COMMENTATIONES MATHEMATICAE Vol. 48, No. 1 (2008), 103-112

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On existence of solutions of a quadratic Urysohn integral equation on an unbounded interval

Abstract. We show that $\omega_0(X) = \lim_{T\to\infty} \lim_{\varepsilon\to 0} \omega^T(X, \varepsilon)$ is a measure of noncompactness defined on some subsets of the space $C(\mathbb{R}_+) = \{x : \mathbb{R}_+ \to \mathbb{R},\ x \text{ continuous}\}$ furnished with the distance defined by the family of seminorms $|x|_n$. Moreover, using a technique associated with the measures of noncompactness, we prove the existence of solutions of a quadratic Urysohn integral equation on an unbounded interval. This measure allows to obtain theorems on the existence of solutions of a integral equations on an unbounded interval under a weaker assumptions then the assumptions of theorems obtained by applying two-component measures of noncompactness.

2000 Mathematics Subject Classification: Primary 45G10, Secondary 47H30.

Key words and phrases: Quadratic Urysohn integral, measure of noncompactness, Tichonov fixed point theorem.

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1. Introduction. Integral equations of various types play an important role in many branches of functional analysis and their applications in the theory of elasticity, mathematical physic and engineering (see [1, 2, 9, 10, 11, 12, 13, 15]). Many autors have investigated the existence of solutions of integral equations on an unbounded interval with the help of some two-component measures of noncompactness in the Banach space $BC(\mathbb{R}_+)$ [4, 5, 6, 7, 8, 14, 16, 17]. This approach seems to be too restrictive. In this paper, at firstly, we show that well-known mapping $\omega_0(X)$ is the measure of noncompactness on some subsets of the space $C(\mathbb{R}_+)$ consisting of all real functions defined and continuous on $\mathbb{R}_+$, equipped with the family of seminorms $|x|_n$.

Next, we investigate the problem of the existence of solutions of the quadratic Urysohn integral equation on unbounded interval having the form

$$x(t) = a(t) + f(t, x(t)) \int_0^\infty u(t, s, x(s))ds\ ,\ t \in \mathbb{R}_+ = [0, \infty)\ . \tag{1}$$

The method used in our considerations depends on the Tichonov fixed point principle and suitable conjunction of the previously mentioned the measure $\omega_0(X)$ and the space $C(\mathbb{R}_+)$.

2. Notation. For further purposes, we collect in this section a few auxiliary results which will be needed in the sequel.

Consider

$$C(\mathbb{R}_+) = \{x : \mathbb{R}_+ \to \mathbb{R},\ x \text{ continuous}\},$$

equipped with the family of seminorms $|x|_n = \sup\{|x(t)| : t \in [0,n]\},\ n \geq 1$.
$C(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$d(x,y) = \sup\{2^{-n}\frac{|x-y|_n}{1+|x-y|_n} :\ n \in \mathbb{N}\}.$$

It is known that $C(\mathbb{R}_+)$ is a locally convex space.

Let us recall two facts:

(A) a sequence (x_n) is convergent to x in $C(\mathbb{R}_+)$ if and only if (x_n) is uniformly convergent to x on compact subsets of $\mathbb{R}_+$,

(B) a family $A \subset C(\mathbb{R}_+)$ is relatively compact if and only if for each $T > 0$, the restrictions to $[0,T]$ of all functions from A form an equicontinuous and uniformly bounded set.

If X is a subset of $C(\mathbb{R}_+)$, then $\overline{X}, \mathrm{conv}X, \mathrm{Conv}X$ denote the closure, convex hull and convex closure of X, respectively. We use the symbols λX and $X+Y$ to denote the algebraic operations on sets.

The family of all nonempty subsets of $C(\mathbb{R}_+)$ consisting of functions uniformly bounded on $\mathbb{R}_+$ will be denoted by $\mathfrak{M}_C$, i.e.

$$\mathfrak{M}_C = \{X \subset C(\mathbb{R}_+) : X \neq \emptyset \text{ and } \sup\{|x(t)| : x \in X,\ t \geq 0\} < \infty\},$$

while subfamily of $\mathfrak{M}_C$ consisting of all relatively compact sets is denoted by $\mathfrak{N}_C$.

