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# Monte Carlo Studies of the $p$-Spin Models on Scale-Free Hypernetworks

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Results of Monte Carlo simulations of $p$-spin models on scale-free hypernetworks are presented. The hypernetworks are obtained using the preferential attachment algorithm, the spins are located in the nodes and the hyperedges connecting $p$ nodes correspond to non-zero ferromagnetic interactions involving $p$ spins. Such models show high degeneracy of the ground state: apart from the ferromagnetic state, depending on the parameters of the preferential attachment algorithm leading to different topologies of the obtained hypernetworks, there are several or even infinitely many disordered (glassy) states with the same energy. For various network topologies quantities such as the specific heat or magnetic susceptibility show maxima as functions of the temperature, which suggests the occurrence of the glassy or ferromagnetic phase transition. The models under study may serve as a starting point for modelling various forms of cooperation in social and economic sciences involving many-body rather than two-body interactions.

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## 1. Introduction

The concept of hypernetworks generalizes that of networks: in networks pairs of nodes are connected by edges, while in hypernetworks groups of more than two nodes are connected by hyperedges [1]. As in the case of networks, the topology of connections in hypernetworks may be complex: e.g., scale-free hypernetworks are characterized by distributions of hyperdegrees (number of hyperedges $d_i$ attached to a given edge $i$) obeying a power scaling law $P(d_i) \propto d_i^{-\alpha}$ [2]. The nodes connected by hyperedges may represent elements of different kind (e.g., users, resources and tag labels applied collaboratively by the users to the resources), but also similar units (e.g., co-authors of papers). The latter possibility leads to the idea of systems of interacting units (spins, agents, etc.) on complex hypernetworks, with each hyperedge corresponding to many-body interaction among units in nodes connected by it (rather than two-body interactions between all pairs of units in nodes connected by this hyperedge), which may be useful in modelling some forms of collaboration in social and economic sciences. Thus, it is interesting to investigate simple examples of such systems, namely $p$-spin models [3] on complex hypernetworks (with hyperedges representing $p$-spin exchange interactions), which could provide useful ideas for socio- and econophysics, as the study of the Ising model did for the social impact theory [4]. In particular, in this contribution ferromagnetic $p$-spin models with $p = 3$ on scale-free hypernetworks are investigated, and it is argued that they exhibit ferromagnetic or glassy phase transition with decreasing temperature.

## 2. Model

Scale-free hypernetworks can be constructed as evolving hypernetworks using algorithm [2] similar to the Barabási–Albert one applied to obtain scale-free networks [5]. The algorithm starts with $p$ nodes connected by one hyperlink. In each step of the construction $m$ new nodes are added to the hypernetwork ($m < p$) which are then connected by one hyperlink to $p - m$ randomly chosen existing nodes according to the following preferential attachment rule: the probability to add a new hyperlink to the existing node $i$ is proportional to the hyperdegree $d_i$ of this node, $p_i = d_i(\sum_j d_j)^{-1}$, where the summation runs over all existing nodes. The algorithm stops after a desired number $N$ of nodes in the hypernetwork is reached. The distribution of hyperdegrees in the resulting hypernetwork obeys a power scaling law, $P(d) \propto d^{-\alpha}$, with the exponent $\alpha$ dependent on $p$ and $m$; e.g., $\alpha = m + 2$ for $m = p - 1$ [2].

A general Hamiltonian for the $p$-spin model is

$$H = -\sum_{\{i_1,i_2,\ldots i_p\}} J_{i_1,i_2,\ldots i_p} s_{i_1} s_{i_2} \ldots s_{i_p}, \qquad (1)$$

with $N$ two-state spins, $s_i = \pm 1, i = 1, 2, \ldots N$ and the summation running over all $\binom{N}{p}$ groups of $p$ spins (each group is counted only once). The exchange integrals $J_{i_1,i_2,\ldots i_p}$ define the topology of connections and strength of interactions. For example, models with all-to-all as well as short-range $p$-spin interactions and $J_{i_1,i_2,\ldots i_p}$ picked randomly from a Gaussian distribution (quenched disorder) were used as models for spin glasses [6–9]. The ferromagnetic models with $p = 4$ were considered on three- and two-dimensional regular cubic and square lattices, respectively, with $J_{i_1,i_2,i_3,i_4} = J$ for spins located at the corners of each plaquette and $J_{i_1,i_2,i_3,i_4} = 0$ otherwise (the plaquette model); in the three-dimensional case ferromagnetic and glassy phase transitions (possibly first-order) as well as metastability were observed [10–14].

