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A Generalization of a Theorem of Móricz and Rhoades on Weighted Means

Abstract. In this paper,we prove a theorem which gives an equivalent formulation of summability by weighted mean methods. The result of Hardy [1] and that of Móricz and Rhoades [2] are special cases of this theorem. In this context, it is important to note that the result of Móricz and Rhoades is valid even without the assumption $\frac{p_n}{P_n} \to 0$ as $n \to \infty$.

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1. Introduction. For the sake of completeness, we recall the following. Given an infinite matrix $A = (a_{nk})$, $n, k = 0, 1, 2, \ldots$ and a sequence $x = \{x_k\}$, $k = 0, 1, 2, \ldots$, by the A-transform of $x = \{x_k\}$, we mean the sequence $Ax = \{(Ax)_n\}$,

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \ldots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = \ell$, we say that $x = \{x_k\}$ is A-summable or summable A to ℓ. If $\lim_{n\to\infty} (Ax)_n = \ell$, whenever $\lim_{k\to\infty} x_k = \ell$, we say that A is regular. The following theorem is well-known (see [3], Theorem II.1, pp. 11-12).

Theorem 1.1 *$A = (a_{nk})$ is regular if and only if*

(i) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii) $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \ldots$; *and*

(iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$.

An infinite series $\sum_{k=0}^{\infty} x_k$ *is said to be A-summable to* ℓ *if* $\{s_n\}$ *is A-summable to* ℓ, *where* $s_n = \sum_{k=0}^{n} x_k$, $n = 0, 1, 2, \dots$.

2. Weighted Means.

DEFINITION 2.1 ([3], p.16) The weighted mean method or $(\bar{N}, p_n)$ method is defined by the infinite matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} \frac{p_k}{P_n}, & k \leqslant n; \\ 0, & k > n, \end{cases}$$

$P_n = \sum_{k=0}^{n} p_k$, $n = 0, 1, 2, \dots$, $P_n \neq 0$, $n = 0, 1, 2, \dots$.

THEOREM 2.2 ([3], p.16) *The weighted mean method* $(\bar{N}, p_n)$ *is regular if and only if*

(i) $\sum_{k=0}^{n} |p_k| = O(P_n)$, $n \to \infty$;
and

(ii) $P_n \to \infty$, $n \to \infty$.

REMARK 2.3 $|P_n| \leqslant \sum_{k=0}^{n} |p_k| \leqslant \sum_{k=0}^{n+m} |p_k| \leqslant L|P_{n+m}|$ for some $L > 0$, $m = 0, 1, 2, \dots$; $n = 0, 1, 2, \dots$.

3. Main Result. We now prove the main result, which gives an equivalent formulation of summability by weighted mean methods.

THEOREM 3.1 *Let* $(\bar{N}, p_n)$, $(\bar{N}, q_n)$ *be two regular weighted mean methods. For a given infinite series* $\sum_{k=0}^{\infty} x_k$, *let*

$$b_n = q_n \sum_{k=n}^{\infty} \frac{x_k}{Q_k}, \quad n = 0, 1, 2, \dots.$$

Let $\sum_{n=0}^{\infty} b_n$ *converge to* ℓ*. Then* $\sum_{k=0}^{\infty} x_k$ *is* $(\bar{N}, p_n)$ *summable to* ℓ *if and only if*

$$\sup_n \left[\frac{1}{|P_n|} \sum_{k=1}^{n} \left| \frac{p_k Q_{k+1}}{q_{k+1}} - \frac{p_{k-1} Q_{k-1}}{q_k} \right| \right] < \infty.$$

Proof Let $B_n = \sum_{k=0}^{n} b_k \to \ell$, $n \to \infty$. Now,

$$\begin{aligned} \frac{b_n}{q_n} - \frac{b_{n+1}}{q_{n+1}} &= \sum_{k=n}^{\infty} \frac{x_k}{Q_k} - \sum_{k=n+1}^{\infty} \frac{x_k}{Q_k} \\ &= \frac{x_n}{Q_n}, \end{aligned}$$

so that

$$x_n = Q_n \left(\frac{b_n}{q_n} - \frac{b_{n+1}}{q_{n+1}} \right), \quad n = 0, 1, 2, \ldots.$$