Now, we recall the definition of quantities which will be used in our further investigations. These ones was introduced and studied in [3]. Let $X \in \mathfrak{M}_C$. Fix $T > 0,\ \varepsilon > 0$. Let us denote.

$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| :\ t,s \in [0,T],\ |t-s| \leq \varepsilon\}\ .$$

Further, let us put:

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}\ ,$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X,\varepsilon)\ ,$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X)\ .$$

Let us observe that mapping $\omega_0 : \mathfrak{M}_C \to \mathbb{R}_+$ can be called a *measure of noncompactness* (see [3]) because it satisfies the following conditions:

1^o the family $\ker \omega_0 = \{X \in \mathfrak{M}_C : \omega_0(X) = 0\} = \mathfrak{N}_C$,

2^o $X \subset Y \Rightarrow \omega_0(X) \le \omega_0(Y)$,

3^o $\omega_0(\overline{X}) = \omega_0(\mathrm{Conv} X) = \omega_0(X)$,

4^o $\omega_0(\lambda X + (1-\lambda)Y) \le \lambda\omega_0(X) + (1-\lambda)\omega_0(Y)$ for $\lambda \in [0,1]$,

5^o If (X_n) is a sequence of closed sets from $\mathfrak{M}_C$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, ...)$ and if $\lim\limits_{n\to\infty} \omega_0(X_n) = 0$ then the set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n$ is nonempty.

The condition 1^o is immediately consequence of (B), 2^o is obvious. We will prove 3^o.
Let us fix $X \in \mathfrak{M}_C$ and take $x \in \mathrm{conv}X$. Then $x = \sum_{i=1}^n \alpha_i x_i$ where $x_1, ..., x_n \in X$, $\sum_{i=1}^n \alpha_i = 1$, $\alpha_1, ..., \alpha_n \ge 0$. Hence we get

$$\omega^T(x,\varepsilon) \le \sum_{i=1}^n \alpha_i \omega^T(x_i,\varepsilon) \le \omega^T(X,\varepsilon),$$

$$\omega^T(\mathrm{conv}X,\varepsilon) = \omega^T(X,\varepsilon) \text{ and } \omega_0(\mathrm{conv}X) = \omega_0(X).$$

Further, taking $x \in \overline{X}$ we obtain that there are $x_n \in X$ such that $x_n \to x$ in $C(\mathbb{R}_+)$. Fix $\delta > 0$, there exists n_0, such that

$$|x_n(t) - x(t)| \le \delta \text{ for } n \ge n_0,\ t \in [0,T].$$

Let us notice that

$$|x(t) - x(s)| \le 2\delta + |x_n(t) - x_n(s)| \text{ for } n \ge n_0,$$

Hence we get

$$\omega^T(x,\varepsilon) \le 2\delta + \omega^T(x_n,\varepsilon) \le 2\delta + \omega^T(X,\varepsilon),$$

$$\omega^T(\overline{X},\varepsilon) \le 2\delta + \omega^T(X,\varepsilon),\ \omega^T(\overline{X},\varepsilon) = \omega^T(X,\varepsilon) \text{ and } \omega_0(\overline{X}) = \omega_0(X).$$

Linking the above equalities we obtain

$$\omega_0(\mathrm{Conv}X) = \omega_0(\overline{\mathrm{conv}X}) = \omega_0(\mathrm{conv}X) = \omega_0(X).$$

The proof of condition 4^o is similar and will be omitted.

Now we will prove 5^o. Let us take arbitrary element $x_n \in X_n$. Applying the diagonal method we can choose a subsequence $(x_{1,n})$ of the sequence (x_n) such that $(x_{1,n})$ is convergence on $[0,1] \cap \mathbb{Q}$. For proving of uniformly convergence of $(x_{1,n})$ on $[0,1]$ it is enough to show that $(x_{1,n})$ satisfies Cauchy's condition. Let $\varepsilon > 0$. In virtue of $\lim\limits_{n\to\infty} \omega_0(\{x_{1,n}, x_{1,n+1}, ...\}) = 0$ we derive that there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that if $|t - s| \le \delta,\ t, s \in [0,1]$ then

$$|x_{1,n}(t) - x_{1,n}(s)| \le \frac{\varepsilon}{3} \text{ for } n \ge n_0.$$

Now we put $q_i = \frac{i}{p},\ i = 0, ..., p$ where p is so large that $|q_i - q_{i-1}| \le \delta,\ i = 1, ..., p$.

