Now (4) is true by induction for our particular value of t, and hence each of the four parts has the lower bound $4^t 2^{-5} t$, and the sum (14) itself is at least $4\cdot 4^t 2^{-5} t = 2^{-3} 4^t t$. The same lower bound holds if $h(J, \alpha_j)$ in (14) is replaced by $h(J, \alpha_{j-1})$, $h(J', \alpha_j)$ or $h(J', \alpha_{j-1})$. We now take the sum of (13) over $j = 1, 2, \ldots, 4^{t+1}$, and we obtain

$$2\sum_{j=1}^{4^{t+1}} h(I, \alpha_j) \geqslant \tfrac{1}{4}\sum_{j=1}^{4^{t+1}} z_j + \tfrac{1}{2}4(2^{-3}4^t t) \geqslant 4^{t-1} + 4^{t-1}t = 2\cdot 4^{t+1}\big(2^{-5}(t+1)\big)$$

by (12). Dividing by $2\cdot 4^{t+1}$ and recalling the definition of α_j we obtain (4) with t replaced by $t+1$.

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An application of Minkowski's theorem in the geometry of numbers

by

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In memory of Professor Wacław Sierpiński

The classical result in the geometry of numbers is given by Minkowski's

THEOREM 1. *An n-dimensional closed convex region symmetrical around the origin and of volume not less than 2^n, contains a point other than the origin of every lattice L in n variables of determinant one.*

Very few applications of this theorem are to be found in the usual literature. They are mostly concerned with sums of powers of linear forms or separate linear forms. As problems are rather scarce, I notice another application which may be of interest and which is given by

THEOREM 2. *Let L be a lattice in $2n$ variables $(x_1, \ldots, x_{2n})$ of determinant one. Then the region given by*

$$|x_r| \leqslant a \ \ (r = 1, \ldots, 2n), \qquad \sum_{r=1}^{n} |x_{2r-1} - x_{2r}| \leqslant 2b, \ \ a > b, \tag{1}$$

contains a point other than the origin of L if

$$b^{2n}\left(\frac{1}{n!} - \frac{n}{1\cdot(n+1)!} + \frac{n\cdot n-1}{2!(n+2)!} + \ldots + \frac{(-1)^n}{(2n)!}\right) \geqslant 2^{-n}. \tag{2}$$

The condition $a > b$ is imposed to exclude the case when the lattice L contains a sublattice of determinant one in x_1, x_2; for if $a < b$, a trivial solution may exist in which $x_3 = \ldots = x_{2n} = 0$.

We have to express the condition that the volume V of (1) is $\geqslant 2^{2n}$. Make the substitution

$$\sqrt{2}x_{2r-1} \to x_{2r-1} + x_{2r}, \qquad \sqrt{2}x_{2r} \to x_{2r-1} - x_{2r} \qquad (r = 1, 2, \ldots, n).$$

Then

$$\sum_{r=1}^{n} |x_{2r-1}| \leqslant b\sqrt{2}, \tag{3}$$

$$|x_{2r-1}+x_{2r}| \leqslant a\sqrt{2}, \quad |x_{2r-1}-x_{2r}| \leqslant a\sqrt{2} \quad (r = 1, \ldots, n). \tag{4}$$

Clearly (4) is equivalent to

$$|x_{2r-1}|+|x_{2r}| \leqslant a\sqrt{2} \quad (r = 1, \ldots, n). \tag{5}$$

Make the substitution $x \to a\sqrt{2}x$. Then

$$V = 2^{2n}(a\sqrt{2})^{2n} V_1, \tag{6}$$

where V_1 is the volume of the region

$$\begin{gathered} \sum_{r=1}^{n} x_{2r-1} \leqslant b/a = c,\ x_r \geqslant 0 \quad (r = 1, \ldots, 2n), \\ x_{2r-1}+x_{2r} \leqslant 1 \quad (r = 1, \ldots, n). \end{gathered} \tag{7}$$

Hence, integrating for x_{2r}, we have

$$V_1 = \int \ldots \int (1-x_1)(1-x_3) \ldots (1-x_{2n-1})\, dx_1 \ldots dx_{2n-1}$$

taken over the region

$$0 \leqslant x_{2r-1} \leqslant 1 \ (r = 1, \ldots, n), \quad x_1+x_3+\ldots+x_{2n-1} \leqslant c.$$

Since $c < 1$, the inequality $x_{2r-1} \leqslant 1$ is redundant. Write $x \to cx$. Then

$$V_1 = c^n \int \ldots \int (1-cx_1) \ldots (1-cx_{2n-1})\, dx_1 \ldots dx_{2n-1}, \quad \sum_{r=1}^{n} x_{2r-1} \leqslant 1.$$

Apply Dirichlet's integral

$$\int_{0 \leqslant \Sigma x \leqslant 1} \ldots \int x_1^{l_1-1} \ldots x_s^{l_s-1}\, dx_1 \ldots dx_s = \frac{\Gamma(l_1) \ldots \Gamma(l_s)}{\Gamma(l_1+\ldots+l_s+1)}.$$

We have now

$$V_1 = c^n\left(\frac{1}{n!} - \frac{nc}{(n+1)!} + \frac{n \cdot n-1}{2!}\frac{c^2}{(n+2)!} - \right.$$
$$\left. - \frac{n \cdot n-1 \cdot n-2}{3!}\frac{c^3}{(n+3)!} + \ldots + (-1)^n \frac{c^n}{(2n)!}\right),$$

and so we require

$$V = 2^{3n} a^{2n} V_1 \geqslant 2^{2n},$$

or

$$b^n\left(\frac{a^n}{n!} - \frac{na^{n-1}}{1!(n+1)!} b + \ldots + (-1)^n \frac{b^n}{(2n)!}\right) \geqslant 2^{-n}. \tag{8}$$

We show now that the minimum value of (8) for given b occurs when $a = b$. It suffices to show that if

$$f(u) = \frac{u^n}{n!} - \frac{nu^{n-1}}{1!(n+1)!} + \frac{n \cdot n-1 \cdot u^{n-2}}{2!(n+2)} + \ldots + \frac{(-1)^n}{(2n)!}, \tag{9}$$

then $f(u) > f(v)$ if $u \geqslant v > 1$. Write

$$f(u)-f(v) = \frac{u^n-v^n}{n!} - \frac{n(u^{n-1}-v^{n-1})}{1!(n+1)!} + \ldots \tag{10}$$

The terms in this series taken positively are steadily decreasing. This requires

$$\frac{n \cdot n-1 \cdot \ldots \cdot n-r+1}{r!(n+r)!}(u^{n-r}-v^{n-r}) > \frac{n \cdot n-1 \cdot \ldots \cdot n-r}{(r+1)!(n+r+1)!}(u^{n-r-1}-v^{n-r-1}),$$

or

$$u^{n-r}-v^{n-r} > \frac{n-r}{(r+1)(n+r+1)}(u^{n-r-1}-v^{n-r-1}).$$

The result is obvious since

$$u^{n-r}-v^{n-r} = u(u^{n-r-1}-v^{n-r-1})+v^{n-r-1}(u-v).$$

Hence $f(u) > f(1)$ if $u > 1$ and so (8) is satisfied if

$$b^{2n}\left(\frac{1}{n!} - \frac{n}{1!(n+1)!} + \frac{n \cdot n-1}{2!(n+2)!} + \ldots + (-1)^n \frac{1}{(2n)!}\right) \geqslant 2^{-n}.$$

This completes the proof.

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