6. Let $\delta[\lambda, r+\omega]$ be a fixed invertible ideal in R_d, δ and λ in Z, and let τ, σ be positive integers such that $\delta^2\lambda = \tau\sigma$. We wish to determine the number $g(\tau)$ of divisors in R_d (necessarily invertible) of $\delta[\lambda, r+\omega]$ of norm τ, i.e. the number of ordered pairs T, S of invertible ideals in R_d such that $\delta[\lambda, r+\omega] = TS$, $N(T) = \tau$. For each prime p define integers k_p, l_p, n_p, s_p by: $p^{k_p}\|\delta$, $p^{l_p}\|\lambda$, $p^{s_p}\|(\lambda, \sigma)$; let n_p be the largest integer such that d/p^{2n_p} is in D.

If $n = 0$ in Theorem 3, i.e. if p is not a bad prime, then Theorem 3 becomes a special case of Theorem 1 of [1]. If z is a positive integer let $\chi(z)$ denote the number of invertible ideals in R_d of norm z. By Theorem 3 and Corollary 2 we now have

THEOREM 4. *If there exists a prime p such that $k_p < n_p$ and $2k_p < s_p < l_p$, then $g(\sigma) = g(\tau) = 0$. Otherwise,*

$$g(\sigma) = g(\tau) = \chi\left(\prod p^{\alpha_p}\right), \quad \textit{where} \quad \alpha_p = \min\left(s_p, k_p + \min(n_p, l_p/2)\right).$$

7. It may be of interest to mention that the result in the opening sentence of this article arose from a neat proof, essentially by descent, that if $a+bi$ is a Gauss integer of norm mn (m and n positive integers), then the number of nonassociate divisors of $a+bi$ of norm m is equal to the number of Gauss integers of norm (a, b, m, n). There is a similar result for quaternions, and presumably a corresponding theorem for factoring ideals in generalized quaternion orders. The analogous problem in cubic fields seems to be complicated. In [1] we gave an algorithm which associates the factorizations of an element $r(x_1+x_2\omega)$ in R_d as a product of elements of norms m and n with representations of $e = (r, m, n)$ by an explicitly given binary quadratic form φ of discriminant d. How φ is naturally connected with the given elements was left unclear. Hence it may be worth mentioning that φ can be transformed by an integral transformation into the primitive form associated in [2], Section 3, with the module $[m, r(x_1+x_2\omega)]$.

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Differential rings of meromorphic functions

by

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To the memory of Wacław Sierpiński

1. Introduction. Let $\mathscr{R}$ be a differential ring of analytic functions, that is a ring closed under differentiation. We may assume without loss of a generality that $\mathscr{R}$ contains the constants C and is therefore an algebra over C. If $\mathscr{R}_0$ is a differential subring of $\mathscr{R}$ we can define the ring $\mathscr{L} = \mathscr{R}_0[D]$ of linear differential operators with coefficients in $\mathscr{R}_0$ and consider $\mathscr{R}$ as an $\mathscr{L}$-module.

1.1. DEFINITION. The elements $f_1, f_2, \ldots, f_n$ of $\mathscr{R}$ are *linearly dependent over* $\mathscr{L}$ if there exist $L_1, \ldots, L_n \in \mathscr{L}$ not all 0 so that $L_1f_1 + \ldots + L_nf_n = 0$ and *linearly independent over* $\mathscr{L}$ otherwise. The *dimension of* $\mathscr{R}$ *over* $\mathscr{L}$ is the maximum number of linearly independent elements of $\mathscr{R}$ over $\mathscr{L}$.

We are interested in the following general conjectures:

1.2. CONJECTURE. *If $\mathscr{R}$ is a ring of entire functions which is finite dimensional over $\mathscr{L}$ then $\mathscr{R}$ is 0-dimensional over $\mathscr{L}$. That is, for every $f \in \mathscr{R}$ there exists an $L \in \mathscr{L}^*$ $(= \mathscr{L}\setminus\{0\})$ so that $Lf = 0$.*

The hypothesis that $\mathscr{R}$ be a ring of entire functions is certainly not superfluous since the conjecture in this form does not hold for rings of meromorphic functions (see § 3). However David Cantor has suggested the following two purely algebraic versions of our conjecture.

1.3. CONJECTURE. *Let $\mathscr{R}$ be an abstract differential ring with $D\mathscr{R} = \mathscr{R}$ and define $\mathscr{R}_0$ and $\mathscr{L}$ as before. If $\mathscr{R}$ is finite dimensional over $\mathscr{L}$ then $\mathscr{R}$ is 0-dimensional over $\mathscr{L}$ (at least if $D\mathscr{R}_0 = \{0\}$).*

1.4. CONJECTURE. *Let $\mathscr{R}, \mathscr{R}_0$ and $\mathscr{L}$ be as in Conjecture 1.3 but make the stronger assumption that $L\mathscr{R} = \mathscr{R}$ for every $L \in \mathscr{R}_0[D]$ whose leading coefficient is a unit of $\mathscr{R}_0$. Then, if $\mathscr{R}$ is finite dimensional over $\mathscr{L}$ it is 0-dimensional over $\mathscr{L}$.*

So far we have no algebraic attack on those conjectures. However we were able to show that there is an upper bound on the growth rates of the functions of $\mathscr{R}$ which is consistent with Conjecture 1.2 ([1]).

In § 2 we prove the conjecture for $\mathscr{R}_0 = C$, that is $\mathscr{L}$ the algebra of linear differential operators with constant coefficients.

In § 3 we characterize the rings of meromorphic functions which are finite dimensional over the same $\mathscr{L} = C[D]$ and find them to be rings of functions meromorphic on compact Riemann surfaces where the dimension over $\mathscr{L}$ is given by the number of poles.

2. Differential rings of entire functions. In this section we prove Conjecture 1.2 for rings of entire functions and $\mathscr{L} = C[D]$ (which is the meaning of $\mathscr{L}$ from now on).

