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A REMARK ON MULTIRESOLUTION ANALYSIS OF $L^p(\mathbb{R}^d)$

BY

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A condition on a scaling function which generates a multiresolution analysis of $L^p(\mathbb{R}^d)$ is given.

1. Introduction and results. A family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^p(\mathbb{R}^d)$ is called a *multiresolution analysis* of $L^p(\mathbb{R}^d)$ if

(i) $V_j \subset V_{j+1}$ and $f(x) \in V_j$ if and only if $f(2^{-j}x) \in V_0$;

(ii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^p(\mathbb{R}^d)$;

(iii) there exists a scaling function ϕ with l^p stable integer translates such that

$$V_0 = \Big\{ \sum_{k\in\mathbb{Z}^d} C(k)\phi(\cdot - k) : \sum_{k\in\mathbb{Z}^d} |C(k)|^p < \infty \Big\}.$$

We say that a function ϕ has l^p *stable integer translates* if there exist $0 < A \le B < \infty$ such that

$$A\Big(\sum_{k\in\mathbb{Z}^d} |C(k)|^p\Big)^{1/p} \le \Big\|\sum_{k\in\mathbb{Z}^d} C(k)\phi(\cdot - k)\Big\|_p \le B\Big(\sum_{k\in\mathbb{Z}^d} |C(k)|^p\Big)^{1/p} \tag{1}$$

for every sequence $\{C(k)\} \in l^p$. Hereafter we assume $1 < p < \infty$ and write $L^p = L^p(\mathbb{R}^d) = \{f : \|f\|_p = (\int_{\mathbb{R}^d} |f(x)|^p\,dx)^{1/p} < \infty\}$ and $l^p = l^p(\mathbb{Z}^d) = \{\{C(k)\} : \sum_{k\in\mathbb{Z}^d}|C(k)|^p < \infty\}$. For simplicity we use $\sum$ without index to replace the sum over $\mathbb{Z}^d$. We say that a function ϕ *generates a multiresolution analysis of* L^p if ϕ has l^p stable integer translates and the family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ defined by

$$\begin{aligned} V_j &= \Big\{ \sum C(k)\phi(2^j\cdot - k) : \sum |C(k)|^p < \infty \Big\} \\ &= L^p \text{ closure of } \Big\{ \sum C(k)\phi(2^j\cdot - k) : \{C(k)\} \text{ has finite length} \Big\} \end{aligned} \tag{2}$$

is a multiresolution analysis of L^p. Hereafter we say that a sequence $\{C(k)\}$ has *finite length* if $C(k) \ne 0$ except for finitely many k.

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The multiresolution analysis of L^2 was introduced by Mallat ([3]) and Meyer ([4]), and is well examined since we can use the Fourier transform ([2]). It becomes an important and almost unique scheme for construction of orthonormal bases of wavelets of L^2 which are unconditional bases of L^p $(1 < p < \infty)$ under some conditions. The multiresolution analysis of L^p $(p \neq 2)$ is still meaningful since the construction of bases of wavelets from a multiresolution analysis of L^2 is still a difficult problem in general, and a function ϕ with some decay at infinity which generates a multiresolution analysis of L^2 generates one of L^p also. Define

$$L^p_* = \Big\{f : \int\limits_{[0,1]^d} \Big(\sum |f(x+k)|\Big)^p\, dx < \infty\Big\}.$$

Jia and Micchelli ([1]) proved that ϕ generates a multiresolution analysis of L^p if $\phi \in L^p_*$ has l^p stable integer translates and satisfies the refinement equation

$$\phi(x) = \sum a(k)\phi(2x-k) \tag{3}$$

with the mask $\{a(k)\} \in l^1$. Let ϕ be a distribution having a continuous Fourier transform. We say that the integer translates of ϕ are *globally linearly independent for tempered sequences* if $\widehat{\phi}(\xi+2k\pi)$ is not identically zero on $\mathbb{Z}^d$ for every $\xi \in \mathbb{R}^d$ (cf. [6]).