The convergence of $(x_{1,n})$ on the set $\{q_0, ..., q_p\}$ implies that there exists $n_1 \in \mathbb{N}$ $(n_1 \geq n_0)$ that

$$|x_{1,n}(q_i) - x_{1,m}(q_i)| \leq \frac{\varepsilon}{3} \text{ for } i = 0, ..., p \text{ and } n, m \geq n_1.$$

Let us observe that for arbitrary $t \in [0,1]$ exists $i \leq p$, such that $|t - q_i| \leq \delta$. Linking this facts, we derive the estimate:

$$|x_{1,n}(t) - x_{1,m}(t)| \leq |x_{1,n}(t) - x_{1,n}(q_i)| + |x_{1,n}(q_i) - x_{1,m}(q_i)| + |x_{1,m}(q_i) - x_{1,m}(t)| \leq$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ for } n, m \geq n_1.$$

This shows that $(x_{1,n})$ is uniformly convergent on $[0,1]$. Repeating this reasoning for $[0,k]$ we obtain that there is a subsequence $(x_{k,n})$ of the sequence $(x_{k-1,n})$ such that $(x_{k,n})$ is uniformly convergent on $[0,k]$. Finally, putting $x_n = x_{n,n}$ we infer that (x_n) is uniformly convergent to some $x \in C(\mathbb{R}_+)$ on a compact subsets of $\mathbb{R}_+$, and in virtue of closedness of X_n and (A) we obtain $x \in \bigcap_{i=1}^{\infty} X_n \neq \emptyset$.

3. Main result. Now we will study the existence of solutions of the quadratic Urysohn integral equation (1). Our considerations are situated in the Fréchet space $C(\mathbb{R}_+)$ described in the previous part.

We will consider Eq. (1) under the following assumptions:

(i) $a : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function,

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function,

(iii) the function f satisfies the Lipschitz condition with respect to the second variable i.e. there exists a continuous function $k(t) > 0$ such that

$$|f(t,x) - f(t,y)| \leq k(t)|x - y|$$

for $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$,

(iv) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t,s,x)| \leq g(t,s)h(|x|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

(v) for every $t \geq 0$ the function $s \to g(t,s)$ is integrable on $\mathbb{R}_+$ and the function $t \to \int\limits_0^\infty g(t,s)ds$ is locally bounded on $\mathbb{R}_+$ i.e.

$$\forall_{T>0} \sup_{t \in [0,T]} \int\limits_0^\infty g(t,s)ds < \infty,$$

(vi) the improper integral $\int_0^\infty g(t,s)ds$ is locally uniformly convergent with respect to t i.e.

$$\forall_{\delta>0}\ \forall_{T>0}\ \exists_{S>0}\ \sup_{t\in[0,T]}\int_S^\infty g(t,s)ds<\delta,$$

(vii) the inequality

$$\sup_{t\ge 0}|a(t)|+\Big(r\sup_{t\ge 0}k(t)\int_0^\infty g(t,s)ds+\sup_{t\ge 0}|f(t,0)|\int_0^\infty g(t,s)ds\Big)h(r)\le r$$

has a positive solution r_0 such that

$$h(r_0)\sup_{t\ge 0}k(t)\int_0^\infty g(t,s)ds<1.$$

Remark 3.1 Notice that the condition $h(r_0)\sup_{t\ge 0}k(t)\int_0^\infty g(t,s)ds<1$ is satisfied provided r_0 satisfies the inequality from (vii) and the function $a(t)$ or $t\to f(t,0)$ do not vanish on $\mathbb{R}$.

Now we can formulate our result which generalizes and completes the results obtained earlier in some papers [4, 14].

