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Minimum energy control of positive continuous-time linear systems with bounded inputs

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Abstract: The minimum energy control problem for the positive continuous-time linear systems with bounded inputs is formulated and solved. Sufficient conditions for the existence of solution to the problem are established. A procedure for solving of the problem is proposed and illustrated by a numerical example.

Key words: positive, continuous-time, minimum energy control, bounded inputs, procedure.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [4, 7]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The positive fractional linear systems have been investigated in [6, 9, 10, 17]. Stability of fractional linear 1D discrete-time and continuous-time systems has been investigated in the papers [1, 3, 17] and of 2D fractional positive linear systems in [5]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [11]. The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka [19-24] and for 2D linear systems with variable coefficients in [18]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [22, 23]. The minimum energy control of fractional positive continuous-time linear systems has been addressed in [14] and for descriptor positive discrete-time linear systems in [16].

In this paper the minimum energy control problem for positive continuous-time linear systems with bounded inputs will be formulated and solved.

The paper is organized as follows. In Section 2 the basic definitions and theorems of the positive continuous-time linear systems are recalled and the necessary and sufficient conditions for the reachability of the positive systems are given. The minimum energy control problem of the positive linear systems with bounded inputs is formulated and solved in Section 3. Sufficient conditions for the existence of solution of the problem are established and a procedure for computation of the optimal inputs and the minimum value of the performance index are also presented. Concluding remarks are given in Section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Reachability of positive continuous-time linear systems

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.1}$$

where $x(t)\in\Re^n$ and $u(t)\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

The solution of equation (2.1) has the form

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad x(0) = x_0. \tag{2.2}$$

Definition 2.1. [7] The system (2.1) is called the internally positive if and only if $x(t)\in\Re_+^n$, $t\ge 0$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 2.1. [7] The system (2.1) is internally positive if and only if

$$A\in M_n, \quad B\in\Re_+^{n\times m}, \tag{2.3}$$

where M_n is the set of $n\times n$ Metzler matrices.

Definition 2.2. The positive system (2.1) (or the positive pair (A,B)) is called reachable in time $t\in[0,t_f]$ if for any given final state $x_f\in\Re_+^n$ there exists an input $u(t)\in\Re_+^m$, for $t\in[0,t_f]$ that steers the state of the system from zero initial state $x(0)=0$ to the state x_f, i.e. $x(t_f)=x_f$.

A real square matrix is called monomial if each its row and each its column contains only one positive entry and the remaining entries are zero.

Theorem 2.2. The positive system (2.1) is reachable in time $t\in[0,t_f]$ if and only if the matrix $A\in M_n$ is diagonal and the matrix $B\in\Re_+^{n\times n}$ is monomial.

Proof is similar to the proof given in [14].

3. Minimum energy control problem for positive systems with bounded inputs

3.1. Problem formulation

Consider the positive system (2.1) with $A \in M_n$ and $B \in \Re_+^{n\times n}$ monomial. If the system is reachable in time $t \in [0, t_f]$, then usually there exists many different inputs $u(t) \in \Re_+^n$ that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Among these inputs we are looking for an input $u(t) \in \Re_+^n$ satisfying the condition

$$u(t) < U \in \Re_+^n \text{ for } t \in [0, t_f], \tag{3.1}$$

that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau) Q u(\tau) d\tau, \tag{3.2}$$

where $Q \in \Re_+^{n\times n}$ is a symmetric positive defined matrix and $Q^{-1} \in \Re_+^{n\times n}$.

The minimum energy control problem for the positive continuous-time linear systems (2.1) with bounded inputs can be stated as follows: Given the matrices $A \in M_n$, $B \in \Re_+^{n\times n}$, $U \in \Re_+^n$ and $Q \in \Re_+^{n\times n}$ of the performance index (3.2), $x_f \in \Re_+^n$ and $t_f > 0$, find an input $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ satisfying (3.1) that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (3.2).