In this paper we will use Fourier analysis to prove

THEOREM 1. *Suppose the integer translates of ϕ are l^p stable and globally linearly independent for tempered sequences. If ϕ satisfies the refinement equation* (3) *with $\sum a(k) = 2$ and $\sum |a(k)|^2(1+|k|)^{2l} < \infty$ for some integer $l > d/2$, then ϕ generates a multiresolution analysis of L^p.*

In particular, if ϕ satisfies the refinement equation (3) with $\{a(k)\} \in l^1$, then the spaces V_j defined by (2) satisfy

(i) $V_j \subset V_{j+1}$ and $f(x) \in V_j$ if and only if $f(2^{-j}x) \in V_0$,

since $\sum C(k)\phi(x-k) = \sum(\sum C(l)a(k-2l))\phi(2x-k)$ and

$$\sum \Big|\sum C(l)a(k-2l)\Big|^p \le \Big(\sum |a(k)|\Big)^p \sum |C(l)|^p.$$

Let $1 < q < \infty$. We say that a measurable function m is a *local L^q multiplier* if for every compact set K there exists a constant C_K independent of f such that

$$\|(m\widehat{f}\,)^\vee\|_q \le C_K \|f\|_q$$

for every $f \in L^q$ with $\operatorname{supp} \widehat{f} \subset K$, where $\widehat{f}$ and $f^\vee$ denote the Fourier transform and inverse Fourier transform respectively.

Observe that $\sum |a(k)|^2(1+|k|)^{2l} < \infty$ for some $l > d/2$ implies $\{a(k)\} \in l^1$. Therefore the matter reduces to

THEOREM 2. *Suppose the integer translates of* ϕ *are* l^p *stable and globally linearly independent for tempered sequences. Assume* $\widehat{\phi}$ *is a continuous local* L^q *multiplier for some* $\infty > q > \max(p, p/(p-1))$. *If* ϕ *satisfies the refinement equation* (3) *with* $\{a(k)\} \in l^1$ *and* $\widehat{\phi}(0) \neq 0$, *then* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ *and* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^p$.

THEOREM 3. *If* ϕ *satisfies the refinement equation* (3) *with* $\sum a(k) = 2$ *and* $\sum |a(k)|^2(1+|k|)^{2l} < \infty$ *for some integer* $l > d/2$, *then* $\widehat{\phi}$ *is a continuous local* L^q *multiplier for* $1 < q < \infty$.

Therefore conditions (i) and (iii) imply (ii) provided $\phi \in L^1$ and the integer translates of ϕ are globally linearly independent for tempered sequences, since $\|(\widehat{\phi}\widehat{f}\,)^\vee\|_q = \|\int \phi(\cdot - y)f(y)\,dy\|_q \le \|\phi\|_1\|f\|_q$ for $1 < q < \infty$. Observe that $L^1 \supset L^p_*$ for any $1 \le p \le \infty$ and the integer translates of $\phi \in L^p_*$ which has l^p stable integer translates must be globally linearly independent for tempered sequences (see also Section 3 below). Hence Theorem 2 improves the result of Jia and Micchelli ([1]). In particular, Theorem 1 is new even when $p = 2$.

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2. Proofs. The proof of Theorem 2 depends on the following two technical lemmas.

LEMMA 1. *Let* m *be a continuous local* L^q *multiplier for some* $q > 2$. *Then for every* x_0 *such that* $m(x_0) \neq 0$ *there exist a compact set* K *and a constant* C *independent of* f *such that* x_0 *is an inner point of* K *and* $\|(m^{-1}\widehat{f}\,)^\vee\|_p \le C\|f\|_p$ *for every* $f \in L^p$ *with* $\operatorname{supp}\widehat{f} \subset K$ *where* $q/(q-1) < p < q$.

Proof. Without loss of generality we assume $x_0 = 0$. Observe that

$$\|(\widetilde{m}\widehat{f}\,)^\vee\|_2 \le C \sup_{B(r)} |\widetilde{m}(x)|\,\|f\|_2$$

for every $f \in L^2$ with $\operatorname{supp}\widehat{f} \subset B(r) = \{x : |x| \le r\}$, where $\widetilde{m}(x) = m(x) - m(0)$ and $r > 0$. Recall that

$$\|(\widetilde{m}\widehat{f}\,)^\vee\|_p \le \|(m\widehat{f}\,)^\vee\|_p + |m(0)|\,\|f\|_p \le C\|f\|_p$$

for every $f \in L^q$ with $\operatorname{supp}\widehat{f} \subset B(r)$. Therefore we get

$$\|(\widetilde{m}\widehat{f}\,)^\vee\|_p \le C(\sup_{B(r)} |\widetilde{m}(x)|)^\theta\|f\|_p$$

for every $f \in L^p$ with $\operatorname{supp}\widehat{f} \subset B(r)$ by the Marcinkiewicz real interpolation between 2 and q or $q/(q-1)$, where $q/(q-1) < p < q$ and $\theta = \theta(p,q) > 0$