Consequently

$$\begin{aligned} s_m &= \sum_{k=0}^{m} x_k \\ &= \sum_{k=0}^{m} Q_k \left(\frac{b_k}{q_k} - \frac{b_{k+1}}{q_{k+1}} \right) \\ &= \sum_{k=0}^{m} Q_k \frac{b_k}{q_k} - \sum_{k=1}^{m+1} Q_{k-1} \frac{b_k}{q_k} \\ &= Q_0 \frac{b_0}{q_0} + \sum_{k=1}^{m} (Q_k - Q_{k-1}) \frac{b_k}{q_k} - Q_m \frac{b_{m+1}}{q_{m+1}} \\ &= b_0 + \sum_{k=1}^{m} q_k \frac{b_k}{q_k} - Q_m \frac{b_{m+1}}{q_{m+1}} \\ &= b_0 + \sum_{k=1}^{m} b_k - Q_m \frac{b_{m+1}}{q_{m+1}} \\ &= \sum_{k=0}^{m} b_k - Q_m \frac{b_{m+1}}{q_{m+1}} \\ &= B_m - Q_m \frac{b_{m+1}}{q_{m+1}}. \end{aligned} \tag{1}$$

By hypothesis $\sum_{k=0}^{\infty} \frac{x_k}{Q_k}$ converges so that

$$\frac{b_n}{q_n} = \sum_{k=n}^{\infty} \frac{x_k}{Q_k} \to 0, n \to \infty.$$

Now,

$$\frac{s_m}{Q_m} = \frac{B_m}{Q_m} - \frac{b_{m+1}}{q_{m+1}}, \quad \text{using (1).}$$

Since $\{B_n\}$ converges, it is bounded so that for some $M > 0$, $|B_n| \leqslant M$, $n = 0, 1, 2, \ldots$. Since $(\bar{N}, q_n)$ is regular, $|Q_n| \to \infty$, $n \to \infty$ so that

$$\left|\frac{B_m}{Q_m}\right| \leqslant \frac{M}{|Q_m|} \to 0, m \to \infty.$$

Thus

$$\frac{s_m}{Q_m} \to 0, m \to \infty.$$

For $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
b_n &= q_n \sum_{k=n}^{\infty} \frac{x_k}{Q_k} \\
&= q_n \lim_{m\to\infty} \sum_{k=n}^{m} \frac{s_k - s_{k-1}}{Q_k} \quad \text{(where } s_{-1} = 0\text{)} \\
&= q_n \lim_{m\to\infty} \left\{ \sum_{k=n}^{m} \frac{s_k}{Q_k} - \sum_{k=n-1}^{m-1} \frac{s_k}{Q_{k+1}} \right\} \\
&= q_n \lim_{m\to\infty} \left\{ \sum_{k=n}^{m-1} \frac{s_k}{Q_k} + \frac{s_m}{Q_m} - \sum_{k=n}^{m-1} \frac{s_k}{Q_{k+1}} - \frac{s_{n-1}}{Q_n} \right\} \\
&= q_n \lim_{m\to\infty} \left\{ \sum_{k=n}^{m-1} \left(\frac{1}{Q_k} - \frac{1}{Q_{k+1}} \right) s_k + \frac{s_m}{Q_m} - \frac{s_{n-1}}{Q_n} \right\} \\
&= -q_n \frac{s_{n-1}}{Q_n} + q_n \sum_{k=n}^{\infty} \left(\frac{1}{Q_k} - \frac{1}{Q_{k+1}} \right) s_k, \quad \text{since } \lim_{m\to\infty} \frac{s_m}{Q_m} = 0 \\
&= -q_n \frac{s_{n-1}}{Q_n} + q_n \sum_{k=n}^{\infty} c_k s_k, \quad \text{where } c_k = \frac{1}{Q_k} - \frac{1}{Q_{k+1}}, \\
&\qquad k = 0, 1, 2, \ldots.
\end{aligned} \tag{2}$$