Theorem 3.2 *Under assumptions (i)-(vii) equation (1) has at least one solution $x=x(t)$ in the space $C(\mathbb{R}_+)$.*

Proof Let $r_0>0$ be a number satisfying the assumption (vii) and define a set

$$B=\{x\in C(\mathbb{R}_+):\ \sup_{t\ge 0}|x(t)|\le r_0\}.$$

Consider the operator U defined on B by the formula

$$(Ux)(t)=a(t)+f(t,x(t))\int_0^\infty u(t,s,x(s))ds,\ t\ge 0\ .$$

At first we show that the function Ux is continuous on $\mathbb{R}_+$.

To do this fix arbitrarily $x\in B,\ T>0$ and $\varepsilon>0$. Next, take arbitrary numbers $t,s\in[0,T]$ such that $|t-s|\le\varepsilon$. Moreover, let $\delta>0$ and $S>0$. Then, keeping in mind our assumptions, we obtain:

$$|(Ux)(t)-(Ux)(s)|\le|a(t)-a(s)|+$$

$$+\left|f(t,x(t))\int_0^\infty u(t,\tau,x(\tau))d\tau-f(s,x(s))\int_0^\infty u(t,\tau,x(\tau))d\tau\right|+$$

$$+\left|f(s,x(s))\int_0^\infty u(t,\tau,x(\tau))d\tau-f(s,x(s))\int_0^\infty u(s,\tau,x(\tau))d\tau\right|\le$$

$$\leq \omega^T(a,\varepsilon) + |f(t,x(t)) - f(s,x(s))| \int\limits_0^\infty |u(t,\tau,x(\tau))| d\tau +$$

$$+ |f(s,x(s))| \int\limits_0^\infty |u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))| d\tau \leq$$

$$\leq \omega^T(a,\varepsilon) + [|f(t,x(t)) - f(t,x(s))| + |f(t,x(s)) - f(s,x(s))|] \int\limits_0^\infty g(t,\tau) h(|x(\tau)|) d\tau +$$

$$+ [|f(s,x(s)) - f(s,0)| + |f(s,0)|] \int\limits_0^\infty |u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))| d\tau \leq$$

$$\leq \omega^T(a,\varepsilon) + \left[k(t)|x(t) - x(s)| + \omega^T_{r_0}(f,\varepsilon)\right] h(r_0) \int\limits_0^\infty g(t,\tau) d\tau +$$

$$+ [k(s)|x(s)| + |f(s,0)|] \int\limits_0^S |u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))| d\tau +$$

$$(2) \qquad +[k(s)|x(s)| + |f(s,0)|] h(r_0) \Big(\int_S^\infty g(t,\tau) d\tau + \int_S^\infty g(s,\tau) d\tau \Big) ,$$

where we denoted

$$\omega^T_d(f,\varepsilon) = \sup\{|f(t,y) - f(s,y)| : \ t,s \in [0,T], \ y \in [-d,d], \ |t-s| \leq \varepsilon\} .$$

Keeping in mind the assumption (vi) we deduce that there exists S so large that the last term of inequality (2) is less then δ i.e.

$$|(Ux)(t) - (Ux)(s)| \leq \omega^T(a,\varepsilon) + \left[k(t)|x(t) - x(s)| + \omega^T_{r_0}(f,\varepsilon)\right] h(r_0) \int\limits_0^\infty g(t,\tau) d\tau +$$

$$(3) \qquad +[k(s)|x(s)| + |f(s,0)|] \int\limits_0^S |u(t,\tau,x(\tau)) - u(s,\tau,x(\tau))| d\tau + \delta .$$

Now, from the above estimate we get:

$$|(Ux)(t) - (Ux)(s)| \leq \omega^T(a,\varepsilon) + k(t)\omega^T(x,\varepsilon) h(r_0) \int\limits_0^\infty g(t,\tau) d\tau +$$

$$(4) \qquad + \omega^T_{r_0}(f,\varepsilon)h(r_0)\int\limits_0^\infty g(t,\tau)d\tau + \sup_{s\le T}(k(s)r_0 + |f(s,0)|)S\omega^{T,S}_{r_0}(u,\varepsilon) + \delta,$$

where, similarly as above, we denoted

$$\omega^{T,S}_d(u,\varepsilon) = \sup\{|u(t,\tau,y) - u(s,\tau,y)| : \ t,s\in[0,T],\ \tau\in[0,S],\ |t-s|\le\varepsilon,\ y\in[-d,d]\}\ .$$

Let us notice that $\omega^T_{r_0}(f,\varepsilon)\to 0$ and $\omega^{T,S}_{r_0}(u,\varepsilon)\to 0$ as $\varepsilon\to 0$, which is a consequence of the uniform continuity of the function f on the set $[0,T]\times[-r_0,r_0]$ and the function u on the set $[0,T]\times[0,S]\times[-r_0,r_0]$, respectively.