([5], p. 21). Furthermore,

$$\|(\widetilde{m}\widehat{f})^\vee\|_p \le \frac{1}{2}|m(0)|\,\|f\|_p$$

holds for every $f \in L^p$ with $\operatorname{supp}\widehat{f} \subset B(r_0)$ when $r_0 > 0$ is chosen small enough by the continuity of $m(x)$. Observe that

$$m^{-1}(x) = m(0)^{-1}\left(1 + \sum_{k=1}^{\infty}\left(\frac{m(0)-m(x)}{m(0)}\right)^k\right).$$

Therefore

$$\begin{aligned}\|(m^{-1}\widehat{f})^\vee\|_p &\le |m(0)|^{-1}\left(\|f\|_p + \sum_{k=1}^{\infty}\left\|\left(\left(\frac{m(x)-m(0)}{m(0)}\right)^k\widehat{f}\right)^\vee\right\|_p\right)\\ &\le 2|m(0)|^{-1}\|f\|_p\end{aligned}$$

for every $f \in L^p$ with $\operatorname{supp}\widehat{f} \subset B(r_0)$ and Lemma 1 is proved.

LEMMA 2. *Let V_j defined by (2) for some $\phi \in L^p$ satisfy $V_j \subset V_{j+1}$ and ψ be any Schwartz function. Then for every $f \in V_0$ there exists $g_j \in V_j$ such that $\|\psi * f - g_j\|_p \to 0$ as $j \to \infty$.*

Proof. Let $g_j(x) = 2^{-jd}\sum_k \psi(2^{-j}k)f(x-2^{-j}k) \in V_j$. Then

$$\begin{aligned}\|g_j - \psi * f\|_p &\le \sum_k \int_{[0,2^{-j}]^d} |\psi(y+2^{-j}k) - \psi(2^{-j}k)|\,dy\,\|f\|_p\\ &\quad + \sum_k 2^{-jd}|\psi(2^{-j}k)|\omega_p(f,2^{-j})\\ &\le C2^{-j}\|f\|_p + C\omega_p(f,2^{-j}) \to 0 \quad \text{as } j \to \infty,\end{aligned}$$

where $\omega_p(f,t) = \sup_{|y|\le t}\|f(\cdot - y) - f(\cdot)\|_p$, and Lemma 2 is proved.

Now we start to prove Theorem 2. First, $Y = \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Let K_0 be a compact set such that for every $\xi \in \mathbb{R}^d$ there exists $\eta \in K_0$ such that $\widehat{\phi}(\eta) \ne 0$ and $(\xi - \eta)/(2\pi) \in \mathbb{Z}^d$ since the integer translates of ϕ are globally linearly independent for tempered sequences. Then for every $\xi_0 \notin K_0$ there exists a Schwartz function ψ such that $\operatorname{supp}\widehat{\psi} \cap K_0 = \emptyset$ and $\widehat{\psi}(\xi_0) = 1$. Let f be any function in Y. Therefore $\psi * f \in Y \subset V_0$ by Lemma 2 and $\widehat{\psi}(\xi)\widehat{f}(\xi) = \tau(\xi)\widehat{\phi}(\xi)$, where $\tau(\xi) = \sum C(k)e^{ik\xi}$ is a 2π-periodic distribution and $\{C(k)\} \in l^p$. Let η_0 be some point in K_0 such that $\widehat{\phi}(\eta_0) \ne 0$ and $(\xi_0 - \eta_0)/(2\pi) \in \mathbb{Z}^d$. Therefore $\tau(\xi) = 0$ on some neighborhood of η_0 and furthermore on some neighborhood of $\eta_0 + 2\pi\mathbb{Z}^d$ since $\widehat{\psi}\widehat{f} = 0$ in some neighborhood of η_0, $\widehat{\phi}(\eta_0) \ne 0$ and τ is 2π-periodic. Hence $\widehat{f}(\xi) = 0$ on some neighborhood of ξ_0 and $\operatorname{supp}\widehat{f} \subset K_0$ for every $f \in Y$. Observe that $f \in Y$ if and only if $f(2^j\cdot) \in Y$ for $j \in \mathbb{Z}$ and any function f with $\operatorname{supp}\widehat{f} = \{0\}$

is a nonzero polynomial. Recall that K_0 is bounded. Therefore $\operatorname{supp}\widehat{f} = \emptyset$, $f = 0$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Second, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^p(\mathbb{R}^d)$. Define

$$X' = \{f \in L^p : \operatorname{supp}\widehat{f} \text{ is compact in } \mathbb{R}^d\}$$

