}^k$. Finally, (1) can be deduced by applying Lemma 2.

Let

$$F_l(x) = \sum_{j=1}^l n_j^{-\theta/8} f_{n_j}(x), \quad Q_l(x) = q_l(x) + n_l^{-\theta/8}(x^{k+1} - \varepsilon_{n_l}^{1+\sigma} x^k),$$

where $q_l(x)$ is the algebraic polynomial of best approximation of degree n_l to $F_{l-1}(x)$, and $\{n_l\}$ is a subsequence of natural numbers chosen by induction: Set $n_1 = N_0$,

$$(3) \qquad n_{l+1} = 2(n_l^{8(k+2+[1/p])/\theta} + [\|F_l^{(M_0)}\|]$$

$$+ [\|F_l^{(k+3)}\|] + [\|F_l^{(k+2+[1/p])}\|^{1/\delta}] + 1)$$

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$(^2)$ Note that $0 < \theta \leq 1$ for each $0 < p \leq \infty$.

for $l = 1, 2, \ldots,$ where $[x]$ is the greatest integer not exceeding x,

$$M_0 = [(1 + \theta/2)(k + 1 + \sigma + 1/p)] + 2,$$

$$\delta = 1 + \left[\frac{1}{p}\right] - \frac{1}{p} - \frac{15\theta}{16} = \frac{1}{16} + \frac{1}{16}\left[\frac{1}{p}\right] - \frac{1}{16p}.$$

It is not difficult to see that

$$(4) \qquad \|F_{l-1} - q_l\| = O(\|F_{l-1}^{(M_0)}\|n_l^{-M_0}).$$

By Lemma 1 and a theorem on simultaneous approximation to continuous functions and their derivatives from D. Leviatan [8],

$$(5) \qquad |q_l^{(k)}(0)| = |F_{l-1}^{(k)}(0) - q_l^{(k)}(0)| = O(\|F_{l-1}^{(k+3)}\|n_l^{-3}).$$

From the expression

$$F_l(x) - Q_l(x) = F_{l-1}(x) - q_l(x) + n_l^{-\theta/8}(f_{n_l}(x) - x^{k+1} + \varepsilon_{n_l}^{1+\sigma}x^k),$$

noticing that

$$\|f_{n_l}(x) - x^{k+1} + \varepsilon_{n_l}^{1+\sigma}x^k\|_{L_{[-1,1]}^p} = \left\|e\varepsilon_{n_l}^{1+\sigma}x^k \exp\left(\frac{\varepsilon_{n_l}^2}{x^2 - \varepsilon_{n_l}^2}\right)\right\|_{L_{[-\varepsilon_{n_l}, \varepsilon_{n_l}]}^p}$$

$$\sim \varepsilon_{n_l}^{k+1+\sigma+1/p},$$

together with (3), (4), we have

$$(6) \qquad \|F_l - Q_l\|_{L_{[-1,1]}^p} \sim n_l^{-\theta/8}\|f_{n_l} - x^{k+1} + \varepsilon_{n_l}^{1+\sigma}x^k\|_{L_{[-1,1]}^p}$$

$$\sim n_l^{-\theta/8}\varepsilon_{n_l}^{k+1+\sigma+1/p}.$$

On the other hand, we see

$$Q_l^{(k)}(0) = q_l^{(k)}(0) - k!n_l^{-\theta/8}\varepsilon_{n_l}^{1+\sigma};$$

thus

$$(7) \qquad |Q_l^{(k)}(0)| \geq C(k)n_l^{-\theta/8}\varepsilon_{n_l}^{1+\sigma} = C(k)n_l^{-1-7\theta/8}$$

follows from (3) and (5).

Now (6) and (7) imply that

$$(8) \qquad \varepsilon_{n_l}^{-k-1/p}\|F_l - Q_l\|_{L_{[-1,1]}^p} \leq C(k)|Q_l^{(k)}(0)|.$$

By a Nikol'skiĭ type inequality for trigonometric polynomials ([3]) (see P. G. Nevai [14, Theorem 1 and the formula in line 11, p. 240] for the case $0 < p < 1$), for any $r \in \Pi_{n_l}$ with $r^{(k)}(0) = 0$, we have

$$(9) \qquad |Q_l^{(k)}(0)| = |Q_l^{(k)}(0) - r^{(k)}(0)| \leq C(k)n_l^{k+1/p}\|Q_l - r\|_{L_{[-1,1]}^p}$$

$$\leq C(k)n_l^{k+1/p}(\|Q_l - F_l\|_{L_{[-1,1]}^p} + \|F_l - r\|_{L_{[-1,1]}^p})$$

([3]) We can apply it to algebraic polynomials in our case simply by making a change of variable.

for $1 \le p \le \infty$, and

(9′) $\quad |Q_l^{(k)}(0)|^p = |Q_l^{(k)}(0) - r^{(k)}(0)|^p \le C(k)n_l^{kp+1}\|Q_l - r\|_{L_{[-1,1]}^p}^p$

$$\le C(k)n_l^{kp+1}(\|Q_l - F_l\|_{L_{[-1,1]}^p}^p + \|F_l - r\|_{L_{[-1,1]}^p}^p)$$

for $0 < p < 1$. Combining (6), (8), (9) and (9′), for l large enough, we get

(10) $\quad \|F_l - r\|_{L_{[-1,1]}^p} \ge C(k)n_l^{-k-1/p}\varepsilon_{n_l}^{-k-1/p}\|F_l - Q_l\|_{L_{[-1,1]}^p}$

$$\ge C(k)n_l^{-k-1/p-\theta/8}\varepsilon_{n_l}^{1+\sigma} = C_k n_l^{-k-1-\alpha},$$

where

$$\alpha = \frac{1}{p} + \frac{7\theta}{8} = \frac{7}{8} + \frac{7}{8}\left[\frac{1}{p}\right] + \frac{1}{8p} < 1 + \left[\frac{1}{p}\right].$$