In this contribution the ferromagnetic $p$-spin models are considered on scale-free hypernetworks built using the above-mentioned preferential attachment algorithm, with the spins located at the nodes and $J_{i_1,i_2,\ldots i_p} = J > 0$ if the nodes $i_1, i_2, \ldots i_p$ are connected by a hyperlink, and $J_{i_1,i_2,\ldots i_p} = 0$ otherwise. For simplicity investigation is constrained to the case $p = 3$, with $m = 1$ or $m = 2$. It should be noted that for $p$ odd the models in general do not possess time-reversal symmetry since flipping all spins does not lead to a state with the same energy. Besides, for $p > 2$ the ground state is degenerate. It is easy to check that for $m = 1$ the spin configuration of the system in the ground state can be deduced from the spin configuration in any hyperlink. For example for $p = 3$, $m = 1$ there are four different spin configurations which minimize energy in one hyperlink (one with all spins up and three with one spin up and two spins down).

As a result, for each realization of the hypernetwork there are also four different ground states of the 3-spin model: the ferromagnetic state with all spins up and the magnetization $M = N^{-1}\sum_{i=1}^{N} s_i = 1$ and three glassy states with $|M| < 1$ (Fig. 1). In contrast, for $m > 1$ the ground state can be highly degenerate.

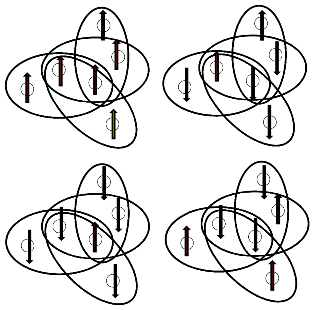

Fig. 1. Four degenerate ground states of a ferromagnetic 3-spin model on an exemplary hypernetwork with $m = 1$ and $N = 6$ nodes; ellipses surrounding groups of three nodes denote hyperedges.

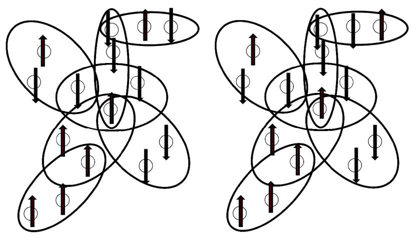

Fig. 2. Two exemplary degenerate ground states of a ferromagnetic 3-spin model on an exemplary hypernetwork with $m = 2$ and $N = 15$ nodes which differ by orientations of two spins in the uppermost hyperedge.

For example for $p = 3$, $m = 2$ there are always two different configurations of the two spins in the two "new" nodes which minimize energy in the hyperlink, depending on the orientation of the spin in the "old" node (Fig. 2). Thus the degeneracy of the ground state grows with $N$ as $2^{N/2}$ and almost all states with minimum energy are disordered, apart from the ferromagnetic one with $M = 1$. Hence in both above-mentioned cases the low-temperature phase can be either ferromagnetic or, more probably, disordered (glassy) and the magnetization is not a valid order parameter. Thus in order to detect the possible phase transition one has to rely on the presence of the maxima of the specific heat or the magnetic susceptibility as functions of the temperature, as in the case of the plaquette model [10–14].

## 3. Results

Below results of standard Monte Carlo (MC) simulations using Glauber dynamics of the two above-mentioned models are presented. The magnetic susceptibility is evaluated as $\chi_M(T) \propto \frac{\partial M}{\partial H} \approx [M(T, H = \Delta H) - M(T, H = 0)]/\Delta H$, with the field $\Delta H \to 0$ applied upwards and the magnetizations obtained from simulations with ferromagnetic initial conditions, or as $\chi_M \propto \beta\overline{(\langle M^2\rangle - \langle M\rangle^2)}$, where $\beta = T^{-1}$, the brackets denote time average and the bar denotes averaging over different realizations of the hypernetwork. Similarly, the specific heat is evaluated as $c \propto -\beta^2 \frac{\partial E}{\partial \beta}$, where $E$ is energy per hyperlink, or $c \propto \beta^2\overline{(\langle E^2\rangle - \langle E\rangle^2)}$. In the case of ferromagnetic initial conditions both sets of results for $\chi_M$, $c$ are usually in good agreement with each other; for random initial conditions the magnetic susceptibility evaluated from the fluctuations of magnetization is significantly lower than for the ferromagnetic initial conditions, though it shows qualitatively similar behaviour as a function of temperature, and the specific heat remains practically the same.

