DOI: 10.5604/01.3001.0015.0525
Volume 106 • Issue 1 • May 2021

Journal of Achievements in Materials and Manufacturing Engineering

International Scientific Journal published monthly by the World Academy of Materials and Manufacturing Engineering

Nonlinear homogeneous dynamical system of fully cracked concrete beam

G. Mohan [a,*], U.K. Pandey [b]

[a] Research Scholar, Department of Civil Engineering, National Institute of Technology, Hamirpur, H.P India

[b] Assistant Professor, Department of Civil Engineering, National Institute of Technology, Hamirpur, H.P India

* Corresponding e-mail address: govindmohan@nith.ac.in

ORCID identifier: https://orcid.org/0000-0002-3813-742X (G.M.)

ABSTRACT

Purpose: Purpose of this paper is the theoretical formulation of elasto-dynamics behaviour of fully cracked concrete beams for two-degree of freedom (DOF) dynamical system. Such modelled system is demonstrated by a new class of two-DOF conservative, fully-nonlinear systems known as nonlinear homogeneous dynamical (NHD) systems.

Design/methodology/approach: The theoretical formulation of Two-DOF dynamical system has been developed by using the fundamental concept of structural analysis. Further, the behaviour of such class of conservative dynamical system has been concluded by using MATLAB.

Findings: Findings can be distinguished into two aspects. First, when subjected to service loads, reinforced concrete structures possess tension cracks. Due to load variation with time leads the breathing action (opening-closing-reopening) of existing cracks and the behaviour of such structures are nonlinearly-elastic. Second, such class of fully nonlinear dynamical system possess dependence of stiffness coefficients on nodal displacements.

Research limitations/implications: The mechanism behind such class of fully nonlinear dynamical system is not yet fully explored.

Practical implications: Practical implications are related to the structural response of fully cracked concrete beam to different earthquake excitation, especially to understand the response of such nonlinear homogeneous elasto-dynamical system with two-DOF concrete beam.

Originality/value: Elasto-dynamics of fully cracked concrete beams.

Keywords: Non-linear dynamics, Two-DOF systems, Homogeneous dynamical systems, Bilinearity ratio, Cracked concrete beams, Fixed beam

Reference to this paper should be given in the following way:

G. Mohan, U.K. Pandey, Nonlinear homogeneous dynamical system of fully cracked concrete beam, Journal of Achievements in Materials and Manufacturing Engineering 106/1 (2021) 5-19. DOI: https://doi.org/10.5604/01.3001.0015.0525

ANALYSIS AND MODELLING

© Copyright by International OCSCO World Press. All rights reserved. 2021

1. Introduction

During their service life, reinforced concrete structures are subjected to various types of loadings, such as static loads, introduced by dead and live loads and dynamic loads, introduced by earthquakes, wind, blast, impact, movement of vehicles, operating equipment, etc. Reinforced concrete structures are designed to withstand both static as well as dynamic loadings without any structural failure. The forcing function consists of sustained constant load and variable dynamic load which causes considerable inertial effects. The behaviour of reinforced concrete structures under dynamic loads depends on their mass along with stiffness properties, while the stiffness characteristics depend solely on the nodal load ratio. Static analysis is currently a routine practise as most of the codal recommendations and advance programs are available, whereas dynamic analysis is time consuming and complicated systems which required additional information related to system. Thus, researches are required to understand the dynamic structural characteristics of such systems for their satisfactory performance.

The majority of the research (dynamic response) has been performed on simply supported beams or cantilever beams, which is one of the most important research gaps contained in the available literature. But, the fixed beam problem is more realistic than other type of structural member and the behaviour of such type of structural member considered as fully cracked concrete beam is not yet exploded. Depending upon the position of point of contra-flexure, such class of new system is evolving with multiple natural frequencies. To prevent resonance, we don't consider such multiple natural frequencies in most of the design code. This paper considers the influence of configurational non-linearity of a fixed beam due to cracking.

The present article aims to investigate the elasto-dynamics of two-DOF nonlinear homogeneous dynamical (NHD) system. The fixed concrete beams which are more realistic, modelled as fully cracked section to derive theoretical formulation of NHD system. The expressions for displacements and flexibility matrix coefficients were derived from expressions of complementary energy. Assuming lumped masses and without damping, the equations of motion have been obtained. The theoretical significance of the proposed conservative nonlinear homogeneous mechanical (NHM) system has been brought out in this paper.

2. Literature survey

All elastic and inelastic structural aspects are used in seismic design and study of reinforced concrete structures because of the high-risk potential of earthquakes. For most of elastic seismic design and analysis, it is assumed that concrete structures are linear elastic and the gross sections are considered for stiffness calculation without recognizing the presence of reinforcement and cracking. In comparison, nonlinear or inelastic design and research aims to integrate the effects of inelastic features such as reinforcement yield, concrete cracking, and de-bonding, etc. [1-4]. For each of the above dimensions, either the response spectra or the time-history analysis can be used to determine the structural response. The design seismic response spectrum is an elastic continuum of acceleration which is influenced by factors such as structural safety and simplicity of design engineers' implementation. By using the response spectra, the structural response can be calculated by knowing the natural time span, the damping ratio and the structure's base data. In contrast, the study of time-history analysis can be carried out by evaluating the dynamic response of concrete structures to site-specific site excitations of the seismic base. The elastic response of concrete structures so obtained can be altered by using the response reduction factor to integrate the effect of being nonlinear and inelastic [5,6].

The study of reinforced concrete structures is carried out in traditional approach by assuming that they are linear-elastic. In this traditional approach of linear-elastic aspect for dynamic analysis of reinforced concrete structures, the member’s stiffness value is fully based on a gross concrete section, which thus disregards both the presence of reinforcement and tensile cracks that are consistent during working loads. The elastic response thus obtained is modified by the suitably applied corresponding response reduction factor in order to study the elasto-plastic behaviour. On the other hand, the concrete structures do not demonstrate such elasto-plastic behaviour under the action of earthquake of lesser intensity than that of design earthquake intensity. In such situations, the elastic response must be optimized [6-9].

The study of reinforced concrete structures in modern methodology is to forecast the behaviour of actual concrete structures at all times. Since the problem chosen is very complex, various approximations or simplification are added, along with several empirical constants. Therefore, the analysis is based on analytical, quantitative, and incremental approaches and it is all-inclusive in nature. Coupled elasto-plastic-cum-damage constitutive models are in scope which imitates the aspect of plain concrete behaviour such as yielding, cracking, loss of stiffness, plastic deformations, hardening/softening, etc. The constitutive equations are used which implement the behaviour of reinforcement bars as well as the bond-slip interface phenomenon. Using the finite element paradigm, these constitutive models are implemented to predict the quantitative behaviour of numerically defined boundary limits that are subject to numerically specified load backgrounds [10,11]. Contrary to the above mentioned theoretical continuum approach; certain structure-theoretical models were also developed [12].

Concrete structures are well known to be cracked at isolated sections, often under service loads or even when exposed to moderate earthquakes. The perceptual structure pattern and depth of cracking will depend on the context of the load history. The creation and enhancement of these isolated flexural cracks with an increase in the loads applied make these concrete members nonlinear and inelastic. As crack forming is a nonlinear and inelastic operation, but the opening-closing-reopening (breathing nature) of the existing cracks make the concrete structures nonlinear-elastic. Extensive research to investigate the effect of one or more discreetly positioned breathing cracks on the dynamic behaviour of elastic beams [13-17]. On the other hand, concrete possess very low flexural strength. In fact, the presence of uncertainty complicates the actual concrete structures with a low tensile strength, along with the discernible presence of the cracks. In addition, complexity reaches the next level as caused by the incorporation of the tension stiffening effect. If the concrete is a non-tension solid, these complexities can be avoided. In this context, a static theory was proposed for these nonlinear-elastic concrete structures which deal with their quasi-static behaviour under the action of applicable loads [18]. These structures were known as homogeneous mechanical systems.

Concrete structures with distinct cracks often belong to the same class of mechanical devices as long as no further cracking occurs. Despite expectations of material linearity and slight displacement, these concrete structures were found to be nonlinear-elastic. This approach simplifies the extremely complex problem of evaluation of dynamic response of discrete cracked structures. Real concrete structures are literally recognised as nonlinear dynamic MDOF systems. However, the nonlinear theory of dynamic systems had previously been confined primarily to SDOF framework as the bilinear oscillators, duffing oscillator, van der pol oscillator, etc [19-21]. Bilinear dynamical systems which had been also investigated under bilinear homogenous dynamical systems [22-25] and dynamic of hysteretic mechanical systems [26], etc whose stiffness depends upon the perception of vibration amplitude and the dynamic behaviour of such class of bilinear elastic SDOF homogeneous system has been observed.

Now several more reports are documented dealing with nonlinear dynamic systems in MDOF. For example, two-DOF coupled systems with a cubic force-displacement relationship [27-32] and a system with potential functions up to the degree four in the form of homogeneous polynomials [33]. Many sophisticated, nonlinear theory related to normal mode, internal resonance phenomenon and auto-parametric resonance were successfully developed to demonstrate the response under loading [27, 34-37]. Such systems modelled as spring and dashpot arrangements that bind together with lumped masses. It has been observed that the system behaviour is increasingly unstable and unpredictable with an increase in the energy component by showing sub-harmonic resonances, bifurcations and chaos, high frequency effects, etc. [36-40].