Define

$$f(x) = \sum_{j=1}^{\infty} n_j^{-\theta/8} f_{n_j}(x).$$

It is clear that $f \in C_{[-1,1]}^k \cap \overline{\Delta}^k$. For any $r \in \Pi_{n_l}$ with $r^{(k)}(0) = 0$,

$$\|f - r\|_{L_{[-1,1]}^p} \ge \|F_l - r\|_{L_{[-1,1]}^p} - 2\left\| \sum_{j=l+1}^{\infty} n_j^{-\theta/8} f_{n_j} \right\|$$

for $1 \le p \le \infty$, and

$$\|f - r\|_{L_{[-1,1]}^p}^p \ge \|F_l - r\|_{L_{[-1,1]}^p}^p - 2\left\| \sum_{j=l+1}^{\infty} n_j^{-\theta/8} f_{n_j} \right\|^p$$

for $0 < p < 1$. In any case, applying (10) we have

$$\|f - r\|_{L_{[-1,1]}^p} \ge C(k)(n_l^{-k-1-\alpha} - n_{l+1}^{-\theta/8}) \ge C(k)(n_l^{-k-1-\alpha} - n_l^{-k-2-[1/p]}),$$

thus

(11) $\qquad\qquad \widetilde{e}_{n_l}^{(k)}(f)_p \ge C(k)n_l^{-k-1-\alpha}.$

At the same time, in view of Lemma 2 and (3), (6), when $1 \le p \le \infty$,

(12) $\quad \omega_{k-m+2+[1/p]}(f^{(m)}, n_l^{-1})_p \le \|F_{l-1}^{(k+2+[1/p])}\|n_l^{-k+m-2-[1/p]}$

$$+ n^{-\theta/8}\omega_{k-m+2+[1/p]}\left(f_{n_l}^{(m)}(x) - \frac{(k+1)!}{(k-m+1)!}x^{k-m+1} \right.$$

$$+ \left. \frac{k!}{(k-m)!}\varepsilon_{n_l}^{1+\sigma}x^{k-m}, n_l^{-1} \right) + O\left(\sum_{j=l+1}^{\infty} n_j^{-\theta/8} \right)$$

$$= O(n_l^{-k+m-1-\beta}) + O(n_l^{-\theta/8}\varepsilon_{n_l}^{k-m+1+\sigma+1/p}) + O(n_l^{-k-2-[1/p]}),$$

where $\beta = 1/p + 15\theta/16$.

In a similar way we deal with the case $0 < p < 1$ and get the same result. Take

$$M_n^* = \min\{n^{-k+m-1/p}\varepsilon_n^{-k+m-1/p}, n^{\theta/16}\}.$$

Then from (11) and (12) for sufficiently large l it follows that

$$\widetilde{e}_{n_l}^{(k)}(f)_p/\omega_{k-m+2+[1/p]}(f^{(m)}, n_l^{-1})_p \geq C(k)M_{n_l}^* n_l^m. \quad \blacksquare$$

3. Remark. Let $\Delta^k(r)$, $r \geq 0$, denote the class of functions such that $\Delta_h^k f(x)$ changes its sign exactly r times on the interval $[-1, 1 - kh]$ for sufficiently small $h > 0$. For $f \in L_{[-1,1]}^p \cap \Delta^k(r)$, let

$$E_n^{(k)}(f, r)_p = \min\{\|f - P\|_{L_{[-1,1]}^p}\},$$

where the minimum is taken over all polynomials $P \in \Pi_n$ which are kth comonotone with f, that is, $\Delta_h^k f(x)\Delta_h^k P(x) \geq 0$ for all $x \in [-1, 1 - kh]$ and sufficiently small $h > 0$.

PROBLEM 3. Let $0 < p \leq \infty$, $r \geq 0$. Does there exist a function $f \in C_{[-1,1]}^k \cap \Delta^k(r)$ for $k \geq 1$ such that

$$\limsup_{n \to \infty} E_n^{(k)}(f, r)_p/\omega_{k+2}(f, n^{-1})_p = +\infty?$$

We can also ask a weak form of this question:

PROBLEM 4. Let $0 < p \leq \infty$, $r \geq 0$. Does there exist a function $f \in C_{[-1,1]}^k \cap \Delta^k(r)$ for $k \geq 1$ such that

$$\limsup_{n \to \infty} E_n^{(k)}(f, r)_p/\omega_{k+2+[1/p]}(f, n^{-1})_p = +\infty?$$

When $r = 0$, $p = \infty$, the above questions become Problem 1 we mentioned in the introduction. The present paper has given a positive answer to Problem 4 in case $r = 1$. Since the method used in this paper cannot be easily applied to general cases, the above questions require further investigation for $r \geq 2$.

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