3.2. Problem solution

To solve the problem we define the matrix

$$W = W(t_f, Q) = \int_0^{t_f} e^{A(t_f-\tau)} B Q^{-1} B^T e^{A^T(t_f-\tau)} d\tau. \tag{3.3}$$

From (3.3) and Theorem 2.2 it follows that the matrix (3.3) is monomial if and only if the fractional positive system (2.1) is reachable in time $[0, t_f]$. In this case we may define the input

$$\hat{u}(t) = Q^{-1} B^T e^{A^T(t_f-t)} W^{-1} x_f \text{ for } t \in [0, t_f]. \tag{3.4}$$

Note that the input (3.4) satisfies the condition $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \Re_+^{n\times n} \text{ and } W^{-1} x_f \in \Re_+^n. \tag{3.5}$$

Theorem 3.1. Let the positive system (2.1) be reachable in time $[0, t_f]$ and let $\bar{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$ be an input that steers the state of the positive system (2.1) from $x_0 = 0$ to $x_f \in \Re_+^n$ and satisfies the condition (3.1). Then the input (3.4) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (3.2), i.e. $I(\hat{u}) \le I(\bar{u})$.

The minimal value of the performance index (3.2) is equal to

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{3.6}$$

Proof. If the conditions (3.5) are met then the input (3.4) is well defined and $\hat{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$. We shall show that the input steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Substitution of (3.4) into (2.2) for $t = t_f$ and $x_0 = 0$ yields

$$x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)} B\hat{u}(\tau)d\tau = \int_0^{t_f} e^{A(t_f-\tau)} BQ^{-1}B^T e^{A^T(t_f-\tau)} d\tau W^{-1} x_f = x_f \tag{3.7}$$

since (3.3) holds. By assumption the inputs $\overline{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Hence

$$x_f = \int_0^{t_f} e^{A(t_f-\tau)} B\overline{u}(\tau)d\tau = \int_0^{t_f} e^{A(t_f-\tau)} B\hat{u}(\tau)d\tau \tag{3.8a}$$

or

$$\int_0^{t_f} e^{A(t_f-\tau)} B[\overline{u}(\tau) - \hat{u}(\tau)]d\tau = 0. \tag{3.8b}$$

By transposition of (3.8b) and postmultiplication by $W^{-1}x_f$ we obtain

$$\int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T B^T e^{A^T(t_f-\tau)} d\tau W^{-1} x_f = 0. \tag{3.9}$$

Substitution of (3.4) into (3.9) yields

$$\int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T B^T e^{A^T(t_f-\tau)} d\tau W^{-1} x_f = \int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T Q\hat{u}(\tau)d\tau = 0. \tag{3.10}$$

Using (3.10) it is easy to verify that

$$\int_0^{t_f} \overline{u}(\tau)^T Q\overline{u}(\tau)d\tau = \int_0^{t_f} \hat{u}(\tau)^T Q\hat{u}(\tau)d\tau + \int_0^{t_f} [\overline{u}(\tau) - \hat{u}(\tau)]^T Q[\overline{u}(\tau) - \hat{u}(\tau)]d\tau. \tag{3.11}$$

From (3.11) it follows that $I(\hat{u}) < I(\overline{u})$ since the second term in the right-hand side of the inequality is nonnegative.

To find the minimal value of the performance index (3.2) we substitute (3.4) into (3.2) and we obtain

$$I(\hat{u}) = \int_0^{t_f} \hat{u}^T(\tau) Q\hat{u}(\tau)d\tau = x_f^T W^{-1} \int_0^{t_f} e^{A(t_f-\tau)} BQ^{-1}B^T e^{A^T(t_f-\tau)} d\tau W^{-1} x_f = x_f^T W^{-1} x_f \tag{3.12}$$

since (3.3) holds.

From (3.4) we have

$$\frac{d\hat{u}(t)}{dt} = -EA^T e^{A^T(t_f - t)} F, \tag{3.13a}$$

where

$$E = Q^{-1}B^T, \quad F = W^{-1}x_f. \tag{3.13b}$$

Using (3.13) we may find $t \in [0, t_f]$ for which $\hat{u}(t) \in \Re_+^n$ reaches its maximal value. Note that if all eigenvalues of the matrix A have positive real parts then $\hat{u}(t)$ reaches its maximal value for $t = 0$ and if they have negative real parts then for $t = t_f$.