Currently, many dynamic nonlinear models are increasingly being used to track and classify concrete structures based on damage-based health monitoring [41]. In concrete beams, cracks were formed so that their normal vibration frequency and stiffness would decrease and the observed damping would increase [42,43]. Overall, the researchers measure the impact on the complex characteristics of free and forced vibration executed by concrete beams of the damage in the form of cracking induced by prior loading. Some advanced mathematical techniques were developed in order to relate the observable natural frequency to preload, crack depth or width, deflection, etc [44-46]. However, due to the non-monotonic shift in the super-harmonic frequencies of concrete beams with increased damage, the efficacy of vibration-based damage detection is compromised [46]. Beams with breathing cracks (opening-closing-reopening) have been observed to reveal bilinear and/or nonlinear behaviour [13,15,19, 48-50]. The aim of the experimental investigations on dynamical behavior of concrete beams is to identification, quantification and localization of damage [44-46, 51-53].

The authors had proposed two-DOF nonlinear dynamical systems as first order homogeneous dynamical (FOHD) systems. The proposed theory predicts the dynamical behaviour of the two-DOF cracked, simply supported beam with one side overhang concrete beam [23,54,55]. This class of non-linear conservative dynamic system under loads variation, exhibit partially linear and partially non-linear. In case of linear stage, system exhibit either fully in positive tension flexural rigidity or fully in negative tension flexural rigidity as the sense of loads varies. On the other hand, in case of nonlinear stage, system exhibit partly in positive tension flexural rigidity and partially in negative tension flexural rigidity or vice-versa, for particular sense of loads variation.

As per the contemporary practice, the dynamical analysis involves the linear elastic analysis and elastic response reduction due to structural ductility [56,57]. There is a scarcity of investigations related to their inherent dynamical behaviour. Thus, in the field of structural dynamics, the seismic design framework dominates. In fact, non-linear seismic behaviour of dynamic system means elasto-plastic response. It was found that the system frequency or eigen frequency of reinforced concrete beams was found to decrease with the growth of crack dimensions and observed damping to increase [42,58]. On the other hand, there were no indications that the breathing action (opening-closing-reopening of existing cracks) affects damping and that the decay rate of cracked beams is higher. Further reduces dynamic stiffness due to cracks [42].

Non-linear theory of dynamic systems and techniques has not yet been implemented into common seismic non-linear analyses. As concrete structures are cracked by bending tension even in service loads; the effects of cracking must be included in their elastic analysis process [22,24,25]. Despite some early studies, there is a lack of scientific study into the dynamic conduct of concrete structures. In view of this, attempts have been made to integrate the cracking effects and the second order effects of concrete frame lateral drift [59,60]. For columns in sway and non-sway cracked concrete frames, new slenderness limits were suggested [61]. Static and dynamic instability of the cracked concrete beam-columns has recently been studied, such as buckling, displacement, flutter and parametric resonances [37,62,63]. Nonlinear dynamic behaviour under seismic loading and impact-blast loading is also exploded [64,65].

3. Nonlinear homogeneous system

3.1. Nonlinear homogeneous mechanical system

Consider a fully cracked prismatic fixed beam model of span ‘l’, with two point loads P_1 & P_2 and lumped masses m_1 & m_2 at a distance 'a' from the left end support and ‘c’ from the right support respectively as shown in Figure 1. The beam is subject to small elastic displacements X_1 & X_2 under the action of loads applied P_1 & P_2. This fully cracked fixed beam displays non-linear relationships between moment and curvature. This means that such a modelled system has different stiffness coefficient values which depend on the sense and the relative magnitude of applied loads.

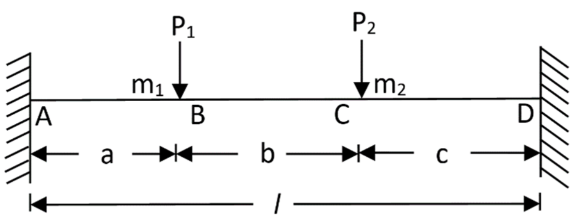

Fig. 1. Representation of modelled fixed beam

As with the fixed beam, the fixed end moment is induced due to externally applied loading, which is induced opposite to that of applied bending moment. For proportional applied load variations, such modelled structures exhibit fully nonlinear mechanical response in the form of discontinuity at origin in their load-displacement relationship. Thus, the section flexural rigidity depends upon the nature of moment, not on the absolute magnitude of the applied flexural moment. This implies that flexural rigidity distribution and the stiffness matrix of the beam depends upon the relative magnitudes and sense of the applied nodal loads.

The beam may be in one of the following possible load combinations depending upon the variation in applied nodal loads:

Condition I $P_1 > 0$, $P_2 > 0$ & $P_2 > P_1$ (1)

Condition II $P_1 > 0$, $P_2 > 0$ & $P_2 = P_1$ (2)

Condition III $P_1 > 0$, $P_2 > 0$ & $P_2 < P_1$ (3)

Condition IV $P_1 > 0$ & $P_2 = 0$ (4)

Condition V $P_1 > 0$ & $P_2 < 0$ (5)

Condition VI $P_1 = 0$ & $P_2 > 0$ (6)

Condition VII $P_1 = 0$ & $P_2 = 0$ (7)

Condition VIII $P_1 = 0$ & $P_2 < 0$ (8)

Condition IX $P_1 < 0$ & $P_2 > 0$ (9)

Condition X $P_1 < 0$ & $P_2 = 0$ (10)

Condition XI $P_1 < 0$, $P_2 < 0$ & $P_2 < P_1$ (11)

Condition XII $P_1 < 0$, $P_2 < 0$ & $P_2 = P_1$ (12)

Condition XIII $P_1 < 0$, $P_2 < 0$ & $P_2 > P_1$ (13)

For any specified set of load combination, the flexural moment distribution of this indeterminate beam can be easily determined. The position of the points of contra-flexure and the corresponding stiffness coefficients of the given beam depend on the relative magnitude of applied loads to each elastically distinguished state. The points of the contra-flexure have different values due to differences in the relative magnitude of the applied loads, which makes the beam to be nonlinear.

In defining the system's flexibility matrix, the following approach has been followed here: first, an expression for the complementary energy function is derived in terms of the nodal loads acting P_1 & P_2 along with incorporation of bi-linearity ratio β. This derivation is based on the assumption that Clapeyron's theorem is also true for this nonlinear mechanical system, to be justified later. Then Castigliano’s theorem applies to this conservative mechanical system and establishes a displacement-force relationship. The tangent flexibility matrix F_{ij} expressions were finally derived [54,66].The breathing action (opening-closing-reopening) of the existing cracks during load variations renders these structures nonlinear-elastic. Under relative magnitude of applied load variations, these structures demonstrate nonlinear mechanical response in the form of discontinuity at origin in their load-displacement relationship. The typical values of bi-linearity ratio β which is defined as the ratio of the two values of flexural rigidity; of common reinforced concrete beams, range up to 10 [22]. In this sense, flexural rigidity values are assumed to be βEI and EI respectively for the respective segment of the beam, which is fully cracked at all sections under positive (sagging) and negative (hogging) flexural moments.

The following expressions are determined for complementary energy Ω, displacements X_i and tangent flexibility coefficients F_{ij} of the elastic beam:

$$\Omega = \int_0^L \frac{M_x^2}{2EI_x} dx \tag{14a}$$

$$X_i = \frac{\partial \Omega}{\partial P_i} \tag{14b}$$

$$\& F_{ij} = \frac{\partial X_i}{\partial P_j} \tag{14c}$$

where i & j = 1, 2 and I_x is the relevant moment of inertia of the section.

The nonlinear systems do not follow the theory of superposition. In reality, the current problem is totally nonlinear homogenous and dynamical system. However, in order to create the displacements corresponding to a specific set of relevant loads, the superposition principle can be invoked only in a computational algorithm for that particular nonlinear mechanical system. First, from the flexural moment distribution the position of the contra-flexure points is calculated. The beam is then expected to be linear elastic and change its flexural-rigidity at the above mentioned points of contra-flexure. Finally, the general expressions for Ω, X_i and F_{ij} are obtained as functions of P_1 & P_2. The displacements can be calculated more specifically for the combined loads by overlaying the corresponding displacements of the separately applied loads P_1 & P_2.The distances s_1, s_2 & s_3 (s_3 exist only in two regions) can correspond to the combination of loads P_1 & P_2, not the individual loads P_1 & P_2.