2.1. THEOREM. *If $\mathscr{R}$ is a ring of entire functions which is finite dimensional over $\mathscr{L}$ then $\mathscr{R}$ is a ring of exponential polynomials.*

Since the conclusion refers only to the individual elements of $\mathscr{R}$ we may restrict attention to the differential subring

$$\langle f\rangle = C[f, f', f'', \ldots]$$

generated by an element f of $\mathscr{R}$. We first prove that f has the correct growth rates, repeating the arguments in [1] for this special case.

2.2. LEMMA. *If $\langle f\rangle$ satisfies the hypothesis of the theorem then f is of finite exponential type.*

2.3. LEMMA. *Let $M(r,f) = \max\limits_{z=r}|f(z)|$, as usual, then for every $\delta > 0$ we have $M(r,f') < M\big(r+M(r,f)^{-\delta}, f\big)^{1+\delta}$.*

Proof. This is an immediate consequence of Cauchy's inequality

$$M(r,f') \leqslant M(r+\varrho, f)/\varrho$$

where we choose $\varrho = M(r,f)^{-\delta}$.

Proof of Lemma 2.2. Let n be the least positive integer so that there exists an $L_n \in \mathscr{L}^*$ with

$$L_n(f^n) = L_{n-1}(f^{n-1}) + \ldots + L_1 f = g \tag{2.4}$$

where $L_1, \ldots, L_{n-1} \in \mathscr{L}$. If $n = 1$ we are finished. If $n > 1$ write

$$L_n = (D-\lambda_1)\ldots(D-\lambda_l) = (D-\lambda_1)^{m_1}\ldots(D-\lambda_k)^{m_k}$$

where $\lambda_1, \ldots, \lambda_k$ are district and solve (2.4) for f^n to get

$$f^n = P_1(z)e^{\lambda_1 z} + \ldots + P_k(z)e^{\lambda_k z} + e^{\lambda_1 z}\int\limits_0^z e^{(\lambda_2-\lambda_1)t_1}\int\limits_0^{t_1}\ldots\int\limits_0^{t_{l-1}} e^{-\lambda_l t_l} g(t_l)\,dt_l\ldots dt_1$$

where $P_i(z)$ is a polynomial of degree $\leqslant m_i - 1$. Thus there exist positive constants (which we generically denote by c) with

$$M(r,f)^n = M(r,f^n) < ce^{cr} + e^{cr}M(r,g) < ce^{cr}M(r,g) \tag{2.5}$$

unless $g = 0$ in which case f^n, and hence f, satisfies the lemma. Now

$$M(r,g) < c\max_{\substack{1\leqslant i\leqslant n-1\\ j\leqslant N}} M\big(r, D^j(f^i)\big)$$

where $N = \max\limits_{1\leqslant i\leqslant n-1}\deg L_i$. Thus by Lemma 2.3 we get

$$\begin{aligned} M(r,g) &< c\max_{1\leqslant i\leqslant n-1} M\big(r+M(r,f^i)^{-\delta}, f^i\big)^{1+N\delta} \\ &= c\max M\big(r+M(r,f)^{-i\delta}, f\big)^{i(1+N\delta)} \\ &< cM\big(r+M(r,f)^{-\delta}, f\big)^{n-1+nN\delta}. \end{aligned} \tag{2.6}$$

If we choose $\delta = 1/(2nN)$ and substitute in (2.5) we get

$$M(r,f)^n < ce^{cr}M\big(r+M(r,f)^{-\delta}, f\big)^{n-1/2}. \tag{2.7}$$

Now, if the lemma does not hold then there exist arbitrarily large r for which the term ce^{cr} on the right of (2.7) satisfies

$$ce^{cr} < M(r,f)^{1/4}. \tag{2.8}$$

Whenever (2.8) holds we get (2.7) to yield

$$M\big(r+M(r,f)^{-\delta}, f\big) > M(r,f)^{n/(n-1/4)} > M(r,f)M(r,f)^{1/4n}. \tag{2.9}$$

Now pick r so that an inequality

$$ce^{c(r+1)} < M(r,f)^{1/4}, \tag{2.10}$$

slightly stronger than (2.8) holds and so that

$$M(r,f)^{\delta/4n} > 2. \tag{2.11}$$

We can now successively use the values $r = r_0, r_1, r_2, \ldots, r_s, \ldots$ where

$$r \leqslant r_s < r_{s+1} = r_s + M(r_s,f)^{-\delta} < r + 1 - \frac{1}{2^{s+1}} \tag{2.12}$$

and

$$M(r_{s+1},f)^\delta > 2M(r_s,f)^\delta \geqslant 2^{s+1}M(r,f)^\delta. \tag{2.13}$$

We get these properties inductively from (2.9) by substitution. Thus

$$r_1 = r + M(r,f)^{-\delta} < r + \tfrac{1}{2},$$
$$M(r_1,f) > M(r,f)M(r,f)^{1/4n} > 2^{1/\delta}M(r,f)$$

which proves (2.12) and (2.13) for $s = 1$. Now assume (2.12) and (2.13) hold for s then

$$r_{s+1} = r_s + M(r_s, f)^{-\delta} < r+1-\frac{1}{2^s} + (2^{s+1}M(r,f)^{\delta})^{-1}$$

$$< r+1-\frac{1}{2^s}+\frac{1}{2^{s+1}} = r+1-\frac{1}{2^{s+1}}$$

and, since r_s satisfies (2.8),

$$M(r_{s+1}, f) > M(r_s, f)M(r_s, f)^{1/4n} \geqslant M(r_s, f)M(r, f)^{1/4n}$$
$$> 2^{1/\delta}M(r_s, f) \geqslant 2^{(s+1)/\delta}M(r, f)$$

which completes the proof of (2.12) and (2.13). However (2.13) implies that

$$M(r+1, f) > M(r_s, f) \geqslant 2^s M(r, f)$$

for all s which is impossible. In other words (2.10) cannot hold for any r large enough to satisfy (2.11).