Now,

$$
\begin{aligned}
B_n &= \sum_{k=0}^{n-1} b_k + b_n \\
&= \sum_{k=0}^{n-1} \frac{b_k}{q_k} q_k + b_n \\
&= \sum_{k=0}^{n-1} q_k \left(\sum_{u=k}^{\infty} \frac{x_u}{Q_u} \right) + b_n \\
&= q_0 \sum_{u=0}^{\infty} \frac{x_u}{Q_u} + q_1 \sum_{u=1}^{\infty} \frac{x_u}{Q_u} + q_2 \sum_{u=2}^{\infty} \frac{x_u}{Q_u} + \cdots + q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + b_n \\
&= (q_0 + q_1 + \cdots + q_{n-1}) \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + q_0 \sum_{u=0}^{n-2} \frac{x_u}{Q_u} + q_1 \sum_{u=1}^{n-2} \frac{x_u}{Q_u} \\
&\qquad + q_2 \sum_{u=2}^{n-2} \frac{x_u}{Q_u} + \cdots + q_{n-2} \frac{x_{n-2}}{Q_{n-2}} + b_n \\
&= Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + b_n + \frac{x_{n-2}}{Q_{n-2}} Q_{n-2} + \frac{x_{n-3}}{Q_{n-3}} Q_{n-3} + \cdots + \frac{x_0}{Q_0} Q_0 \\
&= Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} + b_n + \sum_{k=0}^{n-2} x_k \\
&= s_{n-2} + b_n + Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} \\
&= s_{n-2} + q_n \sum_{u=n}^{\infty} \frac{x_u}{Q_u} + Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} \\
&= s_{n-2} + (Q_n - Q_{n-1}) \sum_{u=n}^{\infty} \frac{x_u}{Q_u} + Q_{n-1} \sum_{u=n-1}^{\infty} \frac{x_u}{Q_u} \\
&= s_{n-2} + Q_n \sum_{u=n}^{\infty} \frac{x_u}{Q_u} + Q_{n-1} \frac{x_{n-1}}{Q_{n-1}} \\
&= s_{n-1} + Q_n \sum_{u=n}^{\infty} \frac{x_u}{Q_u} \\
&= s_{n-1} + Q_n \frac{b_n}{q_n} \\
&= s_{n-1} + Q_n \left[-\frac{s_{n-1}}{Q_n} + \sum_{k=n}^{\infty} c_k s_k \right], \quad \text{using (2)} \\
&= Q_n \sum_{k=n}^{\infty} c_k s_k,
\end{aligned}
$$

so that

$$\frac{B_n}{Q_n} = \sum_{k=n}^{\infty} c_k s_k.$$

Consequently

$$c_n s_n = \frac{B_n}{Q_n} - \frac{B_{n+1}}{Q_{n+1}}, \quad n = 0, 1, 2, \ldots. \tag{3}$$

Let $\{T_n\}$ be the $(\bar{N}, p_n)$ transform of $\{s_k\}$ so that

$$\begin{aligned}
T_n &= \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \\
&= \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{1}{c_k} \left\{ \frac{B_k}{Q_k} - \frac{B_{k+1}}{Q_{k+1}} \right\}, \quad \text{using (3)} \\
&= \frac{1}{P_n} \left[\frac{p_0}{c_0} \frac{B_0}{Q_0} + \sum_{k=1}^{n} \left\{ \frac{p_k}{c_k} - \frac{p_{k-1}}{c_{k-1}} \right\} \frac{B_k}{Q_k} - \frac{p_n}{c_n} \frac{B_{n+1}}{Q_{n+1}} \right] \\
&= \sum_{k=0}^{\infty} a_{nk} B_k,
\end{aligned}$$

where

$$a_{nk} = \begin{cases} \frac{1}{P_n} \frac{p_0}{c_0 Q_0}, & k = 0; \\ \frac{1}{P_n} \left\{ \frac{p_k}{c_k} - \frac{p_{k-1}}{c_{k-1}} \right\} \frac{1}{Q_k}, & 1 \leqslant k \leqslant n; \\ -\frac{1}{P_n} \frac{p_n}{c_n Q_{n+1}}, & k = n+1; \\ 0, & k \geqslant n+2. \end{cases}$$

Note that $\lim_{n\to\infty} a_{nk} = 0$, $k = 0, 1, 2, \ldots$. Also,

$$
\begin{aligned}
\sum_{k=0}^{\infty} a_{nk} &= \sum_{k=0}^{n+1} a_{nk} \\
&= \frac{1}{P_n}\left[\frac{p_0}{c_0 Q_0} + \sum_{k=1}^{n}\left\{\frac{p_k}{c_k} - \frac{p_{k-1}}{c_{k-1}}\right\}\frac{1}{Q_k} - \frac{p_n}{c_n Q_{n+1}}\right] \\
&= \frac{1}{P_n}\left[\frac{p_0}{c_0 Q_0} + \left(\frac{p_1}{c_1} - \frac{p_0}{c_0}\right)\frac{1}{Q_1} + \left(\frac{p_2}{c_2} - \frac{p_1}{c_1}\right)\frac{1}{Q_2}\right. \\
&\qquad \left. + \cdots + \left(\frac{p_n}{c_n} - \frac{p_{n-1}}{c_{n-1}}\right)\frac{1}{Q_n} - \frac{p_n}{c_n Q_{n+1}}\right] \\
&= \frac{1}{P_n}\left[\frac{p_0}{c_0}\left(\frac{1}{Q_0} - \frac{1}{Q_1}\right) + \frac{p_1}{c_1}\left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)\right. \\
&\qquad \left. + \frac{p_2}{c_2}\left(\frac{1}{Q_2} - \frac{1}{Q_3}\right) + \cdots + \frac{p_n}{c_n}\left(\frac{1}{Q_n} - \frac{1}{Q_{n+1}}\right)\right] \\
&= \frac{1}{P_n}\left[\frac{p_0}{c_0}c_0 + \frac{p_1}{c_1}c_1 + \cdots + \frac{p_n}{c_n}c_n\right] \\
&= \frac{1}{P_n}(p_0 + p_1 + \cdots + p_n) \\
&= \frac{1}{P_n}P_n \\
&= 1, \quad n = 0, 1, 2, \ldots,
\end{aligned}
$$