3.2. Nonlinear homogeneous dynamical system

The equations of motion for the above mentioned two-DOF modelled NHM system, considered as un-damped and mass-less beam are derived here. Let m_1 & m_2 be the lumped masses and F_1(t) & F_2(t) be the applied forces. For such system, the general expression for the system's potential energy V, strain energy U and kinetic energy T are stated as:

$$V = -F_i X_i \tag{15a}$$

$$U = \frac{1}{2} K_{ij} X_i X_j \tag{15b}$$

$$\& T = \frac{1}{2} m_i \dot{X}_i^2 \tag{15c}$$

where i = 1, 2 & j = 1, 2 also, F_i = applied nodal forces, X_i = nodal displacements, $\dot{X}_i$ = nodal velocities & K_{ij} = stiffness coefficients.

As discuss above that for NHM systems, the coefficients of the stiffness matrix K_{ij} are homogeneous to nodal displacement of order zero. On applying Euler's theorem, the expression obtained as:

$$\frac{\partial K_{ij}}{\partial X_1} X_1 + \frac{\partial K_{ij}}{\partial X_2} X_2 = 0 \tag{16}$$

The rate of change of potential energy of the loads with respect to time and the rate of change of kinetic energy of the systems with respect to time are as follows:

$$\dot{V} = -F_i \dot{X}_i \tag{17a}$$

$$\& \dot{T} = m_i \dot{X}_i \ddot{X}_i \tag{17b}$$

While, the time rate of strain energy transition is given below this consists of two parts:

$$\dot{U} = K_{ij} \dot{X}_i X_j + \frac{1}{2} \dot{K}_{ij} X_i X_j \tag{18}$$

where the second component $\frac{1}{2}\dot{K}_{ij} X_i X_j$ is the rate of change of strain energy associated with the time differences in the stiffness coefficients.

On expanding the second component of Eq. 18, we have:

$$\frac{1}{2}\dot{K}_{ij} X_i X_j = \frac{\dot{X}_1}{2}\left(X_1 \frac{\partial K_{11}}{\partial X_1} + X_2 \frac{\partial K_{11}}{\partial X_2}\right) + \frac{\dot{X}_1}{2}\left(X_1 \frac{\partial K_{12}}{\partial X_1} + X_2 \frac{\partial K_{12}}{\partial X_2}\right) + \frac{\dot{X}_2}{2}\left(X_1 \frac{\partial K_{21}}{\partial X_1} + X_2 \frac{\partial K_{21}}{\partial X_2}\right) + \frac{\dot{X}_2}{2}\left(X_1 \frac{\partial K_{22}}{\partial X_1} + X_2 \frac{\partial K_{22}}{\partial X_2}\right) \tag{19}$$

By applying Eq. 16 to above equation, we get:

$$\frac{1}{2}\dot{K}_{ij} X_i X_j = 0 \tag{20}$$

Hence, the time rate of strain energy transition becomes:

$$\dot{U} = K_{ij} \dot{X}_i X_j \tag{21}$$

Finally, we observed that there is no time rate of change of strain energy as the temporal fluctuation of the stiffness coefficients takes place. Damaged-elastic concrete modelled as a First Order Homogenous Mechanical (FOHM) system, exhibits same kind of response [67].

Such mechanical systems obey the principle of conservation of energy. This means that the algebraic sum of their rate of change of V, U and T with respect to time must vanish. Hence,

$$\dot{V} + \dot{U} + \dot{T} = 0 \quad (22)$$

On substituting the values of $\dot{V}, \dot{U}$ & $\dot{T}$ in above equation, we get,

$$-F_i\dot{X}_i + K_{ij}\dot{X}_iX_j + m_i\dot{X}_i\ddot{X}_i = 0 \quad (23a)$$

$$m_i\ddot{X}_i + K_{ij}X_j = F_i \quad ; i\&j = 1,2 \quad (23b)$$

As the nodal velocities are arbitrary, equation of motion becomes:

$$m_i\ddot{X}_i + K_{ij}X_j = F_i(t); \quad i \,\&\, j = 1,2 \quad (24)$$

In terms of the mass matrix [M]and instantaneous stiffness matrix [K], the matrix equation of motion becomes:

$$M\ddot{X} + KX = F(t) \quad (25)$$

where $X, \dot{X}, \ddot{X}$ & F represent the nodal amplitude, velocity, acceleration and applied force vectors respectively.

4. Results and discussions

By using the above-mentioned general approach, expressions for Ω, X_i and F_{ij} can easily be derived. In order to demonstrate the above approach and establish a constitutive identity of the beam, a concrete beam with the following numerical, geometric and mechanical information has been chosen:

$$a = 4\,m, \qquad b = 2\,m, \qquad c = 4\,m \;\&\; \beta = 8$$

For above specified geometrical details along with applied set of loading, the beam can be in any of the following ten elastically-distinct states:

State I $P_1 \geq 0$ & $P_2 \geq -0.389\, P_1$ (26)

State II $P_1 > 0$ & $-0.389\, P_1 > P_2 \geq -0.667\, P_1$ (27)

State III $P_1 > 0$ & $-0.667\, P_1 > P_2 \geq -1.5\, P_1$ (28)

State IV $P_1 > 0$ & $-1.5\, P_1 > P_2 \geq -2.572\, P_1$ (29)

State V $P_1 \geq 0$ & $P_2 < -2.572\, P_1$ (30)

State VI $P_1 < 0$ & $P_2 \leq -0.389\, P_1$ (31)

State VII $P_1 < 0$ & $-0.667\, P_1 \leq P_2 > -0.389\, P_1$ (32)

State VIIIP$_1 < 0$ & $-1.5\, P_1 \leq P_2 > -0.667\, P_1$ (33)

State IX $P_1 < 0$ & $-2.572\, P_1 \leq P_2 > -1.5\, P_1$ (34)

State X $P_1 < 0$ & $P_2 > -2.572\, P_1$ (35)

The obtained expressions for first two elastically distinct states are given in Appendix 1. For all nonlinear regions, it is obvious from the above that denominator disappearing which includes infinitely high complementary energy levels, nodal movements and tangent flexibility coefficients. The nonlinear states are characterised by the reliance of the positions of the points of contra-flexure on the loads applied. The distance s_1, s_2 & s_3 depends, to be exact, on the load ratio in those states $\frac{P_2}{P_1}$.

The following mathematical forms can be restated for the above mentioned expressions Ω, X_i and F_{ij} and distance s_1, s_2 & s_3:

$$\Omega = P_1^2 g\left(\frac{P_2}{P_1}\right) \quad (36a)$$

$$X_i = P_1^1 g_i\left(\frac{P_2}{P_1}\right) \quad (36b)$$

$$F_{ij} = P_1^0 g_{ij}\left(\frac{P_2}{P_1}\right) \quad (36c)$$

$$\&\; s_k = P_1^0 g_s\left(\frac{P_2}{P_1}\right) \quad (36d)$$

where g, g_i, g_{ij} & g_s are functions of the load ratio. It should be noted that Ω, X_i and F_{ij} as well as s_k are functional non-negative homogeneous of degree 2, 1, 0 & 0 respectively of the loads ratio $\frac{P_2}{P_1}$ [12]. The present system can be defined as part of FOHM class.

If Euler's theorem for homogeneous functions is extended respectively to Ω & X_i; one is obtained:

$$\frac{\partial\Omega}{\partial P_1}P_1 + \frac{\partial\Omega}{\partial P_2}P_2 = X_1P_1 + X_2P_2 = 2\Omega \quad (37a)$$

$$\frac{\partial X_i}{\partial P_1}P_1 + \frac{\partial X_i}{\partial P_2}P_2 = X_i \quad (37b)$$

In this class of non-linear mechanical systems, the validity of Clapeyron's theorem has been justified by Eq. (37a). For such class of homogeneous dynamical system, the tangent flexibility matrix F_{ij}^T acts as a replacement for secant flexibility matrix F_{ij}^S. The identity of the tangent and secant flexibility matrices was established, i.e., all the tangent-secant-flexibility matrices were identified i.e. $F_{ij}^T = F_{ij}^S$. However, it is clear that for the entire nonlinear states, FOHM systems lack central symmetry in such a way that load inversion does not generate the same absolute energy values, displacement, as well as flexibility coefficients. As a result, flexibility matrix shows a distinct paucity in the passive states [12].

The loads P_1 & P_2 and displacements X_1 & X_2 are divided into 10 elastically distinct conical areas corresponding to the above mentioned ten defined states of the system. Such conical areas were also demonstrated by various researchers [20,54]. Different regions in the load-space and the displacement-space are shown in Figure 2 and Figure 3 respectively. As stated by Eq. (26) to Eq. (35) are defined by the 10 inter-regional boundaries between R-I & R-II, R-II& R-III, R-III & R-IV, R-IV & R-V, R-V & R-VI, R-VI & R-VII, R-VII & R-VIII, R-VIII & R-IX, R-IX & R-X and R-X & R-I.