2.14. Lemma. *If $\langle f\rangle$ satisfies the hypothesis of Theorem 2.1 and is of minimal exponential type then f is a polynomial.*

Proof. We have $\log M(r, f)/r \to 0$. So there exists a sequence $r_s \to \infty$ with

$$\log M(r_s, f)/r_s > \log M(r_s+\varrho, f)/(r_s+\varrho), \qquad \varrho > 0.$$

In other words

(2.15) $$M(r_s+\varrho, f) < M(r_s, f)M(r_s, f)^{\varrho/r_s}.$$

If we have chosen r_s so large that $M(r_s, f) < e^{\varepsilon r}$ for $r \geqslant r_s$ then (2.15) becomes

(2.16) $$M(r_s+\varrho, f) < M(r_s, f)e^{\varepsilon\varrho}$$

and correspondingly

(2.17) $$M(r_s+\varrho, f^k) < M(r_s, f)^k e^{k\varepsilon\varrho}.$$

Applying Cauchy's inequality with $\varrho = j/k\varepsilon$ we get

(2.18) $$M(r_s, D^j(f^k)) < M(r_s+\varrho, f^k)/\varrho^j < c\varepsilon^j M(r_s, f)^k.$$

If we substitute (2.18) in (2.4) then the lowest order derivative of f^n dominates for $r = r_s$ and ε sufficiently small. Thus, if $L_n = a_{nm}D^m + a_{nm+1}D^{m+1}+\ldots$, $a_{nm} \neq 0$, we get

(2.19) $$M(r_s, L_n(f^n)) > \frac{1}{2}|a_{nm}|M(r_s, D^m(f^n)) > cM(r_s, f^n)/r_s^m = cM(r_s, f)^n/r_s^m.$$

For the right side of (2.5) we get

(2.20) $$M\Big(r_s, \sum_{i=1}^{n-1} L_i(f^i)\Big) < cM(r_s, f^{n-1}) = cM(r_s, f)^{n-1}.$$

Comparing (2.19) and (2.20) we get

(2.21) $$M(r_s, f) < cr_s^m$$

for a sequence $r_s \to \infty$ so that δ is a polynomial by Liouville's Theorem.

We may thus assume from now on that f is of finite but nonminimial exponential type. We can therefore consider its Borel transform $F(w)$,

$$F(w) = \sum \frac{a_n}{w^{n+1}} \quad \text{where} \quad f(z) = \sum a_n z^n.$$

Here $F(w)$ is analytic in the complement of a bounded domain $\mathscr{D}$ whose convex hull, $\operatorname{conv}\mathscr{D} = \mathscr{C}$ determines and is determined by the support function

$$|w| = h(-\theta, f) \quad \text{of} \quad \mathscr{C}$$

where

$$h(\theta, f) = \limsup \log|f(re^{i\theta})|/r.$$

Clearly $h(\theta, f^k) = kh(\theta, f)$ and thus, if $F_k(w)$ denotes the Borel transform of f^k with corresponding domain $\mathscr{D}_k$ and $\mathscr{C}_k$ then $\mathscr{C}_k = k\mathscr{C}$.

If we take the Borel transform of (2.4) we get

(2.21′) $$P_n(w)F_n(w) = P_{n-1}(w)F_{n-1}(w)+\ldots+P_1(w)F_1(w),$$

where $P_i(D) = L_i$. Thus the singularities of $F_n(w)$ in $n\mathscr{C}\setminus(n-1)\mathscr{C}$ are poles located at the zeros of $P_n(w)$. This implies that $n\mathscr{C}$ and hence $\mathscr{C}$ is polygonal.

2.22. Lemma. *If $\langle f\rangle$ satisfies the hypothesis of Theorem 2.1 then $f = f_1+f_1^*$ where f_1 is an exponential polynomial and $\mathscr{C}_1^*$, the convex hull of the complement of the domain of analyticity of $F_1^*(w)$ (the Borel transform of f_1^*), is a proper subset of $\mathscr{C}$, containing none of the vertices of $\mathscr{C}$ except possibly* 0.

Proof. Let $\lambda_1, \ldots, \lambda_k$ be the non-zero extreme points of $\mathscr{C}$. Then, by our remarks above the points $n\lambda_1, \ldots, n\lambda_k$ are poles of $F_n(w)$ and

(2.23) $$f^n(z) = P_{n1}(z)e^{n\lambda_1 z}+\ldots+P_{nk}(z)e^{n\lambda_k z}+g_n(z),$$

where $P_{ni}(z)$ are polynomials and the Borel transform $G_n(w)$ of $g_n(z)$ is regular at the points $n\lambda_1, \ldots, n\lambda_k$ so that $\mathscr{C}(G_n)$ is a proper subset of $n\mathscr{C}$ containing none of the non-zero vertices of $n\mathscr{C}$. Analogously we get

(2.24) $$f^{n+1}(z) = P_{n+1,1}(z)e^{(n+1)\lambda_1 z}+\ldots+P_{n+1,k}(z)e^{(n+1)\lambda_k z}+g_{n+1}(z).$$

Dividing f^{n+1} by f^n we get

$$f(z) = q(z)e^{\lambda_j z}+\varphi_j(z); \quad j = 1, 2, \ldots, k \tag{2.25}$$

where $q_j(z)$ is rational and the Borel transform $\Phi_j(w)$ of $\varphi_j(z)$ is analytic at λ_j. Raising (2.25) to the nth power and comparing with (2.23) we get $q_j^n = P_{nj}$ so that q_j is a polynomial and λ_j is a pole of $F(w)$. Thus

$$F(w) = F_1(w)+F_1^*(w)$$

where F_1 is rational with poles $\lambda_1, \ldots, \lambda_k$ and F_1^* is regular outside $\mathscr{C}$ and at $\lambda_1, \ldots, \lambda_k$. Inverting the Borel transform we get the lemma.

If $f_1^* = 0$ we are finished. If not there exists an operator $L_1' = (D-\lambda_1)^{m_1}\ldots(D-\lambda_k)^{m_k} \in \mathscr{L}^*$ so that $L_1'f_1 = 0$ and hence $L_1'f = L_1'f_1^* \in \langle f\rangle$. Thus, by Lemma 2.22 we have either $L_1'f_1^* = 0$ and we are finished or

$$L_1'f_1^* = g_2+g_2^* \tag{2.26}$$

where g_2 is an exponential polynomial and $\mathscr{C}(G_2^*)$ is a subset of $\mathscr{C}(F_1^*)$ containing none of its non-zero extreme points. Inverting L_1' we get

$$f_1^* = f_2+f_2^* \tag{2.27}$$

where f_2 is an exponential polynomial with $\mathscr{C}(F_2) = \mathscr{C}(F_1^*)$ and $\mathscr{C}(F_2^*) \subset \mathscr{C}(F_1^*)$ so that $\mathscr{C}(F_2^*)$ contains none of the non-zero extreme points of $\mathscr{C}(F_1^*)$.