Further observe that in virtue of assumption (v) $\sup\limits_{t\in[0,T]}\int\limits_0^\infty g(t,\tau)d\tau < \infty$. Hence, taking into account the facts established above and free choice of $\delta > 0$ we infer that the function Ux is continuous on the interval $[0,T]$ for any $T > 0$. This implies that Ux is continuous on the whole interval $\mathbb{R}_+$.

Now we show that the mapping U transforms B into itself. In fact, using our assumptions for arbitrarily fixed $t\in\mathbb{R}_+$ we have:

$$|(Ux)(x)| \le |a(t)| + |f(t,x(t))|\int\limits_0^\infty |u(t,s,x(s))|ds \le$$

$$\le |a(t)| + [|f(t,x(t)) - f(t,0)| + |f(t,0)|]\int\limits_0^\infty g(t,s)h(|x(s)|)ds \le$$

$$\le |a(t)| + h(r_0)r_0k(t)\int\limits_0^\infty g(t,\tau)d\tau + h(r_0)|f(t,0)|\int\limits_0^\infty g(t,\tau)d\tau\ .$$

Hence, keeping in mind (vii), we get

$$|(Ux)(t)| \le \sup_{t\ge 0}|a(t)| + h(r_0)r_0\sup_{t\ge 0}k(t)\int\limits_0^\infty g(t,\tau)d\tau + h(r_0)\sup_{t\ge 0}|f(t,0)|\int\limits_0^\infty g(t,\tau)d\tau \le r_0,$$

which means that the mapping U transforms B into itself.

In what follows let us take a nonempty subset X of the set B. Fix $\varepsilon > 0$ and $T > 0$ and take an arbitrary function $x\in X$. Then, using the estimate (4) we obtain:

$$\omega^T(Ux,\varepsilon) \le \omega^T(a,\varepsilon) + k(t)\omega^T(x,\varepsilon)h(r_0)\int\limits_0^\infty g(t,\tau)d\tau +$$

$$+ \omega^T_{r_0}(f,\varepsilon)h(r_0)\int\limits_0^\infty g(t,\tau)d\tau + \sup_{s\le T}(k(s)r_0 + |f(s,0)|)S\omega^{T,S}_{r_0}(u,\varepsilon) + \delta.$$

Hence we get

$$\omega^T(UX,\varepsilon) \le \omega^T(a,\varepsilon) + k(t)\omega^T(X,\varepsilon)h(r_0)\int_0^\infty g(t,\tau)d\tau +$$

$$+ \omega^T_{r_0}(f,\varepsilon)h(r_0)\int_0^\infty g(t,\tau)d\tau + \sup_{s\le T}(k(s)r_0 + |f(s,0)|)S\omega^{T,S}_{r_0}(u,\varepsilon) + \delta,$$

Now, taking into account the properties of the components involved in the above inequality, we have:

$$\omega^T_0(UX) \le h(r_0)\sup_{t\le T} k(t)\int_0^\infty g(t,\tau)d\tau \cdot \omega^T_0(X),$$

$$\omega_0(UX) \le h(r_0)\sup_{t\ge 0} k(t)\int_0^\infty g(t,\tau)d\tau \cdot \omega_0(X). \tag{5}$$

Next, let us consider the sequence of sets (B^n), where $B^1 = \mathrm{Conv}U(B)$, $B^2 = \mathrm{Conv}U(B^1)$ and so on. Observe that all sets of this sequence are nonempty, closed and convex. Moreover, $B^{n+1} \subset B^n$ for $n = 1,2,....$ Further, keeping in mind (5) we get

$$\mu(B^n) \le q^n\mu(B) \ , \tag{6}$$

where we put $q = h(r_0)\sup_{t\ge 0} k(t)\int\limits_0^\infty g(t,\tau)d\tau$. Obviously, in view of (vii) we have that $q < 1$. Apart from this we can calculate that $\omega_0(B) = 2r_0$. In virtue of (6) this implies that $\lim\limits_{n\to\infty} \omega_0(B^n) = 0$. Thus, from the condition 5^o we infer that the set $Y = \bigcap\limits_{n=1}^{\infty} B^n$ is nonempty, closed and convex. Moreover, we deduce that $Y \in \ker\omega_0$. It should be also noted that the operator U maps the set Y into itself.