From the above considerations we have the following procedure for computation the optimal inputs satisfying the condition (3.1) that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimize the performance index (3.2).

Procedure 3.1.

Step 1. Knowing $A \in M_n$ compute e^{At}.

Step 2. Using (3.3) compute the matrix W knowing the matrices A, B, Q for some t_f.

Step 3. Using (3.4) and (3.13) compute the input (3.4) and t_f satisfying the condition (3.1) for given $U \in \Re_+^n$ and $x_f \in \Re_+^n$.

Step 4. Using (3.6) compute the minimal value of the performance index.

Example 3.1. Consider the positive system (2.1) with matrices

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}, \quad a_k > 0, \quad b_k > 0, \quad k = 1,2 \tag{3.14}$$

and the performance index (3.2) with

$$Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad q_k > 0, \quad k = 1, 2. \tag{3.15}$$

Compute the bounded input $\hat{u}(t)$ satisfying

$$\hat{u}(t) = \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} < \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \text{ for } t \in [0, t_f]$$

that steers the state of the system from zero state to

$$x_f = [x_{f1} \ x_{f2}]^T \in \Re_+^2$$

(T denote the transpose) and minimize the performance index.

Using the procedure 3.1 we obtain the following:

Step 1. In this case we have

$$e^{At} = \begin{bmatrix} e^{a_1 t} & 0 \\ 0 & e^{a_2 t} \end{bmatrix}. \tag{3.16}$$

Step 2. Using (3.14), (3.15) and (3.16) we obtain

$$W = \int_0^{t_f} e^{A(t_f-\tau)} BQ^{-1}B^T e^{A^T(t_f-\tau)} d\tau = \int_0^{t_f} e^{A\tau} BQ^{-1}B^T e^{A^T\tau} d\tau = \int_0^{t_f} \begin{bmatrix} b_1^2 q_2^{-1} e^{2a_1\tau} & 0 \\ 0 & b_2^2 q_1^{-1} e^{2a_2\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} \frac{b_1^2 q_2^{-1}}{2a_1}(e^{2a_1 t_f}-1) & 0 \\ 0 & \frac{b_2^2 q_1^{-1}}{2a_2}(e^{2a_2 t_f}-1) \end{bmatrix}. \tag{3.17}$$

Step 3. Using (3.4), (3.14), (3.15) and (3.17) we obtain

$$\hat{u}(t) = Q^{-1}B^T e^{A^T(t_f-t)} W^{-1} x_f$$

$$= \begin{bmatrix} q_1^{-1} & 0 \\ 0 & q_2^{-1} \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ b_2 & 0 \end{bmatrix}^T \begin{bmatrix} e^{a_1(t_f-t)} & 0 \\ 0 & e^{a_2(t_f-t)} \end{bmatrix} \begin{bmatrix} \frac{b_1^2 q_2^{-1}}{2a_1}(e^{2a_1 t_f}-1) & 0 \\ 0 & \frac{b_2^2 q_1^{-1}}{2a_2}(e^{2a_2 t_f}-1) \end{bmatrix}^{-1} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} \tag{3.18}$$

$$= \begin{bmatrix} \frac{2a_2}{b_2} e^{a_2(t_f-t)}(e^{2a_2 t_f}-1)^{-1} x_{f2} \\ \frac{2a_1}{b_1} e^{a_1(t_f-t)}(e^{2a_1 t_f}-1)^{-1} x_{f1} \end{bmatrix}.$$

The minimal value of t_f satisfying the condition (3.1) can be found from the inequality

$$\begin{bmatrix} \frac{2a_2}{b_2} e^{a_2(t_f-t)}(e^{2a_2 t_f}-1)^{-1} x_{f2} \\ \frac{2a_1}{b_1} e^{a_1(t_f-t)}(e^{2a_1 t_f}-1)^{-1} x_{f1} \end{bmatrix} < \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \text{ for } t \in [0, t_f]. \tag{3.19}$$

From (3.19) we have

$$e^{2a_2 t_f} - \frac{2a_2 x_{f2}}{b_2 U_1} e^{a_2 t_f} - 1 > 0, \quad e^{2a_1 t_f} - \frac{2a_1 x_{f1}}{b_1 U_2} e^{a_1 t_f} - 1 > 0. \tag{3.20}$$