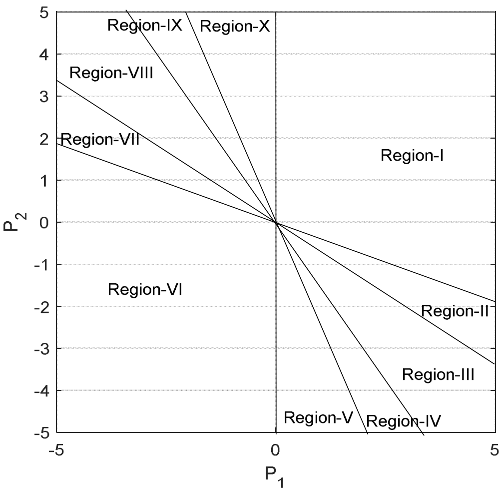

Fig. 2. Elastically-distinct regions in load space

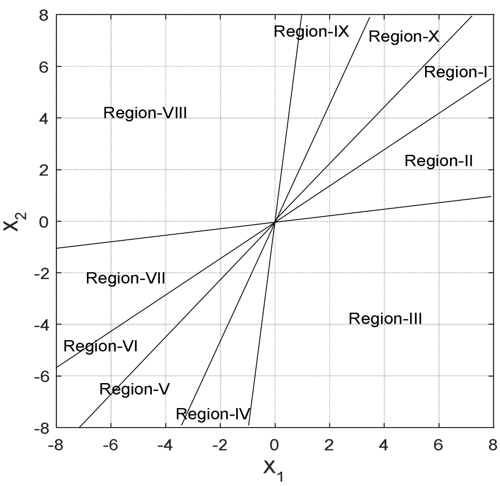

Fig. 3. Elastically-distinct regions in displacement space

To summarize, the coefficients for the flexibility of the nodal elastic forces P_1 & P_2 are zero order homogeneous (ZOH) function, which is in turn first order homogeneous (FOH) function of the nodal displacement X_1 & X_2. ZOH functions of nodal displacements can also be said to be flexibility coefficients. The instantaneous stiffness matrix is derived simply as the inverse of the instantaneous flexibility matrix. Therefore, the stiffness coefficients are also known to be ZOH functions of the nodal displacements. In addition, complementary strain energy can be regarded as a homogeneous function of order two of nodal displacements. Under proportional loading, the load-displacement relationship is linear, so straight inter-regional boundaries of the load-space as well as displacement space are obtained.

The flexibility and stiffness coefficients differ with load ratios as shown below in Figure 4 and Figure 5 for a unit load P_1 acting downwards and upward respectively and the fluctuation in P_2 is considered. There is a sudden change in the coefficients of flexibility and stiffness as the system moves from one state to another state. From the Figure 4a and Figure 5a it was observed that a flat horizontal line does not indicate that their exist a linear zone but, in such the variation is quite small with fluctuation in P_2 which is further clear in Figure 4b and Figure 5b.

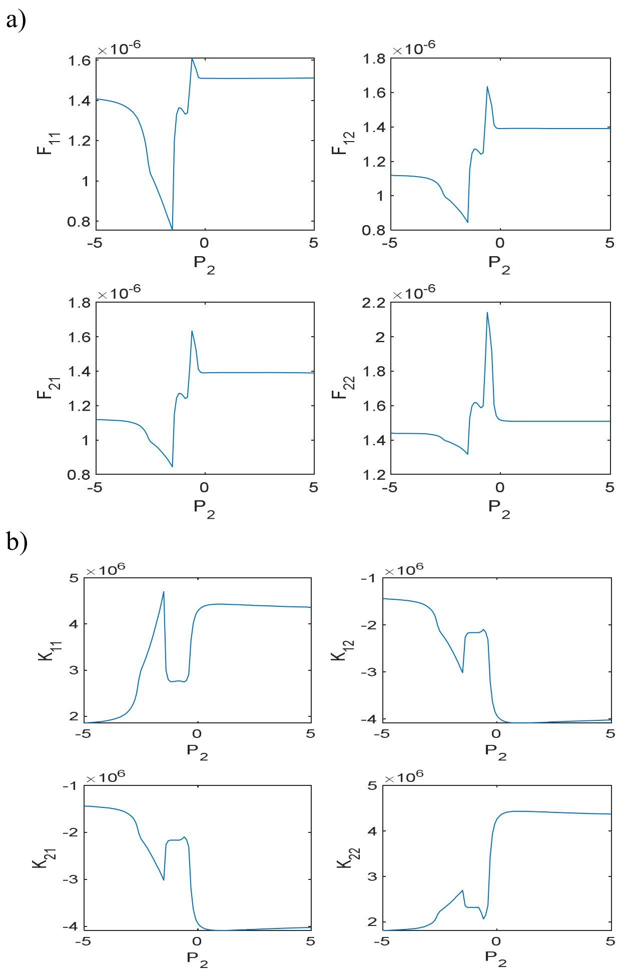

Fig. 4. Variation of coefficients: $P_1 \geq 0$: a) flexibility, b) stiffness

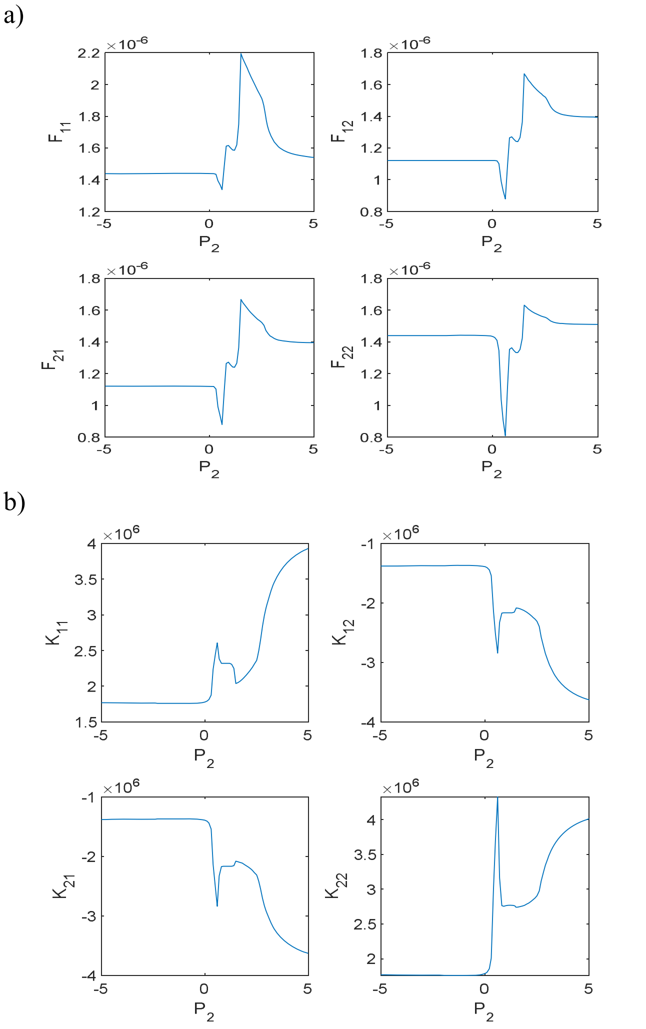

Fig. 5. Variation of coefficients: $P_1 < 0$: a) flexibility, b) stiffness

Other aspect is the polar angle which is defined as the inverse of tangent with reference to the load space ratio. Mathematically, Polar angle is defined as:

$$\theta = \tan^{-1}\left(\frac{P_2}{P_1}\right) \tag{38}$$

Such coefficients profiles vary suddenly as the inter-regional boundaries reaches. Such variation with respect to polar angle is plotted in the Figure 6.

The complementary energy Ω have been observed to vary continuously with polar angle as shown in Figure 7. Authors have been observed from the Figure 7 that the peaks show the differences due to distinct paucity, as the system moves in the passive state. Similar type of behaviour has also been observed for modular damaged elastic concrete which is modelled as a FOHM system [67].