Continuing this process we get $f_i^* = f_{i+1}+f_{i+1}^*$ where F_{i+1} is an exponential polynomial and $\mathscr{C}(F_{i+1}^*) \subset \mathscr{C}(F_i^*)$ so that $\mathscr{C}(F_{i+1}^*)$ contains none of the non-zero extreme points of $\mathscr{C}(F_i^*) = \mathscr{C}(F_{i+1})$. If this process ends in a finite number of steps we are finished. If not then $F(w)$ has an infinite number of poles. Let $\mathscr{D}^*$ be the complement of the domain of meromorphy of F and $\mathscr{C}^* = \operatorname{conv}\mathscr{D}^*$.

2.28. Lemma. *Every non-zero extreme point of $\mathscr{C}^*$ is a limit point of poles of F. If $\mathscr{C}^* = \{0\}$ then, obviously, 0 is a limit point of poles of F.*

Proof. Let $p \neq 0$ be an extreme point of $\mathscr{C}^*$ which is not a limit point of poles of F. Let l be a line of support of $\mathscr{C}^*$ through p. Then on one side of l there is only a finite number of poles of F. So there exists a polynomial $P(w)$ so that $P(w)F(w)$ is analytic on one side of l. The corresponding function $g = P(D)f \in \langle f\rangle$. Thus by Lemma 2.22 the point p is a pole of $G(w) = P(w)F(w)$ and hence a pole of F and hence $p \notin \mathscr{C}^*$.

We can now complete the proof of Theorem 2.1. Let l be a supporting line of $\mathscr{C}^*$ at an extreme point p of $\mathscr{C}^*$ so that F is meromorphic with infinitely many poles on one side, $\mathscr{S}$, of l. Let λ be an extreme point of $\mathscr{C}$ in $\mathscr{S}$ at which $\mathscr{C}$ has a line of support parallel to l. Then $F_n(w)$ has the point $(n-1)\lambda+p$ as a limit point of poles which are exterior to $(n-1)\mathscr{C}$. In other words $F_n(w)$ has infinitely many poles in a domain in which $F, F_2, \ldots, F_{n-1}$ are analytic. This contradicts equation (2.4).

3. Differential rings of meromorphic functions. It is no longer true that rings of meromorphic functions which are finite dimensional over $\mathscr{L}$ are necessarily 0-dimensional. In fact a non-entire meromorphic function does not satisfy a nontrivial linear differential equation with constant coefficients. Since the proper setting for these rings will turn out to be compact Riemman surfaces we shall give first examples in these terms.

3.1. Definition. Let $\mathscr{M}$ be a compact Riemann surface. By a *derivation*, D, we mean an operator which acts locally like a differentiation operator on the analytic functions of $\mathscr{M}$. If g = genus $\mathscr{M} \neq 1$ then D will have one singular point (where its vector field vanishes) p_0 (point at ∞). The operator algebra $\mathscr{L} = C[D]$ is now defined in terms of such a D.

3.2. Theorem. *Let $\mathscr{R}$ be a ring of functions meromorphic on the compact Riemann surface $\mathscr{M}$ whose poles other than p_0 are subsets of $\{p_1, \ldots, p_n\}$ then $\mathscr{R}$ is n-dimensional over $\mathscr{L}$.*

Proof. The dimension of $\mathscr{R}$ over $\mathscr{L}$ is $\geqslant n$ since there exist n functions $f_1, f_2, \ldots, f_n \in \mathscr{R}$ so that f_j is regular except for a pole at p_j. Since applying a non-zero operator $L_j \in \mathscr{L}^*$ cannot cancel a pole, except at p_0, it follows that $\sum L_j f_j$ has poles a p_j whenever $L_j \neq 0$.

On the other hand, given two functions $f, g \in \mathscr{R}$ with poles p_j, say in terms of a local coordinate z, $p_j = z_j$, $D = \dfrac{d}{dz}$

$$f = \frac{a_m}{(z-z_j)^m}+\ldots+\frac{a_1}{z-z_j}+\ldots; \qquad g = \frac{b_n}{(z-z_j)^n}+\ldots+\frac{b_1}{z-z_j}+\ldots$$

with $m \leqslant n$ we get

$$k = a_m(-m)(-m-1)\ldots(-m-n+1)g-D^{n-m}f = \frac{c_{n-1}}{(z-z_j)^{n-1}}+\ldots$$

repeating this process we find operators $L_1, L_2 \in \mathscr{L}^*$ so that $L_1 f+L_2 g$ is regular at p_j.

We can now prove the theorem by induction on n. If $n = 0$ then $f \in \mathscr{R}$ is regular except for a possible pole at p_0. Since D is singular at p_0 we get $D^k f$ regular on $\mathscr{M}$ for some k. Thus $D^k f$ = const and $D^{k+1} f = 0$. Hence $\mathscr{R}$ is 0-dimensional over $\mathscr{L}$.

Now assume the theorem true for $n-1$. Given $n+1$ functions $f_1, \ldots, f_{n+1} \in \mathscr{R}$ they are either all regular at p_n, in which case they are linearly dependent over $\mathscr{L}$ by the induction hypothesis, or without loss of generality we may assume that f_{n+1} has a pole at p_n. In the latter case there exist operators $L_1, \ldots, L_n \in \mathscr{L}^*$ and $L_1', \ldots, L_n' \in \mathscr{L}$ so that $g_j = L_j f_j+L_j' f_{n+1}$ is regular at p_n for $j = 1, 2, \ldots, n$. Thus the g_j are linearly dependent over $\mathscr{L}$ by the induction hypothesis and this dependence yields a dependence of $f_1, \ldots, f_{n+1}$ over $\mathscr{L}$.

From now on we shall again think of functions meromorphic in the Gauss plane with D the ordinary differentiation operator.