and $X = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. Therefore the matter reduces to $X' \subset X$ since X' is dense in L^p and X is a closed subspace of L^p. Let f be any function in X'. Recall that $\widehat{\phi}(0) \neq 0$. Therefore there exists a positive integer j_1 such that $|2^{-j_1}y| \leq \pi/2$ and $|\widehat{\phi}(2^{-j_1}y)| \geq \frac{1}{2}|\widehat{\phi}(0)|$ for any $y \in \operatorname{supp}\widehat{f}$. Since $\widehat{\phi}$ is a local L^q multiplier for some $q > \max(p, p/(p-1))$, by Lemma 1 we get $\tau(\xi)^\vee = (\widehat{\phi}(2^{-j_1}\xi)^{-1}\widehat{f}(\xi))^\vee \in L^p$, where we set $\tau(\xi) = \widehat{\phi}(2^{-j_1}\xi)^{-1}\widehat{f}(\xi)$. Furthermore, $\sum |\tau^\vee(2^{-j_1}k)|^p < \infty$ by the Shannon sampling theorem which says that the L^p norm of a function f whose Fourier transform is supported in $[-\pi/2, \pi/2]^d$ is equivalent to the l^p norm of the sampling values of f at the integer lattice points. Let

$$g(x) = 2^{j_1 d}\sum_k \tau^\vee(2^{-j_1}k)\phi(2^{j_1}x - k) \in V_{j_1} \subset X$$

and ψ be some Schwartz function such that $\widehat{\psi} = 1$ on $\operatorname{supp}\widehat{f}$ and $\operatorname{supp}\widehat{\psi} \subset \{|y| \leq 2^{j_1-1}\pi\}$. Then

$$(\widehat{\psi * g})(\xi) = \widehat{\psi}(\xi)\widehat{g}(\xi) = \widehat{\psi}(\xi)\Big(\sum_k \tau^\vee(2^{-j_1}k)e^{i2^{-j_1}\xi k}\Big)\widehat{\phi}(2^{-j_1}\xi) = \widehat{f}(\xi)\,,$$

$f = \psi * g \in X$ by Lemma 2 and Theorem 2 holds true.

The proof of Theorem 3 depends on the following lemma.

LEMMA 3. *If $D^\alpha m \in L^2_{\mathrm{loc}}$ for all $|\alpha| \leq l$ and some integer $l > d/2$, then m is a continuous local L^q multiplier for all $1 < q < \infty$, where $L^q_{\mathrm{loc}} = \{f : \int_K |f(x)|^q < \infty$ for every compact set $K\}$, and $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $|\alpha| = \sum_{i=1}^d |\alpha_i|$.*

The proof of Lemma 3 follows from the Marcinkiewicz multiplier theorem ([5], p. 96).

Now we start to prove Theorem 3. Let ϕ satisfy the refinement equation (3) with the mask $\{a(k)\}$. Define $H(\xi) = \frac{1}{2}\sum_k a(k)e^{ik\xi}$. Observe that $\widehat{\phi}(\xi) = \prod_{j=1}^\infty H(\xi/2^j)\widehat{\phi}(0)$ and

$$\begin{aligned} D^\alpha\widehat{\phi}(\xi) = &\sum_{\substack{\alpha_1+\ldots+\alpha_s=\alpha \\ \alpha_i\in\mathbb{Z}^d,\alpha_i\neq 0}} C_{\alpha,\alpha_1,\ldots,\alpha_s}\sum_{j_1,\ldots,j_s}\prod_{m=1}^s D^{\alpha_m}H\Big(\frac{\xi}{2^{j_m}}\Big) \\ &\times 2^{-j_m|\alpha_m|}\prod_{j\neq j_1,\ldots,j_s} H\Big(\frac{\xi}{2^j}\Big)\,. \end{aligned}$$

Therefore the matter reduces to proving $\sum_{j_1,\ldots,j_s}\prod_{m=1}^s |D^{\alpha_m}H(\xi/2^{j_m})| \times 2^{-j_m|\alpha_m|} \in L^2_{\rm loc}$, or there exist C and $\varepsilon>0$ independent of $j_1,\ldots,j_s$ for every $R\ge 1$ and every α with $\alpha_m\ne 0$ and $|\alpha|=\sum_{m=1}^s|\alpha_m|\le l_0=[d/2]+1$, such that

$$\int\limits_{|x|\le R}\bigg(\prod_{m=1}^s\bigg|D^{\alpha_m}H\bigg(\frac{\xi}{2^{j_m}}\bigg)\bigg|2^{-j_m|\alpha_m|}\bigg)^2\,dx\le C2^{-\varepsilon(j_1+\ldots+j_s)}\,,$$