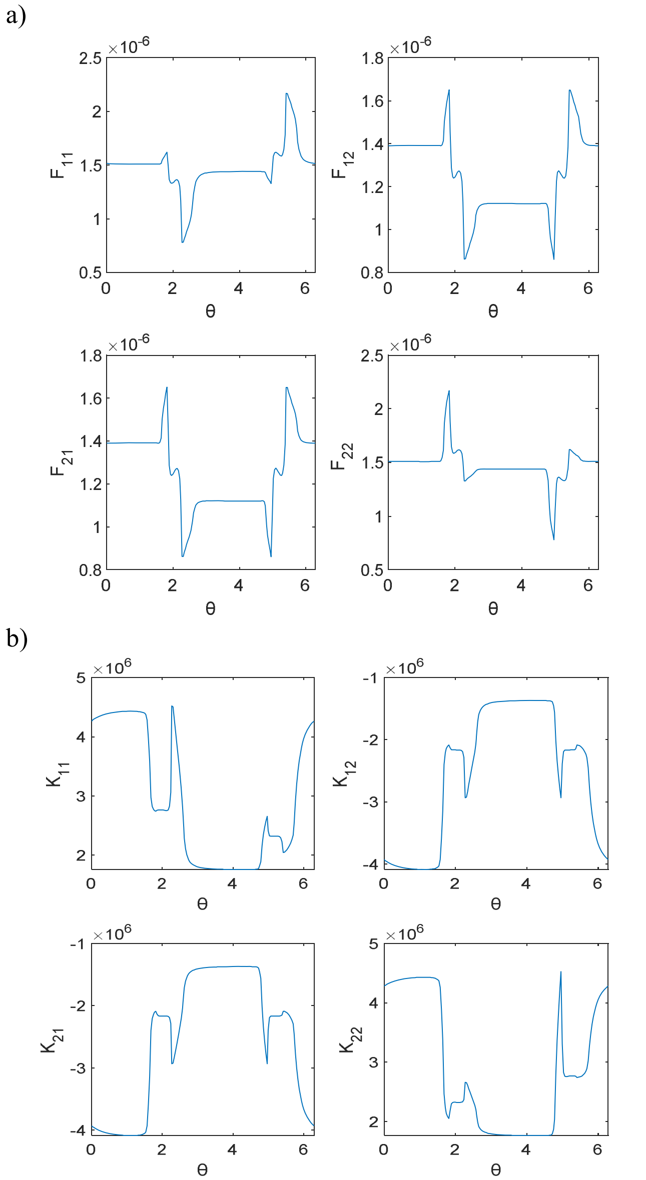

Fig. 6. Variation of coefficients with polar angle: a) flexibility, b) stiffness

It is observed that, the damping coefficients are also a homogeneous function of order zero of the nodal displacements X_1 & X_2 and so of the elastic forces P_1 & P_2 as that of stiffness coefficients. Ten elastically distinct regions in the load space and the displacement space correspond to different values of the damping coefficients C_{ij} along with different modal frequencies ω_1 & ω_2 of the corresponding un-damped systems. Figures 8(a) and 8(b), respectively, display fluctuations between the positives and negatives of unit load P_1 along with fluctuation in load P_2.

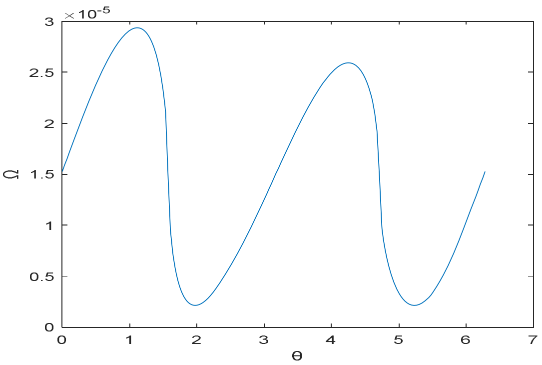

Fig. 7.Variation of energy with polar angle

The constants a_1 & a_2 were calculated by taking two modal damping ratios into consideration, each at a value of 0.05. The damping coefficients and modal frequencies, like the flexibility and stiffness coefficients, differ continuously in non-linear areas and show inter-regional discontinuity. The entire information related to ten distinct elastically states of NHD system are tabulated in Appendix 2.

5. Theoretical significance

In order to understand the two-DOF nonlinear system, many researcher deals with the necessary expression for stiffness coefficient, flexibility coefficient and damping coefficient which are a function of nodal amplitudes [28-35, 68]. This paper uses the classical theory of the structures to evaluate a specific two-DOF cracked concrete beam to derive the equation of motion of the proposed NHD systems. NHD systems are a class of FOHD system that differs from non-linear homogeneous dynamical (NLHD) systems [20]. NLHD systems are Single-degree-of-freedom (SDOF) systems, which are distinguished by the division into four conical linear regions of the space state. The two-DOF systems proposed are the space for the nodal displacement divided into ten non-linear, dynamically conical areas. Of course, some preliminary results for the specific system have been obtained and are ready for contact with numerical integration techniques alone. Obviously, constructing the FOHD systems general theory represents an attractive area of research for dynamics.

To summarize, such NHD systems are non-negative homogeneous function of order two, one and zero of the nodal displacements corresponding to strain energy, nodal elastic forces and instantaneous elastic coefficients respectively. Such NHD systems share many features of linear systems, such as the validity of the Clapeyron's theorem and the identity of the tangent and secant stiffness matrices. On the other hand, the Duffing Oscillator and van der Pol Oscillator are SDOF system, which depend on the stiffness and damping coefficients which further depends upon displacement amplitude respectively. The NHD systems suggested in this paper are two-DOF systems in which

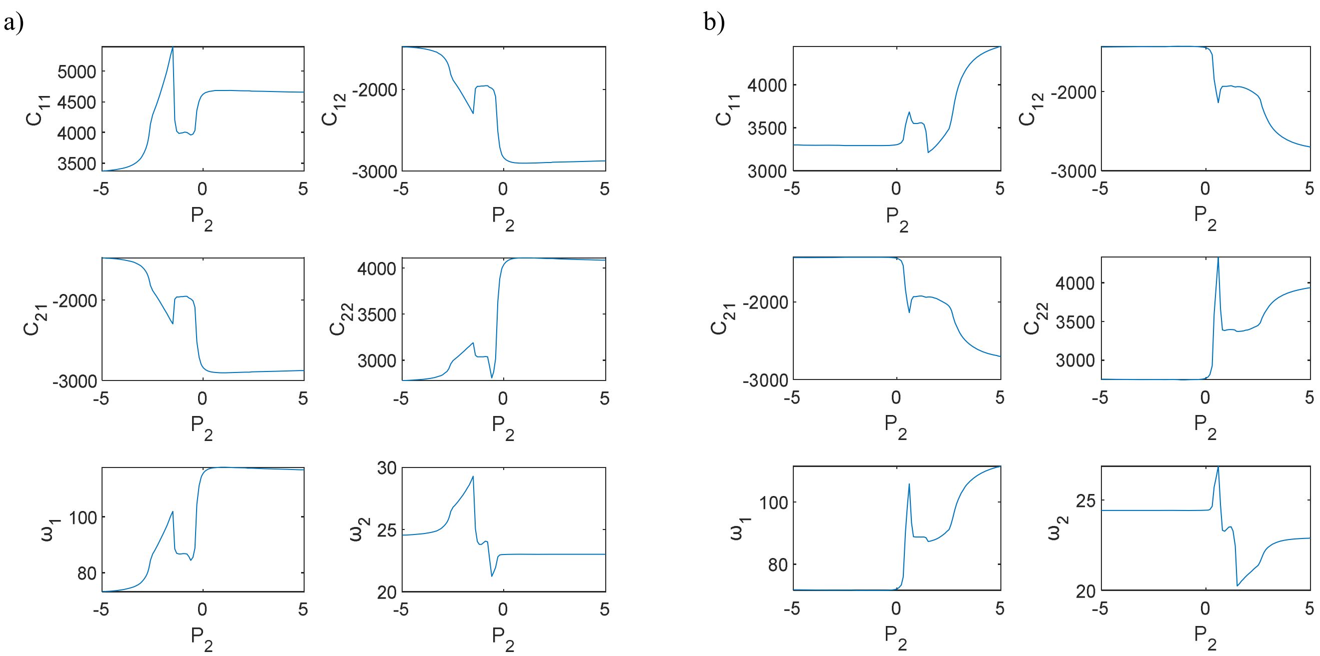

Fig. 8. Variation of damping coefficients and modal frequencies: a) $P_1 \geq 0$, b) $P_1 < 0$

both of the stiffness and the damping coefficients are related to nodal displacement as ZOH functions. Thus, the NHD systems are not supposed to demonstrate the autonomous oscillation and limit cycles correlate with negative damping factor of van der Pol oscillator. The fixed beam problem is more commonly used in civil engineering and such behaviour of fully cracked concrete beam is not exploded. As, the new system is evolving with multiple natural frequencies depending upon the position of point of contra-flexure. So, in most of the design code we do not consider such multiple natural frequency to avoid the resonance.

6. Conclusions

This paper introduces a new class of two-DOF conservative dynamic, fully non-linear systems known as NHD systems. The equation of motion has come from a potential function for a specific mechanical system, namely a classically damped cracked concrete beam which supports lumped masses. The tangent and secant stiffness matrices coefficients are the same for these systems. The time variance rates of the stiffness coefficients during motion do not affect the system response. Similar to Duffing and van der Pol oscillators, the NHD systems possess dependence of the stiffness and damping coefficients upon nodal displacements. The NHD systems proposed are believed to represent a definite contribution to the theory of nonlinear dynamics. It was also determined that modelled two-DOF fully cracked concrete beams behave as NHD system. The scientific validity and functional importance of the proposed model is of tremendous value.

Acknowledgements

The National Institute of Technology Hamirpur, India, supported the work presented in this paper.

Additional information

The authors would like to express their gratitude to the Ministry of Human Resource Development (MHRD) of India for providing financial support for this research.

References

[1] Y. L. MO, Dynamic behavior of concrete structures, Elsevier, Amsterdam, 1994.
DOI: https://doi.org/10.1016/C2009-0-09176-4

[2] A.K. Chopra, Dynamics of structures: theory and applications to earthquake engineering, Prentice Hall Englewood Cliffs, New Jersey, 1995.

[3] G.G. Penelis, A.J. Kappos, Earthquake-resistant concrete structures, CRC press, Florida, 1997.