so that $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$. By hypothesis, $B_k \to \ell$, $k \to \infty$. In view of Theorem 1.1, $T_n \to \ell$, $n \to \infty$, i.e., $\sum_{n=0}^{\infty} x_n$ is $(\bar{N}, p_n)$ summable to ℓ if and only if

$$
\sup_n \frac{1}{|P_n|}\left[\left|\frac{p_0}{c_0 Q_0}\right| + \left|\frac{p_n}{c_n Q_{n+1}}\right| + \sum_{k=1}^{n}\left|\frac{1}{Q_k}\left\{\frac{p_k}{c_k} - \frac{p_{k-1}}{c_{k-1}}\right\}\right|\right] < \infty. \tag{4}
$$

Using Remark 2.3,

$$
\begin{aligned}
&|P_n| \leqslant L|P_{n+m}|, |Q_n| \leqslant L|Q_{n+m}|, \text{ for some } L > 0, m = 0, 1, 2, \ldots; \\
&\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, 2, \ldots.
\end{aligned} \tag{5}
$$

However,

$$
\begin{aligned}
\left|\frac{p_n}{P_n c_n Q_{n+1}}\right| &\leqslant L\left|\frac{p_n}{P_n c_n Q_n}\right|, \quad \text{using (5)}\\
&= \frac{L}{|P_n Q_n|}\left|\frac{p_n}{c_n}\right|\\
&= \frac{L}{|P_n Q_n|}\left|\sum_{k=1}^{n}\left(\frac{p_k}{c_k}-\frac{p_{k-1}}{c_{k-1}}\right)+\frac{p_0}{c_0}\right|\\
&\leqslant \frac{L^2}{|P_n|}\left|\sum_{k=1}^{n}\frac{1}{Q_k}\left(\frac{p_k}{c_k}-\frac{p_{k-1}}{c_{k-1}}\right)+\frac{p_0}{c_0 Q_0}\right|,\\
&\qquad \text{since } |Q_k| \leqslant L|Q_n|, k \leqslant n, \text{ using (5) again}\\
&\leqslant \frac{L^2}{|P_n|}\left[\sum_{k=1}^{n}\left|\frac{1}{Q_k}\left(\frac{p_k}{c_k}-\frac{p_{k-1}}{c_{k-1}}\right)\right|+\left|\frac{p_0}{c_0 Q_0}\right|\right].
\end{aligned}
$$

Thus (4) is equivalent to

$$
\sup_n \frac{1}{|P_n|}\left[\sum_{k=1}^{n}\left|\frac{1}{Q_k}\left\{\frac{p_k}{c_k}-\frac{p_{k-1}}{c_{k-1}}\right\}\right|\right] < \infty. \tag{6}
$$

Now,

$$
\begin{aligned}
\frac{p_k}{c_k}-\frac{p_{k-1}}{c_{k-1}} &= \frac{p_k}{\frac{1}{Q_k}-\frac{1}{Q_{k+1}}}-\frac{p_{k-1}}{\frac{1}{Q_{k-1}}-\frac{1}{Q_k}}\\
&= \frac{p_k Q_k Q_{k+1}}{q_{k+1}}-\frac{p_{k-1} Q_k Q_{k-1}}{q_k}
\end{aligned}
$$

so that (6) can be written as

$$
\sup_n \frac{1}{|P_n|}\left[\sum_{k=1}^{n}\left|\frac{1}{Q_k}\left\{\frac{p_k Q_k Q_{k+1}}{q_{k+1}}-\frac{p_{k-1} Q_k Q_{k-1}}{q_k}\right\}\right|\right] < \infty,
$$

$$
i.e., \sup_n \frac{1}{|P_n|}\left[\sum_{k=1}^{n}\left|\frac{p_k Q_{k+1}}{q_{k+1}}-\frac{p_{k-1} Q_{k-1}}{q_k}\right|\right] < \infty,
$$

completing the proof of the theorem. ∎

Important Remark 3.2 The result of Hardy [1] and that of Móricz and Rhoades [2] are particular cases of Theorem 3.1. In this context, it is important to note that the result of Móricz and Rhoades is valid even without the assumption $\frac{p_n}{P_n} \to 0$, $n \to \infty$.

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