Solving the inequalities (3.20) with respect to t_f we obtain

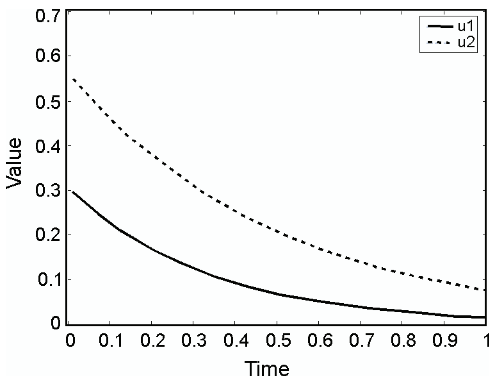
$$t_f > \frac{1}{a_2} \ln\left[\frac{a_2 x_{f2}}{b_2 U_1} + \sqrt{\left(\frac{a_2 x_{f2}}{b_2 U_1}\right)^2 + 1} \right], \quad t_f > \frac{1}{a_1} \ln\left[\frac{a_1 x_{f1}}{b_1 U_2} + \sqrt{\left(\frac{a_1 x_{f1}}{b_1 U_2}\right)^2 + 1} \right], \tag{3.21a}$$

and

$$t_f = \max\left\{\frac{1}{a_2}\ln\left[\frac{a_2x_{f2}}{b_2U_1}+\sqrt{\left(\frac{a_2x_{f2}}{b_2U_1}\right)^2+1}\right],\ \frac{1}{a_1}\ln\left[\frac{a_1x_{f1}}{b_1U_2}+\sqrt{\left(\frac{a_1x_{f1}}{b_1U_2}\right)^2+1}\right]\right\}. \quad (3.21b)$$

For example for $a_1 = 2$, $a_2 = 3$, $b_1 = b_2 = 1$, $U_1 = U_2 = 1$ and $x_f = [1\ 1]^T$ from (3.18) we obtain $\hat{u}_1(t)$ and $\hat{u}_2(t)$ for $t \in [0,1]$ shown on Figure 3.1.

Fig. 3.1. Values of optimal input at time $t \in [0,1]$



Note that $\hat{u}_1(t)$ and $\hat{u}_2(t)$ reaches their maximum values for $t = 0$ since the eigenvalues a_1, a_2 of A are positive.

From (3.21) for the same data we obtain

$$\frac{1}{a_2}\ln\left[\frac{a_2x_{f2}}{b_2U_1}+\sqrt{\left(\frac{a_2x_{f2}}{b_2U_1}\right)^2+1}\right] = \frac{1}{3}\ln[3+\sqrt{10}] = 0.6061,$$
$$\frac{1}{a_1}\ln\left[\frac{a_1x_{f1}}{b_1U_2}+\sqrt{\left(\frac{a_1x_{f1}}{b_1U_2}\right)^2+1}\right] = \frac{1}{2}\ln[2+\sqrt{5}] = 0.7218 \quad (3.22a)$$

and

$$t_f = \max\{0.6061,\ \ 0.7218\} = 0.7218. \quad (3.22b)$$

Step 4. The minimal value of the performance index (3.6) is equal to

$$I(\hat{u}) = x_f^T W^{-1} x_f = [x_{f1}\ \ x_{f2}]\begin{bmatrix} \frac{b_1^2 q_2^{-1}}{2a_1}(e^{2a_1t_f}-1) & 0 \\ 0 & \frac{b_2^2 q_1^{-1}}{2a_2}(e^{2a_2t_f}-1)\end{bmatrix}^{-1}\begin{bmatrix} x_{f1} \\ x_{f2}\end{bmatrix}$$
$$= \frac{2a_1q_2}{b_1^2}(e^{2a_1t_f}-1)^{-1}x_{f1}^2 + \frac{2a_2q_1}{b_2^2}(e^{2a_2t_f}-1)^{-1}x_{f2}^2. \quad (3.23)$$

4. Concluding remarks

Necessary and sufficient conditions for the reachability of the positive continuous-time linear systems have been established (Theorem 2.2). The minimum energy control problem for the positive continuous-time linear systems with bounded inputs has been formulated and solved. Sufficient conditions for the existence of a solution to the problem has been given (Theorem 3.1) and a procedure for computation of optimal input satisfying the condition (3.1) and the minimal value of performance index has been proposed. The effectiveness of the procedure has been demonstrated on the numerical example. The presented method can be extended to positive discrete-time linear systems and to fractional positive continuous-time and discrete-time linear systems with bounded inputs.