3.3. Definition. Let $\mathscr{R}$ be ring of meromorphic functions. For each $f\in\mathscr{R}$ let $\mathscr{P}(f)$ denote the set of poles of f (without regard to multiplicities) and let $\mathscr{P}(\mathscr{R}) = \bigcup_{f\in\mathscr{R}}\mathscr{P}(f)$. A set $\mathscr{P}_0$ is a *minimal pole set* if $\mathscr{P}_0 = \mathscr{P}(f_0)$ for some $f_0\in\mathscr{R}$ and $\mathscr{P}(g)\subsetneqq\mathscr{P}_0$, $g\in\mathscr{R}$ implies that g is entire ($\mathscr{P}(g) = \emptyset$).

3.4. Lemma. *Let $\mathscr{R}$ be a ring of mermorphic functions which is n-dimensional over $\mathscr{L}$. Then the sets $\mathscr{P}(f)$ satisfy the following strong descending chain condition. Given any set $\mathscr{S}$ and any sequence $f_1, f_2, \ldots \in\mathscr{R}$ so that the sequence $\mathscr{S}_k = \mathscr{S}\cap\mathscr{P}(f_k)$ satisfies $\mathscr{S}_1\supset\mathscr{S}_2\supset\ldots\mathscr{S}_k\supset\ldots$ then $\mathscr{S}_{k+1} = \mathscr{S}_k$ with at most $n+1$ exceptions. If $f_1, \ldots, f_n\in\mathscr{R}$ are linearly independent over $\mathscr{L}$ then*

$$\mathscr{P}(\mathscr{R}) = \mathscr{P}(f_1)\cup\ldots\cup\mathscr{P}(f_n).$$

In particular $\mathscr{P}(\mathscr{R})$ is denumerable.

Proof. Assume that the sequence $f_1, f_2, \ldots$ contains a subsequence $g_1, \ldots, g_{n+1}, g_{n+2}$ so that

$$\mathscr{S}\cap\mathscr{P}(g_1)\supsetneqq\mathscr{S}\cap\mathscr{P}(g_2)\supsetneqq\ldots\supsetneqq\mathscr{S}\cap\mathscr{P}(g_{n+1})\neq\emptyset$$

then there exist points $p_1, \ldots, p_{n+1}$ so that p_j is a pole of $g_1, \ldots, g_j$ but not of $g_{j+1}, \ldots, g_{n+1}$. Now assume $L_1g_1+\ldots+L_{n+1}g_{n+1} = 0$ with $L_j\in\mathscr{L}$ and k the least index for which $L_k\neq 0$. Then L_kg_k has a pole at P_k while $L_{k+1}g_{k+1}+\ldots+L_{n+1}g_{n+1}$ does not, contrary to hypothesis.

If $f_1, \ldots, f_n$ are linearly independent over $\mathscr{L}$ in $\mathscr{R}$ then for every $f\in\mathscr{R}$ there exists an $L\in\mathscr{L}^*$ so that $Lf = L_1f_1+\ldots+L_nf_n$, $L_j\in\mathscr{L}$ and hence

$$\mathscr{P}(f) = \mathscr{P}(Lf)\subset\mathscr{P}(L_1f_1)\cup\ldots\cup\mathscr{P}(L_nf_n) = \mathscr{P}(f_1)\cup\ldots\cup\mathscr{P}(f_n).$$

3.5. Lemma. *Let f be meromorphic with $0<|\mathscr{P}(f)|<\infty$ and so that the differential ring $\langle f\rangle = \mathscr{R}$ is finite dimensional over $\mathscr{L}$. Then f is rational.*

Proof. We distinguish two cases.

Case I. $\mathscr{R}$ contains nonconstant entire functions. By Theorem 2.1 such a function $g(z)\in\mathscr{R}$ is an exponential polynomial and hence there exists an $L\in\mathscr{L}$ so that $Lg = z$ or so that $Lg = e^{\lambda z}$ for some $\lambda\neq 0$.

Subcase (i). $z\in\mathscr{R}$. Then $(z-p_1)^{m_1}\ldots(z-p_n)^{m_n}f = h(z)$ is entire where m_j is the multiplicity of the pole p_j of f. Thus by Theorem 2.1, $h(z)$ is an exponential polynomial. If $h(z)$ is a polynomial we are finished. If not, then $\mathscr{R}$ contains a function

$$\varphi(z) = q_1(z)e^{\mu_1 z}+\ldots+q_l(z)e^{\mu_l z},$$

where the q_j are nonintegral rational functions and not all $\mu_j = 0$. However $\langle\varphi\rangle$ is not finite dimensional over $\mathscr{L}$. To see this, let $|\mu_1|\leqslant|\mu_2|\leqslant\ldots\leqslant|\mu_l|$.

Then for any $L_N\in\mathscr{L}^*$ we have

$$L_N(\varphi^N) = q_l^* e^{N\mu_l z}+\ldots$$

where q_l^* is a nonvanishing rational function and the terms not written have growth rate less than $e^{(1-\delta)N|\mu_l|r}$ along the ray $\arg z = -\arg\mu_l$ for some $\delta>0$. Thus

$$|L_N(\varphi^N)(re^{-i\arg\mu_l})|>cr^{-c}e^{N|\mu_l|r}$$

for all large r, while for all $L_1, \ldots, L_{N-1}\in\mathscr{L}$ we have

$$\left|\left(L_1(\varphi)+\ldots+L_{N-1}(\varphi^{N-1})\right)(re^{-i\arg\mu_l})\right|<cr^c e^{(N-1)|\mu_l|r}.$$

Hence $\varphi, \varphi^2, \ldots, \varphi^N$ are linearly independent over $\mathscr{L}$ for all N.