[4] L. Davenne, F. Ragueneau, J. Mazars, A. Ibrahimbegovic, Efficient approaches to finite element analysis in earthquake engineering, Computers & Structures 81/12 (2003) 1223-1239.
DOI: https://doi.org/10.1016/S0045-7949(03)00038-5

[5] T. Paulay, M.J.N. Priestley, Seismic design of reinforced concrete and masonry buildings, John Wiley & Sons, New York, 1992.

[6] IS 1893 (Part 1): 2002, Indian standard code of practice - criteria for earthquake resistant design of structures, New Delhi, 2002.

[7] F. Stangenberg, Nonlinear dynamic analysis of reinforced concrete structures, Nuclear Engineering and Design 29/1 (1974) 71-88.
DOI: https://doi.org/10.1016/0029-5493(74)90099-5

[8] R.W. Clough, J. Penzien, Dynamics of structures, Computers & Structures, Inc., Berkeley, 1995.

[9] IS 456: 2000, Indian standard plain and reinforced concrete – code of practice, New Delhi, 2000.

[10] J. Lemaitre, Coupled elasto-plasticity and damage constitutive equations, Computer Methods in Applied Mechanics and Engineering 51/1-3 (1985) 31-49. DOI: https://doi.org/10.1016/0045-7825(85)90026-X

[11] B.M. Luccioni, D.E. López, R.F. Danesi, Bond-slip in reinforced concrete elements, Journal of Structural Engineering 131/11 (2005) 1690-1698. DOI: https://doi.org/10.1061/(ASCE)0733-9445(2005)131:11(1690)

[12] G.S. Benipal, Rational mechanics of reinforced concrete beams, I.I.Sc-Bangalore, Bangalore, 1994.

[13] A.D. Dimarogonas, Vibration of cracked structures: A state of the art review, Engineering Fracture Mechanics 55/5 (1996) 831-857.
DOI: https://doi.org/10.1016/0013-7944(94)00175-8

[14] S.M. Cheng, X.J. Xu, W. Wallace, A.S.J. Swamidas, Vibrational response of a beam with a breathing crack, Journal of Sound and Vibration 225/1 (1999) 201-208.
DOI: https://doi.org/10.1006/jsvi.1999.2275

[15] M. Kisa, J. Brandon, The effects of closure of cracks on the dynamics of a cracked cantilever beam, Journal of Sound and Vibration 238/1 (2000) 1-18. DOI: https://doi.org/10.1006/jsvi.2000.3099

[16] T.G. Chondros, A.D. Dimarogonas, J. Yao, Vibration of a beam with a breathing crack, Journal of Sound and Vibration 239/1 (2001) 57-67.
DOI: https://doi.org/10.1006/jsvi.2000.3156

[17] N.V. Hung, V.U.V. Hung, T.B. Viet, The effect of crack width on the service life of reinforced concrete structures, IOP Conference series: Earth & Environmental Science 143 (2018) 012044. DOI: https://doi.org/10.1088/1755-1315/143/1/012044

[18] G.S. Benipal, A study on the non-linear elastic behavior of reinforced concrete structural elements under normal loading, Indian Institute of Technology, Delhi, 1993.

[19] Y.C. Chu, M.H.H. Shen, Analysis of forced bilinear oscillators and the application to cracked beam dynamics, American Institute of Aeronautics and Astronautics Journal 30/10 (1992) 2512-2519. DOI: https://doi.org/10.2514/3.11254

[20] J.A. Inaudi, G. Leitmann, J.M. Kelly, Single degree of freedom nonlinear homogeneous systems, Journal of Engineering Mechanics 120/7 (1994) 1543-1562. DOI: https://doi.org/10.1061/(ASCE)0733-9399(1994)120:7(1543)

[21] J.M.T. Thompson, A.R. Bokaian, R. Ghaffari, Sub-harmonic and chaotic motions of a bilinear oscillator, IMA Journal of Applied Mathematics 31/3 (1983) 207-234. DOI: https://doi.org/10.1093/imamat/31.3.207

[22] U.K. Pandey, G.S. Benipal, Bilinear dynamics of SDOF concrete structures under sinusoidal loading, Advances in Structural Engineering 9/3 (2006) 393-407. DOI: https://doi.org/10.1260%2F136943306777641869

[23] U.K. Pandey, Nonlinear elasto-dynamics of cracked concrete beams, PhD Thesis, Indian Institute of Technology, Delhi, 2008.

[24] U.K. Pandey, G.S. Benipal, Bilinear elastodynamical models of cracked concrete beams, Structural Engineeing and Mechanics 39/4 (2011) 465-498. DOI: https://doi.org/10.12989/sem.2011.39.4.465

[25] U.K. Pandey, G.S. Benipal, Response of SDOF bilinear elasto-dynamical models of cracked concrete beams for El centro earthquake, IES Journal Part A: Civil & Structural Engineeing 6/3 (2013) 222-238. DOI: https://doi.org/10.12989/sem.2011.39.4.465

[26] T.K. Caughey, Sinusoidal excitation of a system with bilinear hysteresis, Journal of Applied Mechanics ASME 27/4 (1960) 640-643. DOI: https://doi.org/10.1115/1.3644075

[27] X. Jian, L. Qishao, H. Kelei, Non linear normal modes and their superposition in a two degrees of freedom asymmetric system with cubic nonlinearities, Applied Mathematics and Mechanics 19/12 (1998) 1167-1177. DOI: https://doi.org/10.1007/BF02456638

[28] T. Pirbodaghi and S. Hoseini, Nonlinear free vibration of a symmetrically conservative two-mass system with cubic nonlinearity, Journal of Computational and Nonlinear Dynamics 5/1 (2010) 011006. DOI: https://doi.org/10.1115/1.4000315

[29] M. Qaisi, A.W. Kilani, Power-series solution for a strongly non-linear two-degree-of-freedom system, Journal of Sound and Vibration 233/3 (2000) 489-494. DOI: https://doi.org/10.1006/jsvi.1999.2833

[30] M.A. Savi, P.M.C.L. Pacheco, Chaos in a two degree of freedom duffing oscillator, Journal of the Brazilian Society of Mechanical Sciences 24/2 (2002) 115-121. DOI: https://doi.org/10.1590/S0100-73862002000200006

[31] A.F. Vakakis, R.H. Rand, Normal modes and global dynamics of a two-degree-of-freedom non-linear system-I. Low energies, International Journal of Non-Linear Mechanics 27/5 (1992) 861-874. DOI: https://doi.org/10.1016/0020-7462(92)90040-E

[32] A.F. Vakakis, R.H. Rand, Normal modes and global dynamics of a two-degree-of-freedom non-linear system-II High energies, International Journal of Non-Linear Mechanics 27/5 (1992) 875-888. DOI: https://doi.org/10.1016/0020-7462(92)90041-5

[33] M. Falconi, E.A. Lacomba, C. Vidal, On the dynamics of mechanical systems with homogeneous polynomial potentials of degree 4, Bulletin of the Brazilian Mathematical Society, New Series 38/2 (2007) 301-333. DOI: https://doi.org/10.1007/s00574-007-0048-z

[34] A. Kozmin, Y. Mikhlin, C. Pierre, Transient in a two-DOF nonlinear system, Nonlinear Dynamics 51 (2008) 141-154. DOI: https://doi.org/10.1007/s11071-007-9198-1

[35] Z. Szabó, A. Lukács, Numerical stability analysis of a forced two-DOF oscillator with bilinear damping, Journal of Computational and Nonlinear Dynamics 2/3 (2007) 211-217. DOI: https://doi.org/10.1115/1.2727487

[36] A. Tondl, T. Ruijgrok, F. Verhulst, R. Nabergoj, Autoparametric resonance in mechanical systems, Cambridge University Press, Cambridge, 2015.

[37] M.R. Sharma, A.K. Singh, G.S. Benipal, Parametric resonance in concrete beam-columns, Latin American Journal of Solids and Structures 11/1 (2014) 925-945. DOI: https://doi.org/10.1590/S1679-78252014000600002

[38] A.H. Nayfeh, L.D. Zavodney, The response of two-degree-of-freedom systems with quadratic non-linearities to a combination parametric resonance, Journal of Sound and Vibration 107/2 (1986) 329-350. DOI: https://doi.org/10.1016/0022-460X(86)90242-7

[39] W. Asrar, Two-degree-of-freedom systems with quadratic non-linearities subjected to parametric and self excitation, Journal of Sound and Vibration 150/3 (1991) 447-456. DOI: https://doi.org/10.1016/0022-460X(91)90897-S