where $[x]$ denotes the integer part of x. Recall that $\sum_k|a(k)|^2(1+|k|)^{2l}<\infty$. Therefore $D^\alpha H\in L^2_{\rm loc}$ for every $|\alpha|\le l_0$. By the Sobolev imbedding theorem ([5], p. 124), $D^\alpha H\in L^{p_\alpha}_{\rm loc}$ for every p_α such that $1/p_\alpha>1/2-(l_0-|\alpha|)/d$. Since

$$\sum_{m=1}^s\bigg(\frac12-\frac{l-|\alpha_m|}{d}\bigg)<\frac12\,,$$

there exist p_{α_m} such that $D^{\alpha_m}H\in L^{p_{\alpha_m}}_{\rm loc}$, $1/r=\sum_{m=1}^s 1/p_{\alpha_m}\le 1/2$ and $d/p_{\alpha_m}<|\alpha_m|$. Therefore

$$\begin{aligned}
&\int\limits_{|x|\le R}\bigg(\prod_{m=1}^s\bigg|D^{\alpha_m}H\bigg(\frac{x}{2^{j_m}}\bigg)\bigg|2^{-j_m|\alpha_m|}\bigg)^2\,dx\\
&\qquad\le C\bigg(\int\limits_{|x|\le R}\bigg(\prod_{m=1}^s\bigg|D^{\alpha_m}H\bigg(\frac{x}{2^{j_m}}\bigg)\bigg|2^{-j_m|\alpha_m|}\bigg)^r\,dx\bigg)^{2/r}\\
&\qquad\le C\prod_{m=1}^s\bigg(\int\limits_{|x|\le R}\bigg|D^{\alpha_m}H\bigg(\frac{x}{2^{j_m}}\bigg)\bigg|^{p_{\alpha_m}}\,dx\bigg)^{2/p_{\alpha_m}}2^{-2j_m|\alpha_m|}\\
&\qquad\le C\prod_{m=1}^s 2^{j_m(-2|\alpha_m|+2d/p_{\alpha_m})}\le C2^{-\varepsilon(j_1+\ldots+j_s)}\,,
\end{aligned}$$

where ε is chosen as $\min(2|\alpha_m|-2d/p_{\alpha_m})$ and Theorem 3 is proved.

3. Remarks. If ϕ has compact support, then

$$\bigg\|\sum_k C(k)\phi(\cdot-k)\bigg\|_p\le B\bigg(\sum_k|C(k)|^p\bigg)^{1/p}\tag{4}$$

holds if and only if $\phi\in L^p$. Jia and Micchelli ([1]) proved that $\phi\in L^p_*$, i.e.

$$\int\limits_{[0,1]^d}\bigg(\sum_k|\phi(x+k)|\bigg)^p\,dx<\infty\,,$$

is a sufficient condition for (4) to hold. Obviously $\phi\in L^p$ is a necessary condition. By the inequalities for Rademacher functions ([5], pp. 104, 276), we know that (4) implies

$$\int_{\mathbb{R}^d} \Big(\sum_k |C(k)|^2 |\phi(x+k)|^2\Big)^{p/2} dx \le C \sum_k |C(k)|^p .$$

Furthermore, we have

$$\begin{aligned}\int_{[0,1]^d} \Big(\sum_k |\phi(x+k)|^2\Big)^{p/2} dx \\ &\le C \lim_{k\to\infty} 2^{-kd} \sum_{|s|\le 2^k} \int_{[0,1]^d} \Big(\sum_{|j|\le 2^{k+1}} |\phi(x+j-s)|^2\Big)^{p/2} dx \\ &\le C \lim_{k\to\infty} 2^{-kd} \int_{\mathbb{R}^d} \Big(\sum_{|j|\le 2^{k+1}} |\phi(x+j)|^2\Big)^{p/2} dx < \infty .\end{aligned}$$

Therefore

$$\int_{[0,1]^d} \Big(\sum_k |\phi(x+k)|^2\Big)^{p/2} dx < \infty ,$$

which is stronger than $\phi \in L^p$ when $p > 2$, is a necessary condition for (4) to hold.

For some functions $\{\phi_s\}_{s=1}^N$, define

$$V_j = \Big\{ \sum_{s=1}^N \sum_k C_s(k)\phi_s(x-k) : \sum_{s=1}^N \sum_k |C_s(k)|^p < \infty \Big\} .$$

Then the corresponding result of Theorem 2 holds (see [1], [6] for the definition of l^p stable integer translates of $\{\phi_s\}_{s=1}^N$).

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