Subcase (ii). $e^{\lambda z}\in\mathscr{R}$, $\lambda\neq 0$. In this case

$$h(z) = (e^{\lambda z}-e^{\lambda p_1})^{m_1}\ldots(e^{\lambda z}-e^{\lambda p_n})^{m_n}f(z)$$

is entire and hence an exponential polynomial. In other words

$$f(z) = \frac{P_1(z)e^{\mu_1 z}+\ldots+P_k(z)e^{\mu_k z}}{(e^{\lambda z}-e^{\lambda p_1})^{m_1}\ldots(e^{\lambda z}-e^{\lambda p_n})^{m_n}}.$$

Since $f(z)$ has only a finite number of poles the numerator, $h(z)$, must vanish at all but a finite number of the points of the arithmetic progressions $z_j = p_j+2l\pi i/\lambda$; $l = 0, \pm 1, \pm 2, \ldots$. However, if an exponential polynomial vanishes at almost all points of such a progression then it vanishes at all of them. In other words f must itself be entire, contrary to hypothesis.

Case II. $\mathscr{R}$ contains no nonconstant entire function. Let $\mathscr{P}_1$ be a minimal set of poles with $p_1\in\mathscr{P}_1$ and choose $g\in\mathscr{R}$ with $\mathscr{P}(g) = \mathscr{P}_1$ having a pole of minimal order at p_1 among all such functions. Then for each $h\in\mathscr{R}$ with $\mathscr{P}(h) = \mathscr{P}_1$ there exists an $L\in\mathscr{L}^*$ so that $h-Lg$ has a pole of lower order than g at p_1 and hence $h-Lg = \text{const}$. In particular the function g^2 satisfies an equation

$$g^2 = Lg+c, \quad L\in\mathscr{L}^*, \quad c\in C. \tag{3.6}$$

Now a meromorphic function g can satisfy (3.6) only if it is rational. This can be seen in many ways. For example, write

$$g(z) = R(z)+g_1(z)$$

where $R(z)$ is the sum of the principal parts of g at its poles and $g_1(z)$ is entire. Substitution in (3.6) yields

$$g_1^2 = LR+Lg_1+c-R^2-2Rg_1,$$

or

$$M(r, g_1)^2<c+c\max_{0\leqslant s\leqslant S}M(r, g_1^{(s)})$$

and this inequality is possible only if $g_1 = \text{const}$ by the same proof as that given for Lemma 2.2.

Thus $\mathscr{R}$ contains the nonconstant rational function g. Now either the poles of f are a subset of $\mathscr{P}_1$ in which case f is of the form $Lg+c$ and hence rational, or f has poles $q_1, \ldots, q_s$ which are not contained in $\mathscr{P}_1$. In the latter case the function

$$\big(g(z)-g(q_1)\big)^{m_1} \ldots \big(g(z)-g(q_s)\big)^{m_s} f$$

will have poles only in $\mathscr{P}_1$ and hence be rational.

3.7. THEOREM. *If $\mathscr{R}$ is a ring of meromorphic functions which is finite dimensional over $\mathscr{L}$ and $\mathscr{R}$ contains a function f with $0 < |\mathscr{P}(f)| < \infty$ then $\mathscr{R}$ is a ring of rational functions and $\mathscr{P}(\mathscr{R})$ is finite.*

Proof. By Lemmas 3.4 and 3.5 we need only prove that $\mathscr{P}(g)$ is finite for every $g \in \mathscr{R}$. Because in that case all g with $\mathscr{P}(g) \neq \emptyset$ are rational and all entire g are polynomials as shown in the proof of Lemma 3.5.

Now assume that there exists a $g \in \mathscr{R}$ with infinite $\mathscr{P}(g)$. Consider the set $\mathscr{S} = \mathscr{P}(g) \setminus \mathscr{P}(f) = \{p_1, p_2, \ldots\}$. Then the functions

$$g_0 = g, \quad g_1(z) = \big(f(z)-f(p_1)\big)^{m_1} g(z), \quad \ldots, \quad g_k(z) = \big(f(z)-f(p_k)\big)^{m_k} g_{k-1}(z)$$

have infinitely descending sets $\mathscr{S}_k = \mathscr{S} \cap \mathscr{P}(g_k)$ contrary to Lemma 3.4.

3.8. LEMMA. *Let $\mathscr{R}$ be a differential ring of meromorphic functions which is finite dimensional over $\mathscr{L}$. If $\mathscr{P}(\mathscr{R})$ is infinite and $\mathscr{R}$ contains non-constant entire functions then $\mathscr{R}$ is a ring of functions rational in $e^{\lambda z}$ for some $\lambda \neq 0$ and $\mathscr{P}(\mathscr{R})$ contains a finite number of points in a period strip of $e^{\lambda z}$.*

Proof. According to Theorem 3.7 we know that $\mathscr{P}(f)$ is infinite for every nonentire $f \in \mathscr{R}$. Let $\mathscr{P}(f) = \{p_1, p_2, \ldots\}$ be a minimal pole set. Since $\mathscr{R}$ contains a nonconstant exponential polynomial we have either $z \in \mathscr{R}$ or $e^{\mu z} \in \mathscr{R}$ for some $\mu \neq 0$. However, in the first case we would have $(z-p_1)^{m_1} f(z) \in \mathscr{R}$ as a function with a smaller infinite pole set than f, contrary to hypothesis. In the second case we have

$$g(z) = (e^{\mu z} - e^{\mu p_1})^{m_1} f(z) \in \mathscr{R}$$

and since $\mathscr{P}(g) \supsetneqq \mathscr{P}(f)$ it follows that g is entire and that the $\mathscr{P}(f) \subset \{p_1 + 2m\pi i/\mu \mid m = 0, \pm 1, \ldots\}$. Since the same argument would be used for any $e^{\nu z} \in \mathscr{R}$ it follows that ν and μ must be commensurable. If $\nu = a\mu/b$ where a, b are relatively prime integers then we have

$$p_2 - p_1 = 2m\pi i/\mu = 2m'\pi i/(a\mu/b)$$

so that $m = bm'/a$ and hence $b \leqslant |m|$. Let B be the maximal denominator for any such ν and set $\lambda = \mu/B$. Then every ν for which $e^{\nu z} \in \mathscr{R}$ is an integral multiple of λ and all entire functions of $\mathscr{R}$ have the form

$$h(z) = p_1(z) e^{n_1 \lambda z} + p_2(z) e^{n_2 \lambda z} + \ldots + p_k(z) e^{n_k \lambda z},$$

where the p_i are polynomials and the n_i are integers. However, if $\deg p_i > 0$ for any i then $h(z)$ is not periodic and the zeros of $h(z)-h(p_1)$ will not contain all of $\mathscr{P}(f)$ contrary to the fact that $\big(h(z)-h(p_1)\big)^m f(z)$ is entire. Thus all entire functions of $\mathscr{R}$ are generalized polynomials (sums of integral powers) of $e^{\lambda z}$.