[40] A. Manevich, L. Manevitch, The mechanics of nonlinear system with internal resonances, Imperial College Press, London, 2005.
[41] S. Voggu, S. Sasmal, Dynamic nonlinearities for identification of the breathing crack type damage in reinforced concrete bridges, Structural Health Monitoring 20/1 (2021) 339-359.
DOI: https://doi.org/10.1177%2F1475921720930990
[42] S. Jerath, M.M. Shibani, Dynamic stiffness and vibration of reinforced concrete beams, American Concrete Institute Journal 82/2 (1985) 196-202.
[43] P.G. Kirmser, The effect of discontinuities on natural frequencies of beams, University of Kansas Science Bulletin, Manhattan, 1944.
[44] K.V.D. Abeele, J.D. Visscher, Damage assessment in reinforced concrete using spectral and temporal nonlinear vibration techniques, Cement and Concrete Research 30/9 (2000) 1453-1464.
DOI: https://doi.org/10.1016/S0008-8846(00)00329-X
[45] W.L. Bayissa, N. Haritos, Experimental investigation into vibration characteristics of a cracked RC T-beam, Proceeding of the Conference of Australian Earthquake Engineeing Society, 2004, 1-6.
[46] J. Maeck, M.A. Wahab, B. Peeters, G.D. Roeck, J.D. Visscher, W.P.D. Wilde, J.M. Ndambi, J. Vantomme, Damage identification in reinforced concrete structures by dynamic stiffness determination, Engineeing Structures 22/10 (2000) 1339-1349.
DOI: https://doi.org/10.1016/S0141-0296(99)00074-7
[47] W.I. Hamad, J.S. Owen, M.F.M. Hussein, Modelling the degradation of vibration characteristics of reinforced concrete beams due to flexural damage, Structural Control and Health Monitoring 22/6 (2015) 939-967. DOI: https://doi.org/10.1002/stc.1726
[48] A.K. Darpe, K. Gupta, A. Chawla, Transient response and breathing behaviour of a cracked Jeffcott rotor, Journal of Sound and Vibration 272/1-2 (2004) 207-243.
DOI: https://doi.org/10.1016/S0022-460X(03)00327-4
[49] S. Orhan, Analysis of free and forced vibration of a cracked cantilever beam, NDT & E International 40/6 (2007) 443-450.
DOI: https://doi.org/10.1016/j.ndteint.2007.01.010
[50] R. Ruotolo, C. Surace, P. Crespo, D. Storer, Harmonic analysis of the vibrations of a cantilevered beam with a closing crack, Computers & Structures 61/6 (1996) 1057-1074.
DOI: https://doi.org/10.1016/0045-7949(96)00184-8
[51] E. Hamed, Y. Frostig, Free vibrations of cracked prestressed concrete beams, Engineering Structures 26/11 (2004) 1611-1621.
DOI: https://doi.org/10.1016/j.engstruct.2004.06.004
[52] S.S. Law, X.Q. Zhu, Nonlinear characteristics of damaged concrete structures under vehicular load, Journal of Structural Engineering 131/8 (2005) 1277-1285. DOI: https://doi.org/10.1061/(ASCE)0733-9445(2005)131:8(1277)
[53] J.M. Ndambi, J. Vantomme, K. Harri, Damage assessment in reinforced concrete beams using eigenfrequencies and mode shape derivatives, Engineeing Structures 24/4 (2002) 501-515. DOI: https://doi.org/10.1016/S0141-0296(01)00117-1
[54] U.K. Pandey, G.S. Benipal, First order homogeneous dynamical systems 1: theoretical formulation, International Journal of Structural Engineering 8/3 (2017) 187-204.
DOI: https://doi.org/10.1504/IJSTRUCTE.2017.086435
[55] U.K. Pandey, G.S. Benipal, First order homogeneous dynamical systems 2: Application to cracked concrete beams, International Journalof Structural Engineering 8/3 (2017) 205-226.
DOI: https://doi.org/10.1504/IJSTRUCTE.2017.086436
[56] S. Timoshenko, Vibration problems in engineering, D Van Nostrand Company, New York, 1937.
[57] G.P. Tilly, Dynamic behaviour of concrete structures, The University of Michigan, Michigan, 1986.
[58] S.A. Neild, P.D. Mcfadden, M.S. Williams, A discrete model of a vibrating beam using a time-stepping approach, Journal of Sound and Vibration 239/1 (2001) 99-121. DOI: https://doi.org/10.1006/jsvi.2000.3158
[59] C. Dundar, I.F. Kara, Three dimensional analysis of reinforced concrete frames with cracked beam and column elements, Engineering Structures 29/9 (2007) 2262-2273.
DOI: https://doi.org/10.1016/j.engstruct.2006.11.018
[60] C. Chan, M. Asce, Q. Wang, S.M. Asce, Nonlinear stiffness design optimization of tall reinforced concrete buildings under service loads, Journal of Structural Engineering 132/6 (2006) 978-990.
DOI: https://doi.org/10.1061/(ASCE)0733-9445(2006)132:6(978)
[61] J. Hellesland, Mechanics and slenderness limits of sway-restricted, Journal of Structural Engineering 134/8 (2008) 1300-1309.
DOI: https://doi.org/10.1061/(ASCE)0733-9445(2008)134:8(1300)
[62] M.R. Sharma, A.K. Singh, G.S. Benipal, Elastic stability of concrete beam-columns. Part I: static stability, International Journal of Structural Stability and Dynamics 17/1 (2017) 1750094. DOI: https://doi.org/10.1142/S0219455417500948
[63] M.R. Sharma, A.K. Singh, G.S. Benipal, Elastic Stability of Concrete Beam-Columns. Part II: Dynamic

Stability, International Journal of Structural Stability and Dynamics 17/1 (2017) 1750095. DOI: https://doi.org/10.1142/S021945541750095X

[64] C. Zhang, G. Gholipour, A.A. Mousavi, Nonlinear dynamic behaviour of simply-supported RC beams subjected to combined impact-blast loading, Engineering Structures 181 (2019) 124-142. DOI: https://doi.org/10.1016/j.engstruct.2018.12.014

[65] G. Nanclares, D. Ambrosini, O. Curadelli, M. Domizio, Nonlinear dynamics anlaysis of a RC bridge subjected to seismic loading, Smart Structures and Systems 26/6 (2020) 765-779. DOI: http://dx.doi.org/10.12989/sss.2020.26.6.765

[66] R. Ramamrutham, R. Narayan, Theory of Structure, Dhanpat Rai Publishing Company, Delhi, 2003.

[67] R.R. Babu, G.S. Benipal, A.K. Singh, Constitutive model for bimodular elastic damage of concrete, Latin American Journal of Solids and Structures 7/2 (2010) 143-166. DOI: https://doi.org/10.1590/S1679-78252010000200003

[68] X.T.C. Man, P.Y. Robin, L.M. McClure, Z. Wang, R.D. Finch, B.H. Jansen, Slot depth resolution in vibration signature monitoring of beams using frequency shift, Journal of the Acoustical Society of America 95/4 (1994) 2029-2037. DOI: https://doi.org/10.1121/1.408666

Appendix 1

The following are required expressions for the first two states that are elastically distinct:

State I

$$s_1 = \frac{(14.4P_1+9.6P_2)}{(6.48P_1+3.52P_2)} \;\&\; s_2 = \frac{(9.6P_1+14.4P_2)}{(3.52P_1+6.48P_2)} \tag{A.1}$$

$$\Omega = \frac{(1.45P_1^4+6.14P_1^3P_2+9.29P_1^2P_2^2+6.14P_1P_2^3+1.45P_2^4)10^3}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)} \tag{A.2}$$

$$X_1 = \frac{(0.66P_1^5+3.77P_1^4P_2+8.00P_1^3P_2^2+7.85P_1^2P_2^3+3.57P_1P_2^4+0.61P_2^5)10^5}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^2} \tag{A.3}$$

$$X_2 = \frac{(0.61P_1^5+3.57P_1^4P_2+7.85P_1^3P_2^2+8.00P_1^2P_2^3+3.77P_1P_2^4+0.66P_2^5)10^5}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^2} \tag{A.4}$$

$$F_{11} = \frac{(0.03P_1^8+0.33P_1^7P_2+1.31P_1^6P_2^2+2.85P_1^5P_2^3+3.67P_1^4P_2^4+2.85P_1^3P_2^5+1.31P_1^2P_2^6+0.33P_1P_2^7+0.03P_2^8)10^9}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^4} \tag{A.5}$$

$$F_{12} = \frac{(0.03P_1^8+0.30P_1^7P_2+1.21P_1^6P_2^2+2.63P_1^5P_2^3+3.38P_1^4P_2^4+2.63P_1^3P_2^5+1.21P_1^2P_2^6+0.30P_1P_2^7+0.03P_2^8)10^9}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^4} \tag{A.6}$$

$$F_{21} = \frac{(0.03P_1^8+0.30P_1^7P_2+1.21P_1^6P_2^2+2.63P_1^5P_2^3+3.38P_1^4P_2^4+2.63P_1^3P_2^5+1.21P_1^2P_2^6+0.30P_1P_2^7+0.03P_2^8)10^9}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^4} \tag{A.7}$$

$$F_{22} = \frac{(0.03P_1^8+0.33P_1^7P_2+1.31P_1^6P_2^2+2.85P_1^5P_2^3+3.67P_1^4P_2^4+2.85P_1^3P_2^5+1.31P_1^2P_2^6+0.33P_1P_2^7+0.03P_2^8)10^9}{6EI(22.8P_1^2+54.4P_1P_2+22.8P_2^2)^4} \tag{A.8}$$