Now we show that U is continuous on the set B.

To do this fix $x \in B$ and take functions $x_n \in B$ such that $x_n \to x$ in $C(\mathbb{R}_+)$. We will show $Ux_n \to Ux$ in $C(\mathbb{R}_+)$. Firstly, let us observe

$$|(Ux)(t) - (Ux_n)(t)| \le |f(t,x(t)) - f(t,x_n(t))|\int_0^\infty |u(t,s,x(s))|ds +$$

$$+ |f(t,x_n(t))|\int_0^\infty |u(t,s,x(s)) - u(t,s,x_n(s))|ds \le k(t)|x(t) - x_n(t)|h(r_0)\int_0^\infty g(t,s)ds +$$

$$+|f(t,x_n(t))|\int_0^\infty |u(t,s,x(s)) - u(t,s,x_n(s))|ds. \tag{7}$$

Fix $T > 0$. In virtue of (A) it is enough to show that $|(Ux)(t) - (Ux_n)(t)| \to 0$ uniformly on $[0,T]$ for $n \to \infty$. Using the assumption (v) we have $\sup\limits_{t\le T} k(t) \int\limits_0^\infty g(t,s)ds < \infty$.
This implies that the first term of the inequality (7) tends to 0 uniformly on $[0,T]$ for $n \to \infty$. Next, lest us observe that

$$\sup_{t\le T} |f(t,x_n(t))| \le \sup_{t\le T}[k(t)r_0 + |f(t,0)|] < \infty.$$

It is enough to show that $\int\limits_0^\infty |u(t,s,x(s)) - u(t,s,x_n(s))|ds \to 0$ uniformly on $[0,T]$.
Let $S > 0$ and $\delta > 0$.

$$\int_0^\infty |u(t,s,x(s)) - u(t,s,x_n(s))|ds \le$$

$$\le \int_0^S |u(t,s,x(s)) - u(t,s,x_n(s))|ds + 2h(r_0)\int_S^\infty g(t,s)ds. \tag{8}$$

Hence, in view of the assumption (vi) we can find S so big that the last term of (8) is less then $\frac{\delta}{2}$ for $t \le T$. Moreover

$$\int_0^S |u(t,s,x(s)) - u(t,s,x_n(s))|ds \le S\omega_{r_0}^{T,S}(u, \sup_{s\le S}|x(s) - x_n(s)|), \tag{9}$$

where

$\omega_d^{T,S}(u,\varepsilon) = \sup\{|u(t,s,x) - u(t,s,y)| :\ t \in [0,T],\ s \in [0,S],\ |x-y| \le \varepsilon,\ x,y \in [-d,d]\}$.

The convergence of (x_n) to x in $C(\mathbb{R}_+)$ implies $\lim\limits_{n\to\infty} \sup\limits_{s\le S} |x(s) - x_n(s)| = 0$. Combining this fact, (9) and the uniform continuity of the function u on the set $[0,T] \times [0,S] \times [-r_0, r_0]$ we infer that

$$\sup_{t\le T} \int_0^S |u(t,s,x(s)) - u(t,s,x_n(s))|ds \le \frac{\delta}{2} \text{ for } n \text{ sufficiently big.}$$

Using above inequalities we obtain

$$\sup_{t\le T} \int_0^\infty |u(t,s,x(s)) - u(t,s,x_n(s))|ds \le \delta \text{ for } n \text{ sufficiently big.}$$

This ends the proof of continuity the mapping $U : B \to B$.

Finally, linking all above established properties of the set Y and the operator $U : Y \to Y$ and using the Tichonov fixed point principle we infer that the operator U has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of the integral equation (1). This completes the proof. ∎

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(Received: 22.12.2007)