Finally let φ be any non-entire function of $\mathscr{R}$ and let $\mathscr{P}(\varphi) = \{q_1, q_2, \ldots\}$. Then the functions

$$\varphi, \quad (e^{\mu z} - e^{\mu q_1})^{n_1} \varphi, \quad (e^{\mu a} - e^{\mu q_1})^{n_1} (e^{\mu z} - e^{\mu q_2})^{\mu_2} \varphi, \quad \ldots \tag{3.9}$$

have decreasing sets of poles. Hence by Lemma 3.4 the functions in (3.9) will be entire from a certain point on. Thus

$$(e^{\mu z} - e^{\mu q_1})^{n_1} \ldots (e^{\mu z} - e^{\mu q_k})^{n_k} \varphi = e^{-n\lambda z} P(e^{\lambda z})$$

for some polynomial P, and φ is a rational function of $e^{\lambda z}$.

Finally let $f_1, \ldots, f_n$ be a maximal set of functions independent over $\mathscr{L}$ in $\mathscr{R}$. Since each of them is rational in $e^{\lambda z}$ it has only a finite number of poles in a period strip of $e^{\lambda z}$. By Lemma 3.4 we have $\mathscr{P}(\mathscr{R}) = \mathscr{P}(f_1) \cup \cup \ldots \cup \mathscr{P}(f_n)$ containing only a finite number of poles in each period strip.

As a result of Lemma 3.8 we can restrict attention from now on to rings containing no non-constant entire functions.

3.10. LEMMA. *Let $\mathscr{R}$ be a ring of meromorphic functions finite dimensional over $\mathscr{L}$ and let $\mathscr{P}_0$ be a minimal pole set consisting of more than one point. If $\mathscr{P}_0 = \mathscr{P}(f)$, $f \in \mathscr{R}$ then f is periodic with periods $p_i - p_j$ for all $p_i, p_j \in \mathscr{P}_0$. In particular every infinite minimal pole set consists of a one-dimensional or two-dimensional lattice of points.*

Proof. Let $\mathscr{P}_0 = \{p_1, p_2, \ldots\}$ and let $f \in \mathscr{R}$ with $\mathscr{P}(f) = \mathscr{P}_0$ have a pole of minimal order at p_1 among all such functions. For every $g \in \mathscr{R}$ with $\mathscr{P}(g) = \mathscr{P}_0$ there exists an $L \in \mathscr{L}^*$ so that $g - Lf$ has a pole of lower order than f at p_1 and hence is regular at p_1 which means $g - Lf$ is entire. Since we assumed that $\mathscr{R}$ contains no non-constant entire functions we get

$$g = Lf + c, \quad L \in \mathscr{L}^*, \quad c \in C \tag{3.11}$$

for all $g \in \mathscr{R}$ with $\mathscr{P}(g) = \mathscr{P}_0$. In particular we have

$$f^2 = (a_0 D^k + a_1 D^{k-1} + \ldots + a_k) f + c, \quad a_0 \neq 0. \tag{3.12}$$

Let

$$f = \sum_{l=-m_1}^{\infty} c_{1l} (z-p_1)^l = \sum_{l=-m_2}^{\infty} c_{2l} (z-p_2)^l; \quad c_{1,-m_1} \neq 0, \quad c_{2,-m_2} \neq 0.$$

We wish to prove that $m_1 = m_2$ and $c_{1l} = c_{2l}$ for all l. Substituting in (3.12) we get at p_1 that $k = m$, and

$$c_{1,-m}^2 = a_0(-m_1)(-m_1-1) \ldots (-2m_1+1) c_{1,-m}.$$

Similarly at p_2 we get $k = m_2$ and

$$c_{2,-m_2}^2 = a_0(-m_2)(-m_2-1)\dots(-2m_2+1)c_{1,-m_2}.$$

Thus $m_1 = m_2 = k$ and

$$c_{1,-m_1} = c_{2,-m_2} = (-1)^k \frac{(2k-1)!}{(k-1)!} a_0.$$

Now assume that $c_{1j} = c_{2j} = c_j$ for all $j < l$ and compare the coefficients of $(z-p_i)^{l-k}$ on both sides of (3.12) for $i = 1, 2$ to get

$$2c_{il}c_{-k} + \dots = a_0 l(l-1)\dots(l-k+1)c_{ik} + \dots$$

where the terms not written are independent of i. Thus

$$a_0 c_{il}[2(-k)(-k-1)\dots(-2k+1) - l\dots(l-k+1)]$$

is independent of i which means that c_{il} is independent of i unless the term in square brackets vanishes. The latter happens only if k is even and $l = 2k$. In that case we look at the equation

$$f^3 = (b_0 D^{2k} + \dots + b_{2k})f + c' \tag{3.13}$$

and by comparing coefficients of $(z-p_i)^{-3k}$ we get

$$c_{-k}^3 = b_0(-k)(-k-1)\dots(-3k+1)c_{-k}$$

so that

$$b_0 = \frac{(3k-1)!}{(k-1)!} c_{-k}^2.$$

Now comparing the constant terms in (3.13) we get

$$3c_{-k}^2 c_{i,2k} + \dots = b_0(2k)!\, c_{i,2k} + \dots$$

where the terms not written are independent of i. Thus

$$c_{-k}^2 c_{i,2k}\left[3 - \frac{(2k)!\,(3k-1)!}{(k-1)!}\right]$$

is independent of i and hence $c_{1,2k} = c_{2,2k}$. Hence in every case $f(z-p_1) = f(z-p_2)$ as was to be proved. Finally all $g \in \mathscr{R}$ with $\mathscr{P}(g) = \mathscr{P}_0$ are of the form $Lf + c$ and hence have the same periodicity as f.