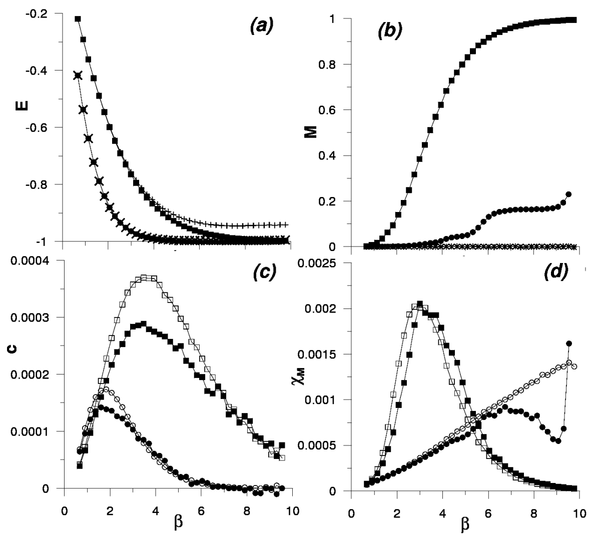


Fig. 3. Results of MC simulations of the 3-spin models on hypernetworks with $N = 10^4$ spins. (a) Energy per hyperlink $E$ vs. inverse temperature $\beta$ for $m = 1$ and ferromagnetic (■) and random (+) initial conditions, and for $m = 2$ and ferromagnetic (•) and random (×) initial conditions; (b) magnetization $M$ vs. $\beta$, symbols as in (a); (c) specific heat $c$ vs. $\beta$ evaluated as $c \propto -\beta^2 \frac{\partial E}{\partial \beta}$ (empty symbols) and $c \propto \beta^2\overline{(\langle E^2\rangle - \langle E\rangle^2)}$ (full symbols) for $m = 1$ (squares) and $m = 2$ (circles); (d) magnetic susceptibility $\chi_M$ vs. $\beta$, symbols as in (c) with $M$ instead of $E$.

For $p = 3$, $m = 1$ the model started with ferromagnetic initial conditions clearly shows phase transition: at low temperature energy decreases to the minimum value

(Fig. 3a), magnetization rises to $M \approx 1$ (Fig. 3b) and both $c$ and $\chi_M$ show maxima at $3 < \beta_c < 4$ (Fig. 3c, d). The low-temperature phase is the ferromagnetic one. If the model is started with random initial conditions, evidence for the phase transition is more vague. At low temperature the system is usually trapped in a disordered (glassy) metastable state with energy higher than that of one of the ground states (Fig. 3a) and with magnetization close to zero (Fig. 3b). Similar trapping of the system in a metastable state was observed in the three-dimensional plaquette model [11]. Energetic barriers separating metastable states from one another and from the ground states are high: for $p = 3$, $m = 1$ the distribution of hyperedges obeys the power scaling law $P(d) \propto d^{-2.5}$, thus large fraction of spins at nodes with high hyperdegrees are connected to many other spins, and changes of the spin configuration leading to the decrease of energy usually require flipping a significant part of spins. As a result, the system can remain in the metastable state practically forever. Both the specific heat and the magnetic susceptibility exhibit maxima as functions of temperature, but the maximum of $\chi_M$ is significantly lower than in the case of the ferromagnetic initial conditions. This suggests that the model with random (paramagnetic) initial conditions shows glassy phase transition at $3 < \beta_c < 4$, possibly to a metastable glassy state.

For $p = 3$, $m = 2$ the models with random and ferromagnetic initial conditions behave in a qualitatively similar way. In both cases at low temperature energy decreases to the minimum value (Fig. 3a) and the specific heat shows maximum at $1.5 < \beta_c < 2$ (Fig. 3c). Direct observation of the time evolution of the system at low temperature reveals that it jumps randomly among many states with energy close to the minimum and with spin configurations which may significantly differ from one another. As a result, at low temperature in the case of ferromagnetic initial conditions the magnetization is non--zero, but much below its initial value $M = 1$ (Fig. 3b), and in the case of random initial conditions it fluctuates significantly around zero. The mean time between jumps increases with decreasing temperature (which results in the gradual increase of $M$ in the case of ferromagnetic initial conditions in Fig. 3b due to deteriorating statistics of the consecutive values of magnetization between jumps), nevertheless, the jumps are quite frequent even at $\beta = 10$. In this model different states are separated by low energetic barriers: the distribution of hyperedges obeys the scaling law $P(d) \propto d^{-4}$, there are only few spins at nodes with high hyperdegree, and jumps between different metastable or ground states can be performed easily by flipping mostly spins at nodes with low hyperdegrees. Thus even at low temperature the system often jumps between many glassy ground and metastable states rather than remains trapped in one of them. This probably explains the divergence of the magnetic susceptibility $\chi_M \propto T^{-1}$ (Fig. 3d) which suggests phase transition at zero temperature. Hence, the model with $p = 3$, $m = 2$ does not show phase transition to a single glassy state at finite $\beta_c$; at low temperature instead of fluctuating around a single ground state the phase trajectory visits a significant part of the state space in the vicinity of the minimum energy manifold.

## 4. Summary

Complex hypernetworks can be found in various real social and economic systems. In this contribution it was shown that systems of units with many-body interactions can be defined on such hypernetworks in a natural way. In particular, the $p$-spin models on scale-free hypernetworks exhibit such phenomena as ferromagnetic or glassy phase transitions, the presence of long-lived metastable states and finite or high degeneracy of the ground state. In real-world systems (society, stock market, etc.) with complex networks of interactions opinions and decisions are usually non-uniform, often randomly distributed and varying in time even if the “social temperature” seems quite low. Such situation can be reproduced by socio- and econophysical models with disordered and possibly degenerate ground states, which may use certain concepts from the $p$-spin models discussed in this contribution.

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