State II

$$s_1 = \frac{(14.4P_1+9.6P_2)}{(6.48P_1+3.52P_2)} \;\&\; s_2 = \frac{(9.6P_1-25.6P_2)}{(3.52P_1-3.52P_2)} \tag{A.9}$$

$$\Omega = \frac{(1.45P_1^4+1.90P_1^3P_2-1.30P_1^2P_2^2-3.55P_1P_2^3-1.37P_2^4)10^3}{6EI\,(22.8P_1^2-10.4P_1P_2-12.4P_2^2)} \tag{A.10}$$

$$X_1 = \frac{(0.66P_1^5 - 0.02P_1^4P_2 - 1.11P_1^3P_2^2 + 0.24P_1^2P_2^3 + 0.94P_1P_2^4 + 0.30P_2^5)10^5}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^2} \quad (A.11)$$

$$X_2 = \frac{(0.58P_1^5 - 0.23P_1^4P_2 - 2.06P_1^3P_2^2 - 0.51P_1^2P_2^3 + 0.87P_1P_2^4 + 0.34P_2^5)10^5}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^2} \quad (A.12)$$

$$F_{11} = \frac{(3.44P_1^8 - 6.30P_1^7P_2 - 3.17P_1^6P_2^2 + 4.54P_1^5P_2^3 - 3.61P_1^4P_2^4 - 3.07P_1^3P_2^5 + 4.04P_1^2P_2^6 + 3.45P_1P_2^7 + 0.68P_2^8)10^7}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^4} \quad (A.13)$$

$$F_{12} = \frac{(3.03P_1^8 - 5.55P_1^7P_2 + 3.81P_1^6P_2^2 + 13.45P_1^5P_2^3 - 3.88P_1^4P_2^4 - 10.86P_1^3P_2^5 - 1.99P_1^2P_2^6 + 1.53P_1P_2^7 + 0.45P_2^8)10^7}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^4} \quad (A.14)$$

$$F_{21} = \frac{(3.03P_1^8 - 5.55P_1^7P_2 + 3.81P_1^6P_2^2 + 13.45P_1^5P_2^3 - 3.88P_1^4P_2^4 - 10.86P_1^3P_2^5 - 1.99P_1^2P_2^6 + 1.53P_1P_2^7 + 0.45P_2^8)10^7}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^4} \quad (A.15)$$

$$F_{22} = \frac{(1.57P_1^8 - 16.09P_1^7P_2 - 3.76P_1^6P_2^2 + 20.53P_1^5P_2^3 + 5.17P_1^4P_2^4 - 8.06P_1^3P_2^5 - 1.62P_1^2P_2^6 + 1.75P_1P_2^7 + 0.52P_2^8)10^7}{6EI\,(22.8P_1^2 - 10.4P_1P_2 - 12.4P_2^2)^4} \quad (A.16)$$

Note: The expression for rest of the eight elastically distinct-states can be calculated similar to the above two states.

Appendix 2

Table A1.
System parameters for different states

S No.	01	02	03	04	05	06
Parameters	F_{11} (m/N)	$F_{12} = F_{21}$ (m/N)	F_{22} (m/N)	K_{11} (N/m)	$K_{12} = K_{21}$ (N/m)	K_{22} (N/m)
R-I	1.511×10^{-6} to 1.554×10^{-6}	1.391×10^{-6} to 1.410×10^{-6}	1.509×10^{-6} to 1.605×10^{-6}	4.366×10^{6} to 2.905×10^{6}	-4.024×10^{6} to -3.202×10^{6}	4.371×10^{6} to 3.437×10^{6}
R-II	1.554×10^{-6} to 1.631×10^{-6}	1.410×10^{-6} to 1.655×10^{-6}	1.605×10^{-6} to 2.181×10^{-6}	2.905×10^{6} to 2.781×10^{6}	-3.202×10^{6} to -2.164×10^{6}	3.437×10^{6} to 2.015×10^{6}
R-III	1.631×10^{-6} to 0.755×10^{-6}	1.655×10^{-6} to 0.845×10^{-6}	2.181×10^{-6} to 1.312×10^{-6}	2.781×10^{6} to 4.701×10^{6}	-2.164×10^{6} to -3.015×10^{6}	2.015×10^{6} to 2.693×10^{6}
R-IV	0.755×10^{-6} to 1.053×10^{-6}	0.845×10^{-6} to 0.995×10^{-6}	1.312×10^{-6} to 1.398×10^{-6}	4.701×10^{6} to 3.101×10^{6}	-3.015×10^{6} to -2.125×10^{6}	2.693×10^{6} to 2.251×10^{6}
R-V	1.053×10^{-6} to 1.408×10^{-6}	0.995×10^{-6} to 1.118×10^{-6}	1.398×10^{-6} to 1.439×10^{-6}	3.101×10^{6} to 1.855×10^{6}	-2.125×10^{6} to -1.442×10^{6}	2.251×10^{6} to 1.815×10^{6}
R-VI	1.438×10^{-6} to 1.395×10^{-6}	1.121×10^{-6} to 0.991×10^{-6}	1.440×10^{-6} to 1.334×10^{-6}	1.769×10^{6} to 1.880×10^{6}	-1.378×10^{6} to -1.540×10^{6}	1.767×10^{6} to 2.006×10^{6}
R-VII	1.395×10^{-6} to 1.335×10^{-6}	0.991×10^{-6} to 0.875×10^{-6}	1.334×10^{-6} to 0.799×10^{-6}	1.880×10^{6} to 2.622×10^{6}	-1.540×10^{6} to -2.851×10^{6}	2.006×10^{6} to 4.353×10^{6}
R-VIII	1.335×10^{-6} to 2.194×10^{-6}	0.799×10^{-6} to 1.631×10^{-6}	0.799×10^{-6} to 1.631×10^{-6}	2.622×10^{6} to 2.037×10^{6}	-2.851×10^{6} to -2.082×10^{6}	4.353×10^{6} to 2.740×10^{6}
R-IX	2.194×10^{-6} to 1.883×10^{-6}	1.667×10^{-6} to 1.515×10^{-6}	1.631×10^{-6} to 1.550×10^{-6}	2.037×10^{6} to 2.401×10^{6}	-2.082×10^{6} to -2.354×10^{6}	2.740×10^{6} to 2.931×10^{6}
R-X	1.883×10^{-6} to 1.541×10^{-6}	1.515×10^{-6} to 1.395×10^{-6}	1.550×10^{-6} to 1.509×10^{-6}	2.401×10^{6} to 3.932×10^{6}	-2.354×10^{6} to -3.630×10^{6}	2.931×10^{6} to 4.013×10^{6}

S No.	07	08	09	10	11
Parameters	C_{11} $(N.s/m)$	$C_{12} = C_{21}$ $(N.s/m)$	C_{22} $(N.s/m)$	ω_1 (rad/s)	ω_2 (rad/s)
R-I	4.658×10^3 to 4.352×10^3	-2.875×10^3 to -2.057×10^3	4.085×10^3 to 3.113×10^3	117.001 to 89.225	23.001 to 22.047
R-II	4.352×10^3 to 3.959×10^3	-2.057×10^3 to -1.979×10^3	3.113×10^3 to 2.805×10^3	89.225 to 86.212	22.047 to 22.111
R-III	3.959×10^3 to 5.401×10^3	-1.979×10^3 to -2.297×10^3	2.805×10^3 to 3.188×10^3	86.212 to 102.002	22.111 to 29.276
R-IV	5.401×10^3 to 4.202×10^3	-2.297×10^3 to -1.853×10^3	3.188×10^3 to 2.972×10^3	102.002 to 89.146	29.276 to 26.558
R-V	4.202×10^3 to 3.370×10^3	-1.853×10^3 to -1.476×10^3	2.972×10^3 to 2.777×10^3	89.146 to 73.117	26.558 to 24.552
R-VI	3.297×10^3 to 3.281×10^3	-1.433×10^3 to -1.533×10^3	2.749×10^3 to 2.926×10^3	71.775 to 75.860	24.423 to 24.646
R-VII	3.281×10^3 to 3.692×10^3	-1.533×10^3 to -2.181×10^3	2.926×10^3 to 4.420×10^3	75.860 to 106.812	24.646 to 26.933
R-VIII	3.692×10^3 to 3.209×10^3	-2.181×10^3 to -1.936×10^3	4.420×10^3 to 3.370×10^3	106.812 to 87.278	26.933 to 20.246
R-IX	3.209×10^3 to 3.489×10^3	-1.936×10^3 to -2.058×10^3	3.370×10^3 to 3.455×10^3	87.278 to 92.001	20.246 to 21.420
R-X	3.489×10^3 to 4.446×10^3	-2.058×10^3 to -2.702×10^3	3.455×10^3 to 3.937×10^3	92.001 to 111.432	21.420 to 22.898

Creative Commons Attribution-NonCommercial-NoDerivatives International 4.0 BY NC ND

© 2021 by the authors. Licensee International OCSCO World Press, Gliwice, Poland. This paper is an open access paper distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International (CC BY-NC-ND 4.0) license (https://creativecommons.org/licenses/by-nc-nd/4.0/deed.en).