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References

[1] Busłowicz M., *Stability of linear continuous time fractional order systems with delays of the retarded type.* Bull. Pol. Acad. Sci. Tech. 56(4): 319-324 (2008).
[2] Dzieliński A., Sierociuk D., Sarwas G., *Ultracapacitor parameters identification based on fractional order model.* Proc ECC'09, Budapest (2009).
[3] Dzieliński A., Sierociuk D., *Stability of discrete fractional order state-space systems.* Journal of Vibrations and Control 14(9/10): 1543-1556 (2008).
[4] Farina L., Rinaldi S., *Positive Linear Systems; Theory and Applications*. J. Wiley, New York (2000).
[5] Kaczorek T., *Asymptotic stability of positive fractional 2D linear systems.* Bull. Pol. Acad. Sci. Tech. 57(3): 289-292 (2009).
[6] Kaczorek T., *Fractional positive continuous-time systems and their Reachability*. Int. J. Appl. Math. Comput. Sci. 18(2): 223-228 (2008).
[7] Kaczorek T., *Positive 1D and 2D systems.* Springer Verlag, London (2001).
[8] Kaczorek T., *Controllability and observability of linear electrical circuits.* Electrical Review 87(9a): 248-254 (2011).
[9] Kaczorek T., *Positivity and reachability of fractional electrical circuits.* Acta Mechanica et Automatica 5(2): 42-51 (2011).
[10] Kaczorek T., *Positive linear systems consisting of n subsystems with different fractional orders.* IEEE Trans. Circuits and Systems 58(6): 1203-1210 (2011).
[11] Kaczorek T., *Practical stability of positive fractional discrete-time linear systems.* Bull. Pol. Acad. Sci. Tech. 56(4): 313-317 (2008).
[12] Kaczorek T., *Reachability and controllability to zero tests for standard and positive fractional discrete-time systems.* Journal Européen des Systèmes Automatisés, JESA, 42(6-8): 769-787 (2008).
[13] Kaczorek T., *Linear Control Systems.* Research Studies Press and J. Wiley, New York (1992).
[14] Kaczorek T., *Minimum energy control of fractional positive continuous-time linear systems.* Submitted to Conf. MMAR (2013).
[15] Kaczorek T., *Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm.* Archive of Control Sciences 21(3): 287-298 (2011).
[16] Kaczorek T., *Minimum energy control of descriptor positive discrete-time linear systems.* Submitted to Compel (2013).
[17] Kaczorek T., *Selected Problems of Fractional Systems Theory.* Springer-Verlag, Berlin (2012).

[18] Kaczorek T., Klamka J., *Minimum energy control of 2D linear systems with variable coefficients.* Int. J. of Control 44(3): 645-650 (1986).

[19] Klamka J., *Controllability of Dynamical Systems.* Kluwer Academic Press, Dordrecht (1991).

[20] Klamka J., *Minimum energy control of 2D systems in Hilbert spaces.* System Sciences 9(1-2): 33-42 (1983).

[21] Klamka J., *Relative controllability and minimum energy control of linear systems with distributed delays in control.* IEEE Trans. Autom. Contr. 21(4): 594-595 (1976).

[22] Klamka J., *Controllability and minimum energy control problem of fractional discrete-time systems.* [in:] *New Trends in Nanotechology and Fractional Calculus*. Eds. Baleanu D., Guvenc Z.B., Tenreiro Machado J.A., Springer-Verlag, pp. 503-509, New York (2010).

[23] Klamka J., *Controllability of dynamical systems-a survey*. Archives of Control Sciences 2(3-4): 281-307 (1993).

[24] Klamka J., *Minimum energy control of discrete systems with delays in control*. International Journal of Control 26(5): 737-744 (1977).