3.14. Lemma. *Under the hypotheses of Lemma 3.10 all functions in $\mathscr{R}$ are of the form $(Lf+c)/P(f)$, $L \in \mathscr{L}$, $c \in \mathbf{C}$, P a polynomial. In particular all functions of $\mathscr{R}$ have the same periodicities as f.*

Proof. Let $g \in \mathscr{R}$. If $\mathscr{P}(g) \subset \mathscr{P}_0$ then we proved that $g = Lf + c$. If not let $\mathscr{S} = \mathscr{P}(g) \setminus \mathscr{P}_0 = \{q_1, q_2, \dots\}$ and consider the functions

$$g_0 = g, \quad g_1(z) = \big(f(z) - f(q_1)\big)^{m_1} g, \quad \dots, \quad g_k(z) = \big(f(z) - f(q_k)\big)^{m_k} g_{k-1}, \quad \dots$$

By Lemma 3.4 we must have $\mathscr{S}_k = \mathscr{S} \cap \mathscr{P}(g_k) = \emptyset$ for some k so that $\mathscr{P}(g_k) \subset \mathscr{P}_0$ and hence

$$P_k(f)g = g_k = Lf + c$$

as was to be proved.

3.15. Lemma. *If $\mathscr{R}$ satisfies the hypotheses of Lemma 3.10 and $\mathscr{P}_0$ is a one-dimensional lattice then there exists a $\lambda \neq 0$ such that the functions of $\mathscr{R}$ are rational functions of $e^{\lambda z}$.*

Proof. According to Lemma 3.14 it suffices to show that the special function f used in the proof of Lemma 3.10 has the desired property. Now f is simply periodic so that we can express f as a meromorphic function F of $u = e^{\lambda z}$ where $2\pi i/\lambda$ is the period of f. Now (3.12) becomes (with $L = P(D)$)

$$F^2(u) = \left(P\left(\frac{d}{dz}\right)F\right)(u) + c = \left(P\left(\lambda u \frac{d}{du}\right)F\right)(u) + c, \tag{3.16}$$

where F is meromorphic with a single pole at $u = e^{\lambda p_1}$. This equation is analogous to (3.6) and leads to the conclusion that F is rational.

We can now sum up the results of this section.

3.17. Theorem. *If $\mathscr{R}$ is a differential ring of meromorphic functions which is finite dimensional over $\mathscr{L}$ but $\mathscr{R}$ is not a ring of entire functions; then $\mathscr{R}$ is a ring of meromorphic functions on a compact Riemann surface $\mathscr{M}$, where $\mathscr{P}(\mathscr{R})$ is finite on that surface with the following three possible cases*

(i) *$\mathscr{P}(\mathscr{R})$ is finite and $\mathscr{R}$ is a ring of rational functions; $\mathscr{M}$ is the Riemann sphere.*

(ii) *$\mathscr{P}(\mathscr{R})$ is the union of a finite number of one-dimensional lattices which are translates of one another and $\mathscr{R}$ is a ring of functions rational in $e^{\lambda z}$. Here $\mathscr{M}$ is the period strip with its boundary lines identified and with two distinct limit points at the two ends.*

(iii) *$\mathscr{P}(\mathscr{R})$ is the union of a finite number of two-dimensional lattices which are translates of one another and $\mathscr{R}$ is a ring of elliptic functions. Here $\mathscr{M}$ is the torus obtained by the usual identification of the edges of a period parallelogram.*

Proof. Case (i) is the content of Theorem 3.7. Case (ii) combines the results of Lemma 3.8 in case $\mathscr{R}$ contains non-constant entire functions and of Lemma 3.15 otherwise. Case (iii) implies that the function f in Lemma 3.10 is elliptic and so by Lemma 3.14 all functions of $\mathscr{R}$ are elliptic with the same periods.

4. Concluding remarks. It would be interesting to investigate the questions raised here for rings of analytic functions of several variables. For examples one could consider differential rings of functions $f(z_1, z_2, \ldots, z_k)$ finite dimensional over $\mathscr{L} = C[D_1, \ldots, D_k]$. It is clear that even for rings of entire functions the situation is more complicated since the product of solutions of linear differential equations with constant coefficients need not satisfy such an equation.

Reference

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Sur les fonctions q-additives ou q-multiplicatives

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Nous considérons ici des fonctions réelles ou complexes définies sur l'ensemble N des entiers $\geqslant 0$.

q étant un entier >1, nous disons que *la fonction f est q-additive si, quel que soit $r \geqslant 1$, on a*

$$f(aq^r+b) = f(aq^r)+f(b) \quad \textit{pour} \quad 1 \leqslant a \leqslant q-1 \ \textit{et} \ 0 \leqslant b < q^r \text{ (1)}.$$

Ceci entraîne évidemment $f(0) = 0$. L'égalité a donc lieu aussi pour $a = 0$.

Un exemple simple de fonction q-additive est fourni par la fonction qui à l'entier $n \geqslant 0$ fait correspondre la somme des chiffres dans la représentation de n dans le système de numération à base q.

Nous disons que *f est q-multiplicative si l'on a $f(0) = 1$ et, quel que soit $r \geqslant 1$,*

$$(1) \qquad f(aq^r+b) = f(aq^r)f(b) \quad \textit{pour} \quad 1 \leqslant a \leqslant q-1 \ \textit{et} \ 0 \leqslant b < q^r.$$

Cette égalité a évidemment lieu aussi pour $a = 0$.

Une fonction q-additive, ou q-multiplicative, est complètement déterminée par ses valeurs pour tous les entiers de la forme aq^r, où $r \geqslant 0$ et $1 \leqslant a \leqslant q-1$, et celles-ci peuvent être égales à des nombres donnés arbitrairement.

En effet, en utilisant le système de numération à base q, on peut écrire, de façon unique,

$$n = \sum_{r=0}^{+\infty} e_r(n)q^r, \quad \text{avec} \quad 0 \leqslant e_r(n) \leqslant q-1 \text{ pour tout } r \geqslant 0.$$

On a d'ailleurs $e_r(n) = 0$ pour $r > \log n/\log q$.

(1) Cette notion a été introduite par A. O. Gelfond (*Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arithmetica, 13, 1968, pp. 259–265).

Gelfond dit que f est "additive dans le système à base q".